

Fractions: multiplying and dividing

In this unit we shall see how to multiply fractions. We shall also see how to divide fractions by turning the second fraction upside down.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

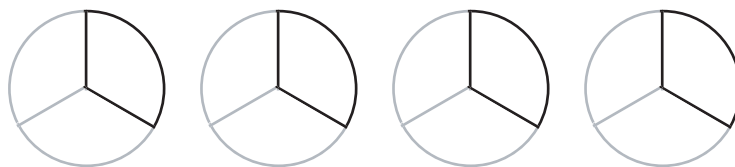
- multiply fractions, including improper fractions and mixed fractions;
- divide fractions, including improper fractions and mixed fractions.

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1. Multiplying fractions

How do we multiply fractions? Let us start with an example. What is $4 \times \frac{1}{3}$? This means 4 lots of one third.



Numerically, we perform the calculation like this:

$$4 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3} = 1\frac{1}{3}.$$

The concept is the same as when we multiply whole numbers, for instance

$$4 \times 5 = 5 + 5 + 5 + 5 = 20.$$

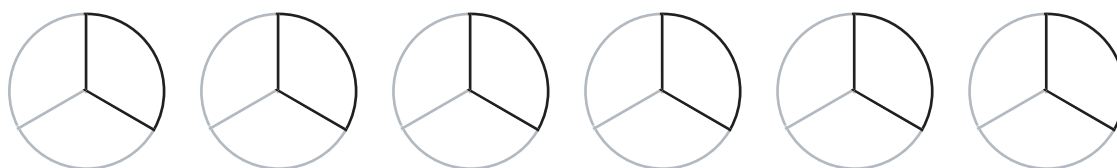
Now when we multiply numbers, we often use the fact that multiplication is ‘commutative’. This means that, for example, $4 \times 5 = 5 \times 4$, so that

$$\begin{aligned} 5 + 5 + 5 + 5 &= 4 + 4 + 4 + 4 + 4, \\ 20 &= 20. \end{aligned}$$

Let us see what this means when we are using fractions. We expect to find that $6 \times \frac{1}{3} = \frac{1}{3} \times 6$. Now

$$6 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3},$$

and $(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3})$ equals $1 + 1 = 2$.



On the other hand, $\frac{1}{3} \times 6$ means one-third of 6, and that is just $6 \div 3$, so that we have 6 wholes split into 3 equal parts, giving us 2 again.



So taking a fraction a whole number of times is the same as taking a fraction of a whole number.

Example

Calculate $5 \times \frac{2}{3}$.

Solution

We obtain

$$5 \times \frac{2}{3} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{10}{3} = 3\frac{1}{3}.$$

This is just the same as two-thirds of 5. Now one-third of 5 is $\frac{5}{3}$, so two-thirds of 5 is $2 \times \frac{5}{3}$ which is $\frac{10}{3}$.

Any whole number can be written as a fraction. For example, 2 can be written as $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, and so on. Any numbers can be used, as long as the numerator is twice the denominator. So, for example,

$$2 \times \frac{3}{4} = \frac{2}{1} \times \frac{3}{4} = \frac{2 \times 3}{1 \times 4} = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}.$$

Example

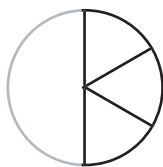
Calculate $7 \times \frac{5}{9}$.

Solution

We obtain

$$7 \times \frac{5}{9} = \frac{7}{1} \times \frac{5}{9} = \frac{7 \times 5}{1 \times 9} = \frac{35}{9} = 3\frac{8}{9}.$$

Now let us look at multiplying fractions by other fractions. For example, what is $\frac{1}{3} \times \frac{1}{2}$? This means one third of one half.



If we take a half and split it into 3, we have $\frac{1}{6}$ of a whole. We can see this if we also divide the other half into 3 pieces. Numerically, we have

$$\frac{1}{3} \times \frac{1}{2} = \frac{1 \times 1}{3 \times 2} = \frac{1}{6}.$$

As you can see, we have obtained the answer $\frac{1}{6}$ by first multiplying together the two numerators to give the numerator of the answer. We have then multiplied together the two denominators to give the denominator of the answer. This is the general technique we shall use.

Let us take another example. What is $\frac{1}{3} \times \frac{2}{5}$? This means that we want to start with two-fifths of a whole, and then take one third of that.



To split $\frac{2}{5}$ into 3 parts, it is easier to split each fifth into 3 parts and take one from each. Splitting each fifth into 3 gives us 15 pieces, so we have $\frac{1}{15}$ from each one fifth section, giving $\frac{2}{15}$:

$$\frac{1}{3} \times \frac{2}{5} = \frac{1 \times 2}{3 \times 5} = \frac{2}{15}.$$

So, as before, we are multiplying the numerators together and then multiplying the denominators together.

