mathcentre

Triangle formulae

mc-TY-triangleformulae-2009-1

A common mathematical problem is to find the angles or lengths of the sides of a triangle when some, but not all of these quantities are known. It is also useful to be able to calculate the area of a triangle from some of this information. In this unit we will illustrate several formulae for doing this.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve triangles using the cosine formulae
- solve triangles using the sine formulae
- find areas of triangles

Contents

1. Introduction

Consider a triangle such as that shown in Figure 1.

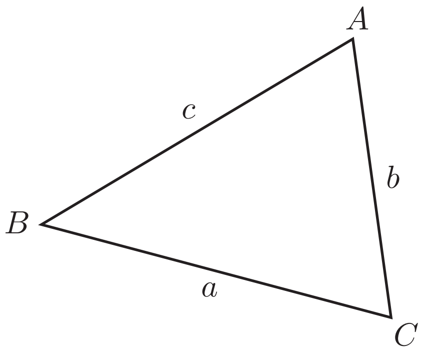

Figure 1. A triangle with six pieces of information: angles at A, B, and C; sides a, b and c.

There are six pieces of information available: angles at A, B and C, and the sides a, b and c. The angle at A is usually written A, and so on. Notice that we label the sides according to the following convention:

side b is opposite the angle B

side c is opposite the angle C

side a is opposite the angle A

Now if we take three of these six pieces of information we will (except in two special cases) be able to draw a unique triangle.

Let's deal first with the special cases.

The first special case

The first special case is when we know just the three angles. Then having drawn one triangle with these angles, we can draw as many more triangles as we wish, all with the same shape as the original, but larger or smaller. All will have the same angles but the sizes of the triangles will be different. We cannot define a unique triangle when we know just the three angles. This behaviour is illustrated in Figure 2 where the corresponding angles in the two triangles are the same, but clearly the triangles are of different sizes.

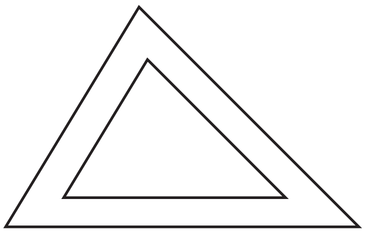

Figure 2. Given just the three angles we cannot construct a unique triangle.

The second special case

There is a second special case whereby if we are given three pieces of information it is impossible to construct a unique triangle. Suppose we are given one angle, A say, and the lengths of two of the sides. This situation is illustrated in Figure 3 (a). The first given side is marked //. The second given side is marked /; this can be placed in two different locations as shown in Figures 3b) and 3c). Consequently it is impossible to construct a unique triangle.

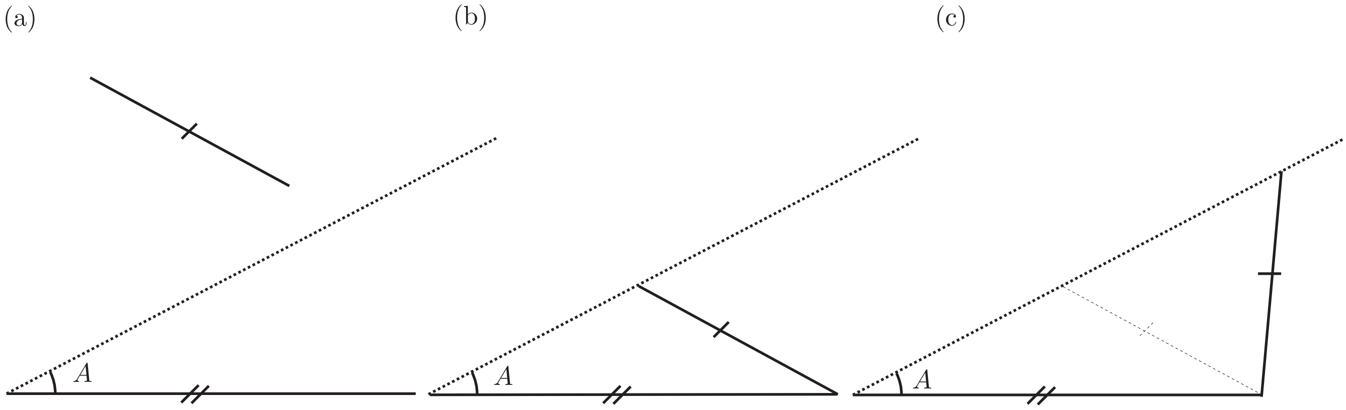

Figure 3. It is impossible to draw a unique triangle given one angle and two side lengths.

