Limits of functions

mc-TY-limits-2009-1

In this unit, we explain what it means for a function to tend to infinity, to minus infinity, or to a real limit, as x tends to infinity or to minus infinity. We also explain what it means for a function to tend to a real limit as x tends to a given real number. In each case, we give an example of a function that does not tend to a limit at all.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- decide whether a function tends to plus or minus infinity, or to a real limit, as x tends to infinity;
- decide whether a function tends to plus or minus infinity, or to a real limit, as x tends to minus infinity;
- decide whether a function tends to a real limit as x tends to a given real number.

Contents

1. The limit of a function as x tends to infinity

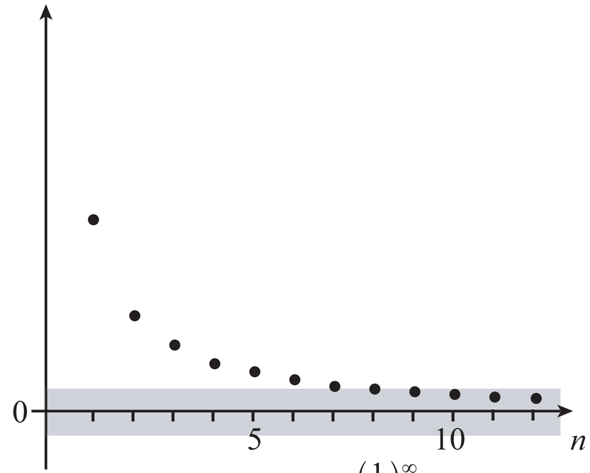
If we have a sequence $(y_n)_{n=1}^{\infty}$, we can say what it means for the sequence to have a limit as n tends to infinity. We write

$$y_n \to l \quad \text{as} \quad n \to \infty$$

if, however small a distance we choose, y_n eventually gets closer to l than that distance, and stays closer. We can also write

$$\lim_{x\to\infty} f(x) = l\,.$$

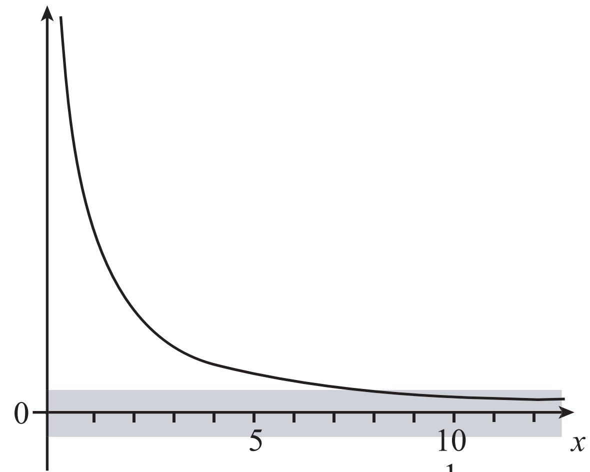
For example, consider the sequence where $y_n = 1/n$. The numbers in this sequence get closer and closer to zero. Whatever positive number we choose, y_n will eventually become smaller than that number, and stay smaller. So y_n eventually gets closer to zero than any distance we choose, and stays closer. We say that the sequence has limit zero as n tends to infinity.



the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$

We define the limit of a function in a similar way. For example, the points of the sequence $(1/n)_{n=1}^{\infty}$ are also points on the graph of the function $f(x) = 1/x$ for $x > 0$. As x gets larger, $f(x)$ gets closer and closer to zero. In fact, $f(x)$ will get closer to zero than any distance we choose, and will stay closer. We say that $f(x)$ has limit zero as x tends to infinity, and we write