Example

Calculate $\frac{2}{5} \times \frac{4}{9}$.

Solution

We obtain

$$\frac{2}{5} \times \frac{4}{9} = \frac{2 \times 4}{5 \times 9} = \frac{8}{45}.$$

Example

Calculate $\frac{2}{3} \times \frac{4}{5}$.

Solution

We obtain

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}.$$

Example

Calculate $\frac{2}{3} \times \frac{9}{10}$.

Solution

We obtain

$$\frac{2}{3} \times \frac{9}{10} = \frac{2 \times 9}{3 \times 10} = \frac{18}{30}.$$

Now in this example we can simplify the result by cancelling, and we would get $\frac{3}{5}$ as our final result. But often it is easier to cancel as we go along. If we do that, we get

$$\frac{2}{3} \times \frac{9}{10} = \frac{2 \times 9}{3 \times 10} = \frac{1 \times 3}{1 \times 5} = \frac{3}{5}.$$

The process is exactly the same if we wish to multiply three fractions rather than two.

Example

Calculate $\frac{1}{2} \times \frac{3}{4} \times \frac{2}{3}$.

Solution

We obtain

$$\frac{1}{2} \times \frac{3}{4} \times \frac{2}{3} = \frac{1 \times 3 \times 2}{2 \times 4 \times 3} = \frac{1 \times 1 \times 1}{1 \times 4 \times 1} = \frac{1}{4}.$$

What happens if we have mixed fractions?

Just as before when dealing with mixed fractions, we turn them into improper fractions first.

Example

Calculate $2\frac{1}{3} \times \frac{3}{4}$.

Solution

We obtain

$$2\frac{1}{3} \times \frac{3}{4} = \frac{2 \times 3 + 1}{3} \times \frac{3}{4} = \frac{7 \times 3}{3 \times 4} = \frac{7 \times 1}{1 \times 4} = \frac{7}{4} = 1\frac{3}{4}.$$

Example

Calculate $1\frac{2}{5} \times 2\frac{5}{6}$.

Solution

We obtain

$$1\frac{2}{5} \times 2\frac{5}{6} = \frac{1 \times 5 + 2}{5} \times \frac{2 \times 6 + 5}{6} = \frac{7 \times 17}{5 \times 6} = \frac{119}{30} = 3\frac{29}{30}.$$



Key Point

To multiply fractions, multiply the numerators together and multiply the denominators together separately. To multiply mixed fractions, turn them into improper fractions first.

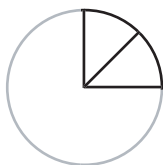
Exercises

1. Perform the following multiplications:

- (a) $6 \times \frac{2}{5}$ (b) $2 \times \frac{5}{9}$ (c) $4 \times \frac{3}{10}$ (d) $\frac{3}{4} \times \frac{7}{11}$
(e) $\frac{3}{5} \times \frac{10}{12}$ (f) $2\frac{1}{5} \times 3\frac{2}{3}$

2. Dividing fractions

We shall now look at what happens when we divide fractions. Let us take $\frac{1}{4}$ and divide it by 2.



$$\frac{1}{4} \div 2 = \frac{1}{8}$$

Dividing by 2 is the same as taking a half, so dividing by 2 and multiplying by a half are the same thing. We could write

$$\frac{1}{4} \div 2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}.$$

What about $\frac{1}{3}$ divided by 4? We would have

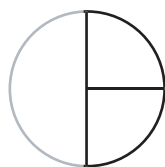
$$\frac{1}{3} \div 4 = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}.$$

And again, for $\frac{1}{2} \div 2$ we would have

$$\frac{1}{2} \div 2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

In all these cases it looks as if, to divide by a number, we can instead multiply the denominator by that number. Another way of saying this is that we write the divisor, that is the number we are dividing by, as a fraction. When we have done this, we turn that fraction upside-down and multiply instead.

Now we can extend this approach to divide by numbers which are themselves fractions rather than whole numbers. For example, what is $\frac{1}{2} \div \frac{1}{4}$? In other words, how many times does a quarter go into a half?



Following the rule, we obtain

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{1 \times 4}{2 \times 1} = \frac{4}{2} = 2$$

which makes sense since $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $\frac{1}{4}$ goes into $\frac{1}{2}$ twice.

Example

Calculate $\frac{1}{3} \div \frac{1}{5}$.

Solution

We obtain

$$\frac{1}{3} \div \frac{1}{5} = \frac{1}{3} \times \frac{5}{1} = \frac{1 \times 5}{3 \times 1} = \frac{5}{3} = 1\frac{2}{3}.$$

Example

Calculate $2 \div \frac{1}{8}$.

