Completing the square

In this unit we consider how quadratic expressions can be written in an equivalent form using the technique known as completing the square. This technique has applications in a number of areas, but we will see an example of its use in solving a quadratic equation.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- write a quadratic expression as a complete square, plus or minus a constant
- solve a quadratic equation by completing the square

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1. Introduction

In this unit we look at a process called **completing the square**. It can be used to write a quadratic expression in an alternative form. Later in the unit we will see how it can be used to solve a quadratic equation.

2. Some simple equations

Example

Consider the quadratic equation $x^2 = 9$. We can solve this by taking the square root of both sides:

$$x = 3 \quad \text{or} \quad -3$$

remembering that when we take the square root there will be two possible answers, one positive and one negative. This is often written in the briefer form $x = \pm 3$.

This process for solving $x^2 = 9$ is very straightforward, particularly because:

- 9 is a 'square number', or 'complete square'. This means that it is the result of squaring another number, or term, in this case the result of squaring 3 or -3.
- x^2 is a complete square - it is the result of squaring x.

So simply square-rooting both sides solves the problem.

Example

Consider the equation $x^2 = 5$.

Again, we can solve this by taking the square root of both sides:

$$x = \sqrt{5} \quad \text{or} \quad -\sqrt{5}$$

In this example, the right-hand side of $x^2 = 5$, is not a square number. But we can still solve the equation in the same way. It is usually better to leave your answer in this **exact** form, rather than use a calculator to give a decimal approximation.

Example

Suppose we wish to solve the equation

$$(x-7)^2 = 3$$

Again, we can solve this by taking the square root of both sides. The left-hand side is a complete square because it results from squaring $x - 7$.

$$x - 7 = \sqrt{3} \quad \text{or} \quad -\sqrt{3}$$

By adding 7 to each side we can obtain the values for x:

$$x = 7 + \sqrt{3} \quad \text{or} \quad 7 - \sqrt{3}$$

We could write this in the briefer form $x = 7 \pm \sqrt{3}$.

Example

Suppose we wish to solve

$$(x+3)^2 = 5$$

Again the left-hand side is a complete square. Taking the square root of both sides:

$$x+3 = \sqrt{5} \quad \text{or} \quad -\sqrt{5}$$

By subtracting 3 from each side we can obtain the values for x:

$$x = -3+\sqrt{5} \quad \text{or} \quad -3-\sqrt{5}$$

Exercises

1. Solve the following quadratic equations

a)	$x^2 = 25$	b)	$x^2 = 10$	c)	$x^2 = 2$	d)	$(x+1)^2 = 9$
e)	$(x+3)^2 = 16$	f)	$(x-2)^2 = 100$	g)	$(x-1)^2 = 5$	h)	$(x+4)^2 = 2$

3. The basic technique

Now suppose we wanted to try to apply the method used in the three previous examples to

$$x^2 + 6x = 4$$

In each of the previous examples, the left-hand side was a **complete square**. This means that in each case it took the form $(x+a)^2$ or $(x-a)^2$. This is not the case now and so we cannot just take the square-root. What we try to do instead is rewrite the expression so that it becomes a complete square - hence the name **completing the square**.

Observe that complete squares such as $(x+a)^2$ or $(x-a)^2$ can be expanded as follows:



Key Point

complete squares:

$$\begin{aligned}(x+a)^2 &= (x+a)(x+a)\\ &= x^2+2ax+a^2\end{aligned}$$

$$\begin{aligned}(x-a)^2 &= (x-a)(x-a)\\ &= x^2-2ax+a^2\end{aligned}$$

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We will use these expansions to help us to complete the square in the following examples.

Example

Consider the quadratic expression

$$x^2 + 6x - 4$$

We compare this with the complete square

$$x^2 + 2ax + a^2$$

Clearly the coefficients of x^2 in both expressions are the same.

We would like to match up the term $2ax$ with the term $6x$. To do this note that $2a$ must be 6, so that $a = 3$.
Recall that

$$(x + a)^2 = x^2 + 2ax + a^2$$

Then with $a = 3$

$$(x + 3)^2 = x^2 + 6x + 9$$

This means that when trying to complete the square for $x^2 + 6x - 4$ we can replace the first two terms, $x^2 + 6x$, by $(x + 3)^2 - 9$. So

$$\begin{aligned} x^2 + 6x - 4 &= (x + 3)^2 - 9 - 4 \\ &= (x + 3)^2 - 13 \end{aligned}$$

We have now written the expression $x^2 + 6x - 4$ as a complete square plus or minus a constant. We have **completed the square**. It is important to note that the constant term, 3, in brackets is half the coefficient of x in the original expression.