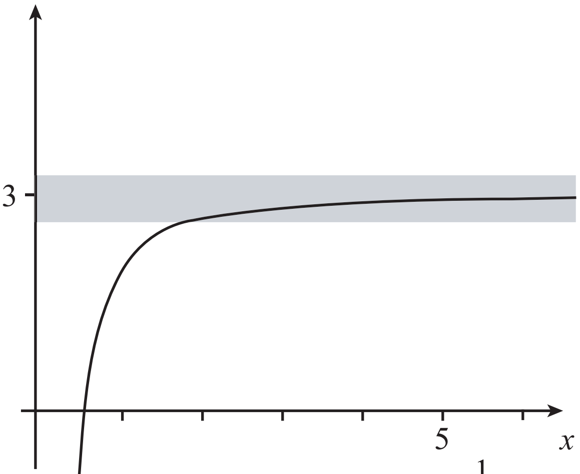
$$f(x) \to 0 \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = 0\,.$$



the function $f(x) = \frac{1}{x}$

Another example of a function that has a limit as x tends to infinity is the function $f(x) = 3 - 1/x^2$ for $x > 0$. As x gets larger, $f(x)$ gets closer and closer to 3. For any small distance, $f(x)$ eventually gets closer to 3 than that distance, and stays closer. So we say that $f(x)$ has limit 3 as x tends to infinity, and we write

$$f(x) \to 3 \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = 3\,.$$

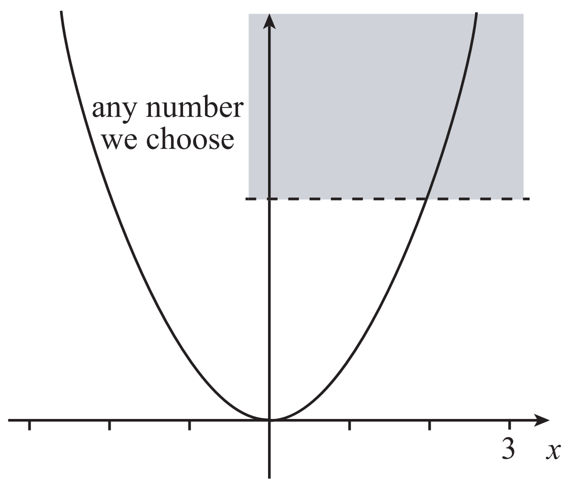


the function $f(x) = 3 - \dfrac{1}{x^2}$

In general, we say that $f(x)$ tends to a real limit l as x tends to infinity if, however small a distance we choose, $f(x)$ gets closer than that distance to l and stays closer as x increases.

Of course, not all functions have real limits as x tends to infinity. Let us look at some other types of behaviour. If we take the function $f(x) = x^2$, we see that $f(x)$ does not get closer to any particular number as x increases. Instead, $f(x)$ just gets larger and larger. At some point, $f(x)$ will get larger than any number we choose, and will stay larger. In this case, we say that $f(x)$ tends to infinity as x tends to infinity, and we write

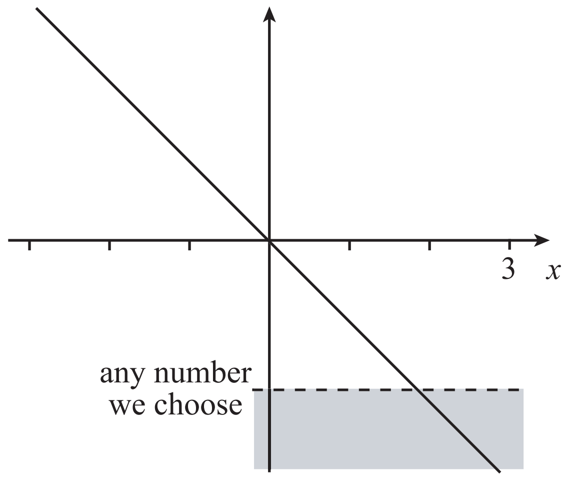
$$f(x) \to \infty \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = \infty\,.$$



the function $f(x) = x^2$

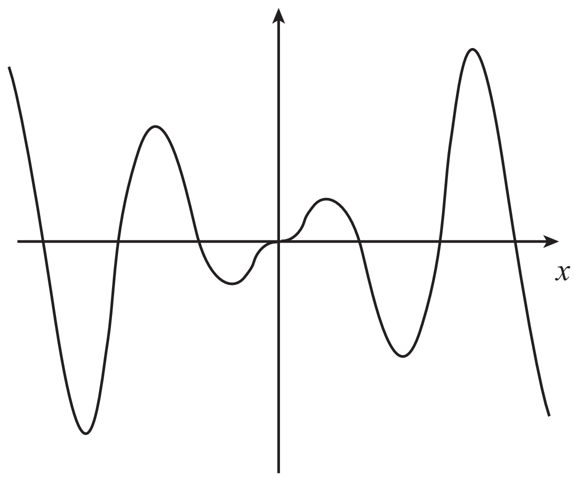
The function $f(x) = -x$ does not have a real limit as x tends to infinity. As x gets larger, this function eventually gets more negative than any number we can choose, and it will stay more negative. In this case, we say that $f(x)$ tends to minus infinity as x tends to infinity, and we write

