# Second Order Ordinary Differential Equations

mccp-richard-2

## Introduction

**Prerequisites:** In order to make the most of this resource, you need to know about trigonometry, differentiation, integration and complex numbers.

We are looking at equations involving a function $y(x)$, its first derivative and second derivative:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \qquad (1)$$

We will only look at equations where the coefficients $a$, $b$ and $c$ are constant; we will not treat in this handout the case of coefficients which are functions of $x$.

## Homogeneous Equations

If $f(x) = 0$ then the ODE is called an homogeneous equation. To solve a second order homogeneous ODE, we look at the **characteristic equation**, obtained by replacing $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and $y$ by $r^2$, $r$ and $1$ in the ODE:

$$ar^2 + br + c = 0$$

We distinguish between 3 cases: the case when the roots of the characteristic equation are distinct and real, complex or equal.

**Case 1: real and distinct roots $r_1$ and $r_2$**

Then the solutions of the homogeneous equation are of the form:

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}$$

The constants $A$ and $B$ can be anything you like if there are no boundary conditions. If you have boundary conditions, e.g. you know that $y(x_0) = \alpha$ and $y'(x_0) = \beta$, then $A$ and $B$ will be uniquely defined by:

$$\begin{aligned} Ae^{r_1 x_0} + Be^{r_2 x_0} &= \alpha \\ Ar_1 e^{r_1 x_0} + Br_2 e^{r_2 x_0} &= \beta \end{aligned}$$

**Example**

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

The characteristic equation is: $r^2 + 5r + 6 = 0$ and the roots are $\frac{-5 \pm \sqrt{25 - 4 \times 6}}{2} = -3$ or $-2$. Therefore the solutions of the ODE are:

$$y(x) = Ae^{-3x} + Be^{-2x}$$



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**Case 2: complex roots**

If the roots are complex then they can be written as $r + js$ and $r - js$ (with j the imaginary number, $j^2 = -1$) and the solutions of the homogeneous equations are of the form:

$$\begin{aligned} y(x) &= e^{rx}\left(Ae^{jsx} + Be^{-jsx}\right) \\ \text{which can also be written as} &= e^{rx}\left(C\cos(sx) + D\sin(sx)\right) \end{aligned}$$



As before, the constants A and B (or C and D) will be defined by the boundary conditions.

**Example**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y = 0$$

The characteristic equation is: $r^2 + 4r + 9 = 0$ and the roots are $\dfrac{-4 \pm \sqrt{16 - 4 \times 9}}{2} = -2 + \sqrt{5}j$ or $-2 - \sqrt{5}j$. Therefore the solutions of the ODE are:

$$\begin{aligned} y(x) &= e^{-2x}\left(Ae^{\sqrt{5}xj} + Be^{-\sqrt{5}xj}\right) \\ \text{or} &= e^{-2x}\left(C\cos(\sqrt{5}x) + D\sin(\sqrt{5}x)\right) \end{aligned}$$

**Case 3: equal roots $r_1$=$r_2$=r**

If the characteristic equation has one root only then the solutions of the homogeneous equation are of the form:

$$y(x) = Ae^{rx} + Bxe^{rx}$$

**Example**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

The characteristic equation is: $r^2 + 4r + 4 = 0$ e.g. $(r + 2)^2 = 0$ and its root is -2. Therefore the solutions of the ODE are:

$$y(x) = Ae^{-2x} + Bxe^{-2x}$$

## Second Order ODEs with Right-Hand Side

If the right-hand side in Equation (1) is not 0, then the solutions can be found as follows:

- First, find the form of the solution of the corresponding homogeneous equation **keeping the constants** $A$ **and** $B$ **as such**: this is called the complementary solution $y_c(x)$;
- Second, find a particular integral of the ODE $y_p(x)$.

Then the solutions of the ODE are of the form: $y(x) = y_c(x) + y_p(x)$. **At this point only**, you may determine the constants $A$ and $B$ from the boundary conditions.

There are two methods to find a particular integral of the ODE: the method of undetermined coefficients and the method of variation of parameters.


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## Undetermined coefficients

This method consists in making an educated guess as to what the particular integral should look like. The following table can be used:



| f(x) | particular integral |
|---|---|
| $k$ | $C$ |
| $kx$ | $Cx + D$ |
| $kx^2$ | $Cx^2 + Dx + E$ |
| $k\sin x$ or $k\cos x$ | $C\cos x + D\sin x$ |
| $k\sinh x$ or $k\cosh x$ | $C\cosh x + D\sinh x$ |
| $e^{kx}$ | $Ce^{kx}$ |
| $e^{rx}$, where r is a root of the characteristic equation | $Cxe^{rx}$ or $Cx^2e^{rx}$ |

The constants $C$ and $D$ are found by 'plugging' the particular integral in the ODE, which will lead to conditions that define $C$ and $D$.