Apart from these two special cases, if we are given three pieces of information about the triangle we will be able to draw it uniquely. There are formulae for doing this which we describe in the following sections.

2. The cosine formulae

We can use the cosine formulae when three sides of the triangle are given.

Key Point

Cosine formulae

When given three sides, we can find angles from the following formulae:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The cosine formulae given above can be rearranged into the following forms:

Key Point

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

If we consider the formula $c^2 = a^2 + b^2 - 2ab\cos C$, and refer to Figure 4 we note that we can use it to find side c when we are given two sides (a and b) and the **included** angle C.

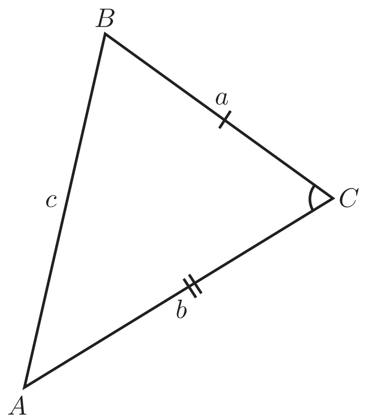

Figure 4. Using the cosine formulae to find c if we know sides a and b and the included angle C.

Similar observations can be made of the other two formulae.

So there are in fact six cosine formulae, one for each of the angles - that's three altogether, and one for each of the sides, that's another three. We only need to learn two of them, one for the angle, one for the side and then just cycle the letters through to find the others.

Exercise 1
Throughout all exercises the standard triangle notation (namely side a opposite angle A, etc.) is used.

1. Find the length of the third side, to 3 decimal places, and the other two angles, to 1 decimal place, in the following triangles

   (a) $a = 1,\ b = 2,\ C = 30°$

   (b) $a = 3,\ c = 4,\ B = 50°$

   (c) $b = 5,\ c = 10,\ A = 30°$

2. Find the angles (to 1 decimal place) in the following triangles

   (a) $a = 2,\ b = 3,\ c = 4$

   (b) $a = 1,\ b = 1,\ c = 1.5$

   (c) $a = 2,\ b = 2,\ c = 3$

3. The sine formulae

We can use the sine formulae to find a side, given two sides and an angle which is NOT included between the two given sides.

Key Point

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumcircle.

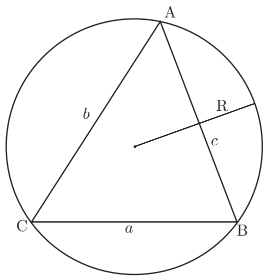

Figure 5. The circumcircle is the circle drawn through the three points of the triangle.

R is the radius of the circumcircle - the circumcircle is the circle that we can draw that will go through all the points of the triangle as shown in Figure 5.

Taking just the first three terms in the formulae we can rearrange them to give

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

and we can use the formulae in this form as well.

4. Some examples of the use of the cosine and sine formulae

Example

Suppose we are given all three sides of a triangle:

$$a = 5, \quad b = 7, \quad c = 10$$

We will use this information to determine angle A using the cosine formula:

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{7^2 + 10^2 - 5^2}{2 \times 7 \times 10} \\ &= \frac{49 + 100 - 25}{140} \\ &= \frac{124}{140} \\ A &= \cos^{-1}\frac{124}{140} = 27.7^\circ \qquad \text{(1 d.p.)}\end{aligned}$$

The remaining angles can be found by applying the other cosine formulae.

Example

Suppose we are given two sides of a triangle and an angle, as follows

$$b = 10, \quad c = 5, \quad A = 120^\circ$$

It's not immediately obvious what information we have been given. In the last Example it was very clear. So we make a sketch to mark out the information we have been given as shown in Figure 6.