$$f(x) \to -\infty \text{ as } x \to \infty\,, \quad \text{or} \quad \lim_{x\to\infty} f(x) = -\infty\,.$$



the function $f(x) = -x$

Some functions do not have any kind of limit as x tends to infinity. For example, consider the function $f(x) = x \sin x$. This function does not get close to any particular real number as x gets large, because we can always choose a value of x to make $f(x)$ larger than any number we choose. However $f(x)$ does not tend to infinity, because it does not stay larger than the number we have chosen, but instead returns to zero. For a similar reason, $f(x)$ does not tend to minus infinity. So we cannot talk about the limit of this function as x tends to infinity.



the function $f(x) = x \sin x$

Key Point

The function $f(x)$ has a real limit l as x tends to infinity if, however small a distance we choose, $f(x)$ gets closer than this distance to l and stays closer, no matter how large x becomes.

The function $f(x)$ tends to infinity as x tends to infinity if, however large a number we choose, $f(x)$ gets larger than this number and stays larger, no matter how large x becomes.

The function $f(x)$ tends to minus infinity as x tends to infinity if, however large and negative a number we choose, $f(x)$ gets more negative than this number and stays more negative, no matter how large x becomes.

Exercise 1

For each of the following functions $f(x)$, find the real limit as $x \to \infty$ if it exists. If it does not exist, state whether the function tends to infinity, tends to minus infinity, or has no limit at all.

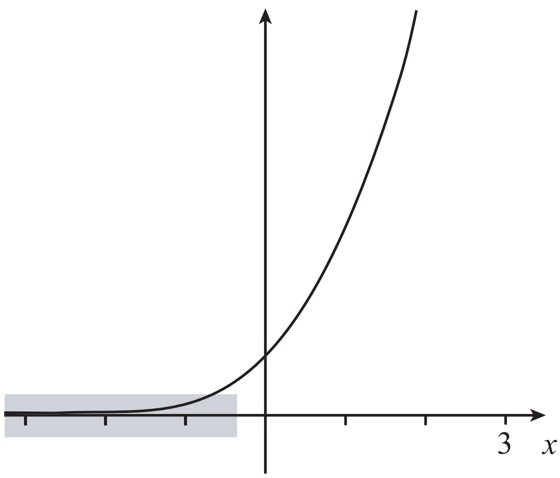
(a) $f(x) = 2x^2 - 3x^3$ (b) $f(x) = \tan x$ (c) $f(x) = \dfrac{x+1}{x-1}$ (d) $f(x) = e^{-x}\sin x$

(e) $f(x) = e^x \cos^2 x$ (f) $f(x) = \tan^{-1} x$

2. The limit of a function as x tends to minus infinity

As well as defining the limit of a function as x tends to infinity, we can also define the limit as x tends to minus infinity. Consider the function $f(x) = e^x$. As x becomes more and more negative, $f(x)$ gets closer and closer to zero. However small a distance we choose, $f(x)$ gets closer than that distance to zero, and it stays closer as x becomes more negative. We say that $f(x)$ has limit zero as x tends to minus infinity, and we write

$$f(x) \to 0 \text{ as } x \to -\infty\,, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = 0\,.$$