Example

Suppose we wish to complete the square for the quadratic expression $x^2 - 8x + 7$.

We want to try to rewrite this so that it takes the form of a complete square plus or minus a constant.

We compare

$$x^2 - 8x + 7 \qquad \text{with the standard form} \qquad x^2 - 2ax + a^2$$

The coefficients of x^2 are the same. To make the coefficients of x the same we must choose a to be 4. Recall that

$$(x - a)^2 = x^2 - 2ax + a^2$$

Then with $a = 4$

$$(x - 4)^2 = x^2 - 8x + 16$$

This means that when trying to complete the square for $x^2 - 8x + 7$ we can replace the first two terms, $x^2 - 8x$, by $(x - 4)^2 - 16$.

So

$$\begin{aligned} x^2 - 8x + 7 &= (x - 4)^2 - 16 + 7 \\ &= (x - 4)^2 - 9 \end{aligned}$$

We have now written the expression $x^2 - 8x + 7$ as a complete square plus or minus a constant. We have completed the square. Again note that the constant term, -4, in brackets is half the coefficient of x in the original expression.

Example

Suppose we wish to complete the square for the quadratic expression $x^2 + 5x + 3$.

This means we want to try to rewrite it so that it has the form of a complete square plus or minus a constant. In the examples we have just worked through we have seen how this can be done by comparing with the standard forms $(x + a)^2$ and $(x - a)^2$. We would like to be able to 'complete the square' without writing down all the working we did in the previous examples. The key point to remember is that the number in the bracket of the complete square is half the coefficient of x in the quadratic expression.

So with $x^2 + 5x + 3$ we know that the complete square will be $\left(x + \frac{5}{2}\right)^2$. This has the same x^2 and x terms as the given quadratic expression but the constant term is different. We must balance the constant term by a) subtracting the extra constant that our complete square has introduced, that is $\left(\frac{5}{2}\right)^2$, and b) putting in the constant term from our quadratic, that is 3.

Putting this together we have

$$x^2 + 5x + 3 = \left(x + \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 3 \qquad (1)$$

To finish off we just combine the two constants

$$-\left(\frac{5}{2}\right)^2 + 3 = -\frac{25}{4} + \frac{12}{4} = -\frac{13}{4}$$

and so

$$x^2 + 5x + 3 = \left(x + \frac{5}{2}\right)^2 - \frac{13}{4}$$

We have now written the expression $x^2 + 5x + 3$ as a complete square plus or minus a constant. We have completed the square. Again note that the constant term, $\frac{5}{2}$, in brackets is half the coefficient of x in the original expression.

The explanation given above is really just an outline of our thought process; when we complete the square in practice we would not write it all down. We would probably go straight to equation (1). The ability to do this will come with practice.

Exercises

2 Completing the square for the following quadratic expressions

a) $x^2 + 2x + 2$ b) $x^2 + 2x + 5$ c) $x^2 + 2x - 1$ d) $x^2 + 6x + 8$
e) $x^2 - 6x + 8$ f) $x^2 + x + 1$ g) $x^2 - x - 1$ h) $x^2 + 10x - 1$
i) $x^2 + 5x + 4$ j) $x^2 + 6x + 9$ k) $x^2 - 2x + 6$ l) $x^2 - 3x + 1$

4. Cases in which the coefficient of x^2 is not 1.

We now know how to complete the square for quadratic expressions for which the coefficient of x^2 is 1. When faced with a quadratic expression where the coefficient of x^2 is not 1 we can still use this technique but we put in an extra step first - we factor out this coefficient.

Suppose we wish to complete the square for the expression $3x^2 - 9x + 50$.

We begin by factoring out the coefficient of x^2, in this case 3. It does not matter that 3 is not a factor of 50; we can still do this by writing the expression as

$$3\left(x^2 - 3x + \frac{50}{3}\right)$$

Now the expression in brackets is a quadratic with coefficient of x^2 equal to 1 and so we can proceed as before. The number in the complete square will be half the coefficient of x, so we will use $\left(x - \frac{3}{2}\right)^2$. Then we must balance up the constant term just as we did before by subtracting the extra constant we have introduced, that is $\left(\frac{3}{2}\right)^2$, and putting in the constant from the quadratic expression, that is $\frac{50}{3}$.

$$3\left\{x^2 - 3x + \frac{50}{3}\right\} = 3\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{50}{3}\right\}$$

The arithmetic to tidy up the constants is a bit messy:

$$\begin{aligned} -\left(\frac{3}{2}\right)^2 + \frac{50}{3} &= -\frac{9}{4} + \frac{50}{3} \\ &= -\frac{27}{12} + \frac{200}{12} \\ &= \frac{173}{12} \end{aligned}$$

So putting all this together

$$x^2 - 3x + \frac{50}{3} = \left(x - \frac{3}{2}\right)^2 + \frac{173}{12}$$

and finally

$$3\left(x^2 - 3x + \frac{50}{3}\right) = 3\left(\left(x - \frac{3}{2}\right)^2 + \frac{173}{12}\right)$$

and we have completed the square.