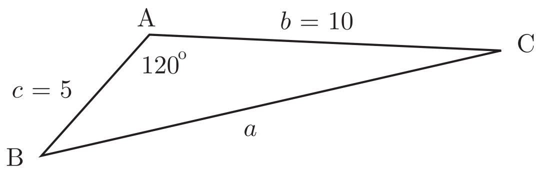

Figure 6. The information given in the Example.

From the Figure we can deduce that we have been given 2 sides and the included angle. We can use the cosine formula to deduce the length of side a.

$$\begin{aligned}a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 10^2 + 5^2 - 2 \times 10 \times 5\cos 120^\circ \\ &= 100 + 25 - 100\cos 120^\circ \\ &= 125 - 100 \times \left(-\frac{1}{2}\right) \\ &= 175 \\ a &= \sqrt{175} = 13.2 \qquad \text{(3 s.f.)}\end{aligned}$$

Now that we have worked out the length of side a, we have three sides. We could use the cosine formulae to find out either one of the remaining angles.

Example

Suppose we are given the following information:

$$c = 8, \quad b = 12, \quad C = 30°$$

Note that we are given two side lengths and an angle which is not the included angle. Referring back to the special cases described in the Introduction you will see that with this information there is the possibility that we can obtain two distinct triangles with this information.

As before we need a sketch in order to understand the information (Figure 7.)

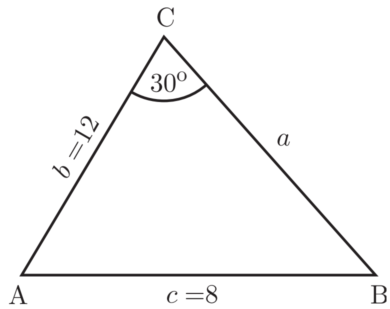

Figure 7. We are given two sides and a non-included angle.

Because we have been given two sides and a non-included angle we use the sine formulae.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Because we are given b, c and C we use the following part of the formula in order to find angle B.

$$\begin{aligned}
\frac{\sin B}{b} &= \frac{\sin C}{c} \\
\frac{\sin B}{12} &= \frac{\sin 30°}{8} \\
\sin B &= \frac{12 \times \sin 30°}{8} \\
&= \frac{12 \times \frac{1}{2}}{8} \\
&= \frac{6}{8} \\
&= \frac{3}{4} \\
&= 0.75 \\
B &= \sin^{-1} 0.75 \\
&= 48.6° \qquad (1 \text{ d.p.})
\end{aligned}$$

Now there is a potential complication here because there is another angle with sine equal to 0.75. Specifically, B could equal $180° - 48.6° = 131.4°$.

In the first case the angles of the triangle are then:

$C = 30°$, $\quad B = 48.6°$, $\quad A = 180° - 78.6° = 101.4°$

In the second case we have:

$C = 30°$, $\quad B = 131.4°$, $\quad A = 180° - 161.4° = 18.6°$.

The situation is depicted in Figure 8. In order to solve the triangle completely we must deal with the two cases separately in order to find the remaining unknown a.

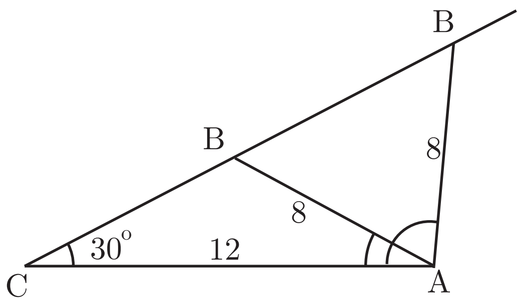

Figure 8. There are two possible triangles.