the function $f(x) = e^x$

In general we write

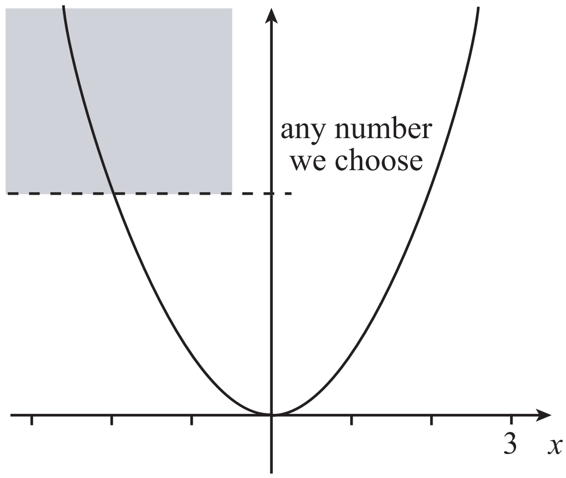
$$f(x) \to l \text{ as } x \to -\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = l$$

if, however small a distance we choose, $f(x)$ eventually gets closer to l than that distance, and stays closer, as x becomes large and negative.

If, as x gets more negative, a function gets larger and stays larger than any number we can choose, we write

$$f(x) \to \infty \text{ as } x \to -\infty\,, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = \infty\,.$$

For example, take the function $f(x) = x^2$ again. We have already seen that it tends to infinity as x tends to infinity. But it also tends to infinity as x tends to minus infinity. As x gets large and negative, the function gets larger than any number we can choose, and stays larger.

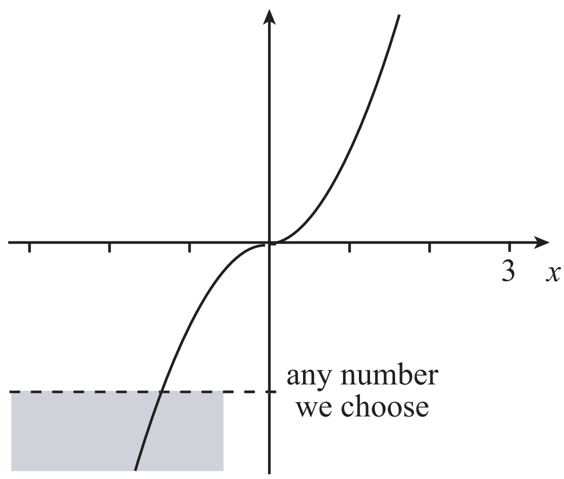


the function $f(x) = x^2$

If, instead, as x gets more negative, a function gets more negative and stays more negative than any number we can choose, we write

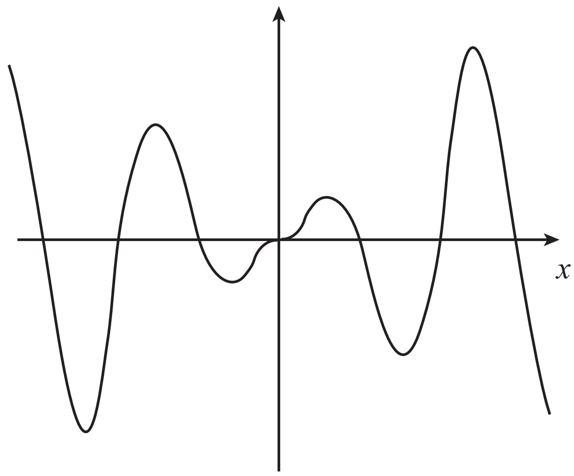
$$f(x) \to -\infty \text{ as } x \to -\infty\,, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty\,.$$

As an example, consider the function $f(x) = x^3$. You can see that, as x gets more and more negative, x^3 becomes more negative than any number we can choose, and stays more negative. So $f(x)$ tends to minus infinity as x tends to minus infinity.



the function $f(x) = x^3$

Some functions do not have any kind of limit as x tends to minus infinity. For example, consider the function $f(x) = x \sin x$ that we saw earlier. This function does not get close to any particular real number as x gets large and negative, because we can always choose a value of x to make $f(x)$ larger than any number we choose. However $f(x)$ does not tend to infinity, because it does not stay larger than the number we have chosen as we decrease x, but instead returns to zero. For a similar reason, $f(x)$ does not tend to minus infinity. So we cannot talk about the limit of this function as x tends to minus infinity.



the function $f(x) = x \sin x$

Key Point

The function $f(x)$ has a real limit l as x tends to minus infinity if, however small a distance we choose, $f(x)$ gets closer than this distance to l and stays closer, no matter how large and negative x becomes.