This is the 'completing the square' form for a quadratic expression for which the coefficient of x^2 is not 1.

Exercises

3 Completing the square for the following quadratic expressions

a) $2x^2 + 4x - 8$ b) $5x^2 + 10x + 15$ c) $3x^2 - 27x + 9$ d) $2x^2 + 6x + 1$
e) $3x^2 - 12x + 2$ f) $15 - 10x - x^2$ g) $24 + 12x - 2x^2$ h) $9 + 6x - 3x^2$

5. Summary of the process

It will be useful if you can get used to doing this process automatically. The method can be summarised as follows:



Key Point

1. factor out the coefficient of x^2 - then work with the quadratic expression which has a coefficient of x^2 equal to 1
2. check the coefficient of x in the new quadratic expression and take half of it - this is the number that goes into the complete square bracket
3. balance the constant term by subtracting the square of the number from step 2, and putting in the constant from the quadratic expression
4. the rest is arithmetic that may often involve fractions

6. Solving a quadratic equation by completing the square

Let us return now to a problem posed earlier. We want to solve the equation $x^2 + 6x = 4$.

We write this as $x^2 + 6x - 4 = 0$. Note that the coefficient of x^2 is 1 so there is no need to take out any common factor.
Completing the square for quadratic expression on the left-hand side:

$$\begin{aligned} x^2 + 6x - 4 &= 0 \\ (x+3)^2 - 9 - 4 &= 0 \quad (1) \\ (x+3)^2 - 13 &= 0 \quad (2) \\ (x+3)^2 &= 13 \\ x + 3 &= \pm\sqrt{13} \\ x &= -3 \pm \sqrt{13} \end{aligned}$$

We have solved the quadratic equation by completing the square.

To produce equation (1) we have noted that the coefficient of x in the quadratic expression is 6 so the number in the 'complete square' bracket must be 3; then we have balanced the constant by subtracting the square of this number, 3^2, and putting in the constant from the quadratic, -4. To get equation (2) we just do the arithmetic which in this example is quite straightforward.

Exercises

4 Use completing the square to solve the following quadratic equations

a) $x^2 + 4x - 12 = 0$ b) $x^2 + 5x - 6 = 0$ c) $10x^2 + 7x - 12 = 0$
d) $x^2 + 4x - 8 = 0$ e) $10 + 6x - x^2 = 0$ f) $2x^2 + 8x - 25 = 0$

Give your answers either as fractions or in the form $p \pm \sqrt{q}$

Answers

1. a) ± 5 b) $\pm\sqrt{10}$ c) $\pm\sqrt{2}$ d) 2, -4
e) 1, -7 f) 12, -8 g) $1 \pm \sqrt{5}$ h) $-4 \pm \sqrt{2}$

2. a) $(x+1)^2 + 1$ b) $(x+1)^2 + 4$ c) $(x+1)^2 - 2$ d) $(x+3)^2 - 1$
e) $(x-3)^2 - 1$ f) $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ g) $\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}$ h) $(x+5)^2 - 26$
i) $\left(x + \frac{5}{2}\right)^2 - \frac{9}{4}$ j) $(x+3)^2$ k) $(x-1)^2 + 5$ l) $\left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

3. a) $2\left[(x+1)^2 - 5\right]$ b) $5\left[(x+1)^2 + 2\right]$ c) $3\left[\left(x - \frac{9}{2}\right)^2 - \frac{69}{4}\right]$
d) $2\left[\left(x + \frac{3}{2}\right)^2 - \frac{7}{4}\right]$ e) $3\left[(x-2)^2 - \frac{10}{3}\right]$ f) $-\left[(x+5)^2 - 40\right]$
g) $-2\left[(x-3)^2 - 21\right]$ h) $-3\left[(x-1)^2 - 4\right]$

4. a) 2, -6 b) 1, -6 c) $-\frac{3}{2}, \frac{4}{5}$
d) $-2 \pm \sqrt{12}$ e) $3 \pm \sqrt{19}$ f) $-2 \pm \sqrt{\frac{33}{2}}$