Case 1. Here $C = 30°$, $B = 48.6°$, $A = 101.4°$. We use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\begin{aligned} a &= \frac{12 \sin 101.4°}{\sin 48.6°} \\ &= 15.7 \quad \text{(1 d.p.)} \end{aligned}$$

Case 2. Here $C = 30°$, $B = 131.4°$, $A = 18.6°$. Again we can use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\begin{aligned} a &= \frac{12 \sin 18.6°}{\sin 131.4°} \\ &= 5.1 \quad \text{(1 d.p.)} \end{aligned}$$

Exercise 2

1. Find the lengths of the other two sides (to 3 decimal places) of the triangles with

   (a) $a = 2$, $A = 30°$, $B = 40°$

   (b) $b = 5$, $B = 45°$, $C = 60°$

   (c) $c = 3$, $A = 37°$, $B = 54°$

2. Find all possible triangles (give the sides to 3 decimal places and the angles to 1 decimal place) with

   (a) $a = 3,\ b = 5,\ A = 32°$

   (b) $b = 2,\ c = 4,\ C = 63°$

   (c) $c = 2,\ a = 1,\ B = 108°$

5. The area of a triangle

We now look at a set of formulae which will give us the area of a triangle. A standard formula is

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}$$

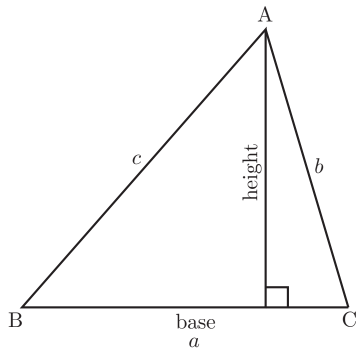

Figure 9. The area of the triangle is $\frac{1}{2}$ × base × height

Let us assume we know the lengths a, b and c, and the angle at B. Consider the right-angled triangle on the left-hand side of Figure 9. In this triangle

$$\sin B = \frac{\text{height}}{c}$$

and so, by rearranging,

$$\text{height} = c \sin B$$

Then from the formula for the area of the large triangle, $\triangle ABC$,

$$\begin{aligned} \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} ac \sin B \end{aligned}$$

Now consider the right-angled triangle on the right-hand side in Figure 9.

$$\sin C = \frac{\text{height}}{b}$$

and so, by rearranging,

$$\text{height} = b \sin C$$

So, the area of the large triangle, $\triangle ABC$, is also given by

$$\text{area} = \frac{1}{2} ab \sin C$$

It is also possible to show that the formula

$$\text{area} = \frac{1}{2} bc \sin A$$

will also give the area of the large triangle.

Key Point

When we are given two sides and the included angle, the area of the triangle can be found from one of the three formulae:

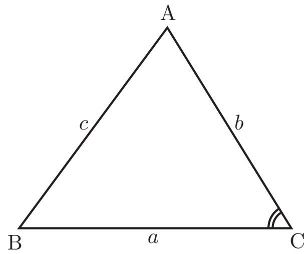

Figure 10.

$$\begin{aligned}\text{area} &= \frac{1}{2}ab\sin C\\ &= \frac{1}{2}bc\sin A\\ &= \frac{1}{2}ca\sin B\end{aligned}$$

These formulae do not work if we are not given an angle. An ancient Greek by the name of Hero (or Heron) derived a formula for calculating the area of a triangle when we know all three sides.

Key Point

Hero's formula:

$$\begin{aligned}\text{area} &= \sqrt{s(s-a)(s-b)(s-c)}\\ \text{where}\qquad s &= \frac{a+b+c}{2} = \text{semi-perimeter}\end{aligned}$$

The semi-perimeter, as the name implies, is half of the perimeter of the triangle.

Example

Suppose we are given the lengths of three sides of a triangle:

$$a = 5 \quad b = 7 \quad c = 10$$

We can use Hero's formula:

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$\begin{aligned} s &= \frac{a+b+c}{2} \\ &= \frac{5+7+10}{2} \\ &= 11 \end{aligned}$$

Then

$$\begin{aligned} \text{area} &= \sqrt{11(11-5)(11-7)(11-10)} \\ &= \sqrt{11 \times 6 \times 4 \times 1} \\ &= \sqrt{264} \\ &= 16.2 \quad (3 \text{ s.f.}) \end{aligned}$$

So the area is 16.2 square units.