The function $f(x)$ tends to infinity as x tends to minus infinity if, however large a number we choose, $f(x)$ gets larger than this number and stays larger, no matter how large and negative x becomes.

The function $f(x)$ tends to minus infinity as x tends to minus infinity if, however large and negative a number we choose, $f(x)$ gets more negative than this number and stays more negative, no matter how large and negative x becomes.

Exercise 2

For each of the functions $f(x)$ from Exercise 1, find the real limit as $x \to -\infty$ if it exists. If it does not exist, state whether the function tends to infinity, tends to minus infinity, or has no limit at all.

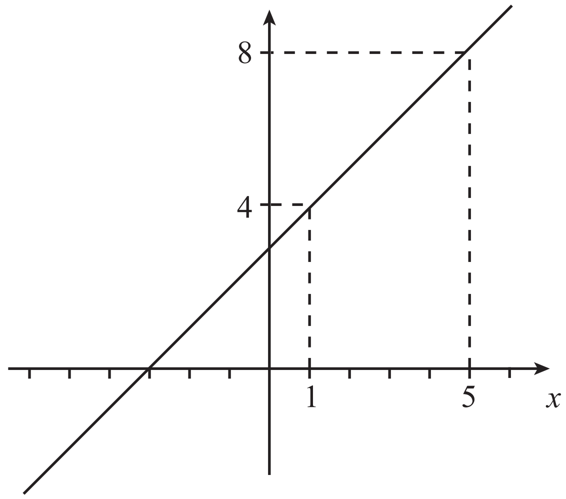
3. The limit of a function as x tends to a real number

There is one more type of limit that we can define for functions. Let us consider the function $f(x) = x+3$. If we choose a number, such as 1, then as x gets closer and closer to that number, $f(x)$ also gets closer and closer to a number, in this case 4. We write

$$f(x) \to 4 \text{ as } x \to 1\,, \quad \text{or} \quad \lim_{x\to 1} f(x) = 4\,.$$

Similarly, $f(x)$ gets closer and closer to 8 as x gets closer and closer to 5. So we write

$$f(x) \to 8 \text{ as } x \to 5\,, \quad \text{or} \quad \lim_{x\to 5} f(x) = 8\,.$$



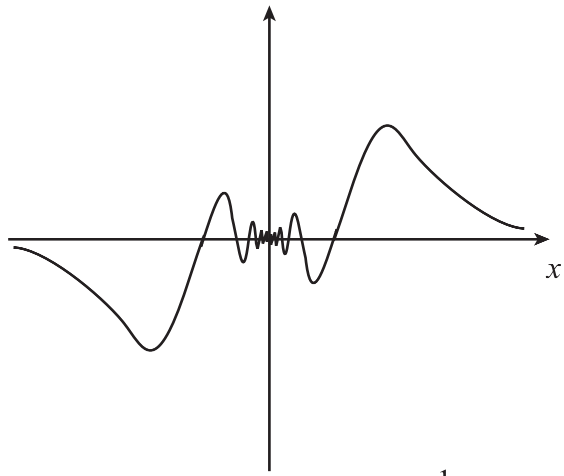
the function $f(x) = x + 3$

Now this definition of a limit might not look very useful. We know that when $x = 1$ then the value of $f(x)$ is 4. And again, when $x = 5$ then the value of (x) is 8. Why would we want to bother looking at what happens when x gets closer and closer to these numbers?

The reason is that we might sometimes have a function that is not defined at a point. For example, consider the graph of the function $x\sin(1/x)$. This function is defined for every value apart from zero, because at $x = 0$ we have a fraction with a zero denominator inside the sine function. Bt if we look at the rest of the graph, we can see that $f(x)$ gets closer and closer to zero as x gets closer and closer to zero. So we write

$$f(x) \to 0 \text{ as } x \to 0\,, \quad \text{or} \quad \lim_{x\to 0} f(x) = 0\,.$$

We might say that $f(0)$ 'ought to equal 0', even though there is no properly defined value for $f(x)$ at zero.