Solution

We obtain

$$2 \div \frac{1}{8} = 2 \times \frac{8}{1} = 2 \times 8 = 16.$$

Example

Calculate $4 \div \frac{1}{3}$.

Solution

We obtain

$$4 \div \frac{1}{3} = 4 \times \frac{3}{1} = \frac{12}{1} = 12.$$

So far, when dividing we have only looked at fractions with a numerator of 1. Let us now look at other fractions. What is $\frac{3}{4} \div 2$? Using the rule, we obtain

$$\frac{3}{4} \div 2 = \frac{3}{4} \div \frac{2}{1} = \frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{4 \times 2} = \frac{3}{8}.$$

So again the process is turn the second fraction, the divisor, upside down, and then to multiply.

Example

Calculate $\frac{3}{5} \div 4$.

Solution

We obtain

$$\frac{3}{5} \div 4 = \frac{3}{5} \div \frac{4}{1} = \frac{3}{5} \times \frac{1}{4} = \frac{3 \times 1}{5 \times 4} = \frac{3}{20}.$$

Example

Calculate $\frac{2}{3} \div \frac{3}{4}$.

Solution

How many times does $\frac{3}{4}$ fit into $\frac{2}{3}$? Since $\frac{3}{4}$ is larger than $\frac{2}{3}$ we should expect the answer to be less than 1. Following the rule, we obtain

$$\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{2 \times 4}{3 \times 3} = \frac{8}{9}.$$

Finally we need to look at how to deal with mixed fractions when dividing.

Example

Calculate $1\frac{2}{3} \div 2\frac{1}{4}$.

Solution

As before, we need to convert mixed fractions to improper fractions before dividing:

$$1\frac{2}{3} \div 2\frac{1}{4} = \frac{1 \times 3 + 2}{3} \div \frac{2 \times 4 + 1}{4} = \frac{5}{3} \div \frac{9}{4}.$$

Now we continue as before by turning the divisor upside down, and multiplying:

$$\frac{5}{3} \div \frac{9}{4} = \frac{5}{3} \times \frac{4}{9} = \frac{20}{27}.$$

Example

Calculate $2\frac{4}{5} \div 4\frac{2}{3}$.

Solution

$$2\frac{4}{5} \div 4\frac{2}{3} = \frac{2 \times 5 + 4}{5} \div \frac{4 \times 3 + 2}{3} = \frac{14}{5} \div \frac{14}{3} = \frac{14}{5} \times \frac{3}{14} = \frac{3}{5}.$$



Key Point

To divide fractions, turn the second fraction upside-down and multiply. To divide mixed fractions, turn them into improper fractions first.

Exercises

2. Perform the following divisions:

(a) $\frac{1}{5} \div 2$ (b) $\frac{1}{6} \div 4$ (c) $\frac{1}{4} \div \frac{1}{2}$ (d) $\frac{1}{3} \div \frac{1}{3}$

(e) $2 \div \frac{1}{5}$ (f) $3 \div \frac{1}{4}$ (g) $\frac{2}{5} \div 3$ (h) $\frac{3}{4} \div 2$

(i) $\frac{3}{4} \div \frac{2}{3}$ (j) $\frac{3}{5} \div \frac{1}{2}$ (k) $4 \div \frac{2}{5}$ (l) $5 \div \frac{2}{3}$

3. Perform the following divisions:

(a) $2\frac{1}{10} \div 1\frac{1}{8}$ (b) $5\frac{1}{4} \div \frac{3}{8}$ (c) $1\frac{1}{3} \div 3\frac{1}{4}$

Answers

1.

(a) $\frac{12}{5}$ or $2\frac{2}{5}$ (b) $\frac{10}{9}$ or $1\frac{1}{9}$ (c) $\frac{6}{5}$ or $1\frac{1}{5}$ (d) $\frac{21}{44}$

(e) $\frac{1}{2}$ (f) $\frac{121}{15}$ or $8\frac{1}{15}$

2.

(a) $\frac{1}{10}$ (b) $\frac{1}{24}$ (c) $\frac{1}{2}$ (d) 1

(e) 10 (f) 12 (g) $\frac{2}{15}$ (h) $\frac{3}{8}$

(i) $\frac{9}{8}$ or $1\frac{1}{8}$ (j) $\frac{6}{5}$ or $1\frac{1}{5}$ (k) 10 (l) $\frac{15}{2}$ or $7\frac{1}{2}$

3.

(a) $\frac{28}{15}$ or $1\frac{13}{15}$ (b) 14 (c) $\frac{16}{39}$