Example

Suppose we wish to find the area of a triangle given the following information:

$$b = 10 \quad c = 5 \quad A = 120°$$

A sketch illustrates this information.

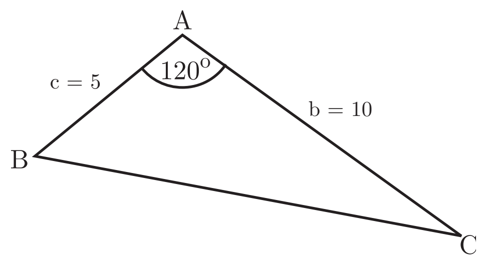

Figure 11. We are given two sides and the included angle.

We are given two sides and the included angle.

$$\begin{aligned} \text{area} &= \frac{1}{2} bc \sin A \\ &= \frac{1}{2} \times 10 \times 5 \times \sin 120 \\ &= 25 \sin 120 \\ &= 21.7 \quad (3 \text{ s.f.}) \end{aligned}$$

So the area is 21.7 square units.

Exercise 3

1. Find the areas of each of the triangles (to 3 decimal places) in Exercise 1, Question 1.
2. Find the areas of each of the triangles (to 3 decimal places) in Exercise 1, Question 2.

6. Summary

Cosine Formulae

for finding an angle using the three sides:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

for finding a side using two sides and the included angle

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Sine Formulae

Use when you are given two sides and the non-included angle, or two angles and a side:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Formulae for the area of a triangle

$$\begin{aligned} \text{area} &= \frac{1}{2}ab \sin C \\ &= \frac{1}{2}bc \sin A \\ &= \frac{1}{2}ca \sin B \end{aligned}$$

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$s = \frac{a+b+c}{2} = \text{ semi-perimeter}$$

Exercise 4

1. Determine the lengths of all the sides (to 3 decimal places), the sizes of all the angles (to 1 decimal place) and the area (to 3 decimal places) of each of the following triangles.

   (a) $a = 5$, $b = 3$, $c = 6$

   (b) $a = 5$, $b = 2$, $C = 42°$

   (c) $c = 3$, $A = 40°$, $B = 60°$

   (d) $b = 2$, $A = 73°$, $B = 41°$

   (e) $a = 5$, $b = 3$, $c = 4$

   (f) $c = 2$, $b = 5$, $B = 78°$

Answers

Exercise 1
1. (a) $c = 1.239$, $A = 23.8°$, $B = 126.2°$ (b) $b = 3.094$, $A = 48.0°$, $C = 82.0°$
(c) $a = 6.197$, $B = 23.8°$, $C = 126.2°$.

2. (a) $A = 29.0°$, $B = 46.6°$, $C = 104.5°$ (b) $A = 41.4°$, $B = 41.4°$, $C = 97.2°$
(c) $A = 41.4°$, $B = 41.4°$, $C = 97.2°$

Exercise 2
1. (a) $b = 2.571$, $c = 3.759$ (b) $a = 7.044$, $c = 6.124$ (c) $a = 1.806$, $b = 2.427$

2. (a) Two possible triangles:
$B = 62.0°$, $C = 86.0°$, $c = 5.647$ and $B = 118.0°$, $C = 30.0°$, $c = 2.833$
(b) $B = 27.0°$, $A = 88.0°$, $a = 4.487$ (c) $b = 2.457$, $A = 23.2°$, $C = 51.9°$

Exercise 3
1. (a) 0.5 (b) 0.587 (c) 12.5
2. (a) 2.905 (b) 0.496 (c) 1.984

Exercise 4
1. (a) $A = 56.3°$, $B = 29.9°$, $C = 93.8°$, Area = 7.483
(b) $c = 3.760$, $A = 117.2°$, $B = 20.9°$, Area = 3.346
(c) $a = 1.958$, $b = 2.638$, $C = 80°$, Area = 2.544
(d) $a = 2.915$, $c = 2.785$, $C = 66°$, Area = 2.663
(e) $A = 90°$, $B = 36.9°$, $C = 53.1°$, Area = 6
(f) $a = 5.017$, $A = 79.0°$, $C = 23.0°$, Area = 4.908