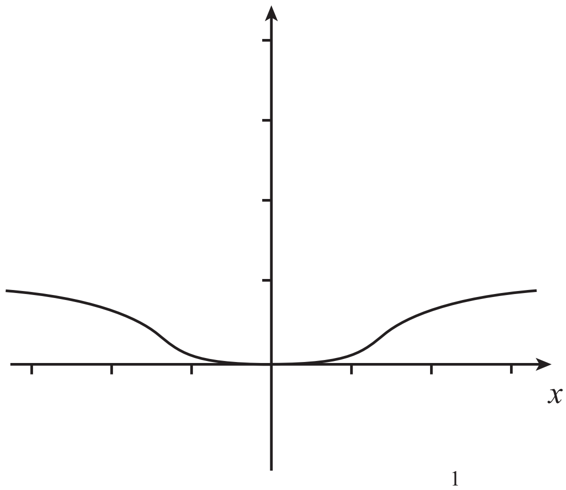


the function $f(x) = x \sin \frac{1}{x}$

So the limit of a $f(x)$ at a point where the function is undefined can be thought of as the value the function should be taking at that point.

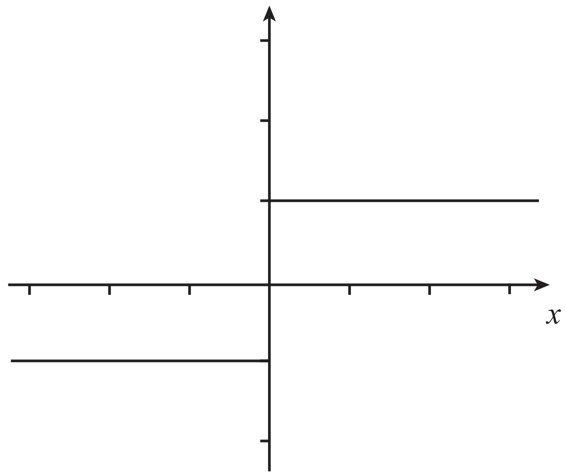
A similar thing happens with the function $f(x) = \mathrm{e}^{-1/x^2}$. This function gets closer and closer to zero as x gets closer and closer to zero. But again the value of the function at $x = 0$ is not defined, because there is a fraction with a zero denominator in the function. Nevertheless, we can see that the limit of the function is 0 as x tends to zero, so we write

$$f(x) \to 0 \text{ as } x \to 0\,, \quad \text{or} \quad \lim_{x\to 0} f(x) = 0\,.$$



the function $f(x) = \mathrm{e}^{\frac{1}{x^2}}$

Some functions do not have limits at certain points. If we take the function $f(x) = |x|/x$ then, for $x > 0$, $f(x) = x/x = 1$. But for $x < 0$, $f(x) = -x/x = -1$. At $x = 0$ the function is undefined, because there is a zero denominator. If x is positive then going closer and closer to zero keeps $f(x)$ at 1. But if x is negative, going closer and closer to zero keeps $f(x)$ at -1. So this function does not have a limit at $x = 0$.



the function $f(x) = \frac{|x|}{x}$

Key Point

The limit of $f(x)$ as x tends to a real number, is the value $f(x)$ approaches as x gets closer to that real number.

Exercise 3

Find each of the following real limits, if they exist. If the real limit does not exist, state whether the function tends to infinity, tends to minus infinity, or has no limit at all.

(a) $\lim_{x\to 0} \sin x$ (b) $\lim_{x\to 0} \sin \frac{1}{x}$ (c) $\lim_{x\to 2} \frac{1}{x-2}$ (d) $\lim_{x\to 2} \frac{1}{(x-2)^2}$

(e) $\lim_{x\to 1} \frac{x^2-3x+2}{1-x}$ (f) $\lim_{x\to \pi/2} \tan x$

Answers

1.

(a) minus infinity (b) no limit (c) 1 (d) 0 (e) no limit (f) $\pi/2$

2.

(a) infinity (b) no limit (c) 1 (d) no limit (e) 0 (f) $-\pi/2$

3.

(a) 0 (b) no limit (c) no limit (d) infinity (e) 1 (f) no limit