**Example**

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2\sin 4x$$

We first find the complementary solution of the ODE. The characteristic equation is $r^2 - 5r + 6 = 0$ and the roots are $\dfrac{5 \pm \sqrt{25 - 4 \times 6}}{2} = 3$ or 2. Therefore the complementary solution is:

$$y_c(x) = Ae^{3x} + Be^{2x}$$

Then, we find a particular integral of the ODE. Since the right-hand side contains a $\sin 4x$, we look for a particular integral in the form $y_p(x) = C\cos 4x + D\sin 4x$. We want $y_p$ to be solution of the ODE so we must have:

$$\frac{d^2y_p}{dx^2} - 5\frac{dy_p}{dx} + 6y_p = 2\sin 4x$$

We have:

$$\frac{dy_p}{dx} = -4C\sin 4x + 4D\cos 4x$$

$$\frac{d^2y_p}{dx^2} = -16C\cos 4x - 16D\sin 4x$$

Putting back in the ODE:

$$(-16C\cos 4x - 16D\sin 4x) - 5(-4C\sin 4x + 4D\cos 4x) + 6(C\cos 4x + D\sin 4x) = 2\sin 4x$$

Re-arranging cos and sin:

$$(-16C - 20D + 6C)\cos 4x + (-16D + 20C + 6D)\sin 4x = 2\sin 4x$$

$$(-10C - 20D)\cos 4x + (-10D + 20C)\sin 4x = 2\sin 4x$$

The last equation must be true for any value of $x$, so we must have:

$$\begin{cases} -10C - 20D = 0 \\ 20C - 10D = 2 \end{cases}$$

So:

$$\begin{cases} C = \frac{2}{25} \\ D = -\frac{1}{25} \end{cases}$$


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So a particular integral of the ODE is $y_p(x) = \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x$ and the general solutions of the ODE are of the form:

$$y(x) = \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x + Ae^{3x} + Be^{2x}$$

**Variation of parameters**



This method is more general and will work for any function $f(x)$ in the right-hand side of Equation (1), although it may look intimidating at first sight! First let's rewrite the complementary solution of the ODE in the form:

$$y_c(x) = Ay_1(x) + By_2(x)$$

with $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$ or $xe^{r_1 x}$ if $r_1 = r_2$ with $r_1$, $r_2$ roots of the characteristic equation

Then a particular integral of Equation (1) is:

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1,y_2)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1,y_2)}dx$$

with $W$ the Wronskian: $W(y_1,y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$

**Example**

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$$

First, let's find the complementary solution of the ODE. The characteristic equation is $r^2 - 2r + 1 = 0$, e.g. $(r-1)^2 = 0$, so there is one root which is 1. The complementary solution is of the form:

$$y_c(x) = Ae^x + Bxe^x$$

To find a particular integral of the ODE, we calculate the Wronskian:

with: $y_1(x) = e^x$ and: $y_2(x) = xe^x$

$$\begin{aligned} W(y_1,y_2) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^x(1+x)e^x - e^x xe^x = e^{2x} \end{aligned}$$

Then a particular integral of the ODE is:

$$\begin{aligned} y_p(x) &= -e^x\int \frac{xe^x}{e^{2x}}\frac{e^x}{x^2+1}dx + xe^x\int \frac{e^x}{e^{2x}}\frac{e^x}{x^2+1}dx \\ &= -e^x\int \frac{x}{x^2+1}dx + xe^x\int \frac{1}{1+x^2}dx \end{aligned}$$

$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) \quad \text{and} \quad \int \frac{1}{1+x^2}dx = \arctan x$$

$$= -e^x \cdot \frac{1}{2}\ln(1+x^2) + xe^x \cdot \arctan x$$

The general solution of the ODE is:

$$y(x) = Ae^x + Bxe^x - \frac{1}{2}e^x\ln(1+x^2) + xe^x\arctan x$$


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**Exercises**

(a) $\dfrac{d^2y}{dx^2} + 7y = 0$

(b) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-2x}$

(c) $3\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - y = 2x - 3$

(d) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 2e^{-2x}$ with $y(0) = 1,\ \dfrac{dy}{dx}(0) = -2$

(e) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2\cos^2 x$

(f) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 4\sinh x$



**Answers**

(a) $y = Ae^{i\sqrt{7}x} + Be^{-i\sqrt{7}x}$
or $y = C\cos(\sqrt{7}x) + D\sin(\sqrt{7}x)$

(b) $y = Ae^{-x} + Bxe^{-x} + e^{-2x}$

(c) $y = Ae^{x} + Be^{-1/3x} - 2x + 7$

(d) $y = e^{-2x}(2 - \cos x)$

(e) $y = (A + Bx)e^{-2x} + \dfrac{1}{4} + \dfrac{1}{8}\sin(2x)$

(f) $y = (A + Bx - x^2)e^{-x} + \dfrac{1}{2}e^{x}$



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