''JUST THE MATHS''

by

A.J. Hobson

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(Average number of pages = 1038 ¸140 = 7.4 per unit)

All units are in presented as .PDF files

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FOREWORD

(Home) (About the Author) (Teaching Units) (Teaching Slides)

In 35 years of teaching mathematics to Engineers and Scientists, I have frequently been made aware (by students) of a common cry for help. "We're coping, generally, with our courses", they may say, "but it's Just the Maths". This is the title chosen for the package herein.

Traditional text-books and programmed learning texts can sometimes include a large amount of material which is not always needed for a particular course; and which can leave students feeling that there is too much to cope with. Many such texts are biased towards the mathematics required for specific engineering or scientific disciplines and emphasise the associated practical applications in their lists of tutorial examples. There can also be a higher degree of mathematical rigor than would be required by students who are not intending to follow a career in mathematics itself.

"Just the Maths" is a collection of separate units, in chronological topic-order, intended to service foundation level and first year degree level courses in higher education, especially those delivered in a modular style. Each unit represents, on average, the work to be covered in a typical two-hour session consisting of a lecture and a tutorial. However, since each unit attempts to deal with self-contained and, where possible, independent topics, it may sometimes require either more than or less than two hours spent on it.

"Just the Maths" does not have the format of a traditional text-book or a course of programmed learning; but it is written in a traditional pure-mathematics style with the minimum amount of formal rigor. By making use of the well-worn phrase, "it can be shown that", it is able to concentrate on the core mathematical techniques required by any scientist or engineer. The techniques are demonstrated by worked examples and reinforced by exercises that are few enough in number to allow completion, or near-completion, in a one-hour tutorial session. Answers to exercises are supplied at the end of each unit of work.

A.J. Hobson
January 2002

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ABOUT THE AUTHOR

Tony Hobson was, until retirement in November 2001, a Senior Lecturer in Mathematics of the School of Mathematical and Information Sciences at Coventry University. He graduated from the University College of Wales, Aberystwyth in 1964, with a BSc. Degree 2(i) in Pure Mathematics, and from Birmingham University in 1965, with an MSc. Degree in Pure Mathematics. His Dissertation for the MSc. Degree consisted of an investigation into the newer styles teaching Mathematics in the secondary schools of the 1960's with the advent of experiments such as the Midland Mathematics Experiment and the School Mathematics Project. His teaching career began in 1965 at the Rugby College of Engineering Technology where, as well as involvement with the teaching of Analysis and Projective Geometry to the London External Degree in Mathematics, he soon developed a particular interest in the teaching of Mathematics to Science and Engineering Students. This interest continued after the creation of the Polytechnics in 1971 and a subsequent move to the Coventry Polytechnic, later to become Coventry University. It was his main teaching interest throughout the thirty six years of his career; and it meant that much of the time he spent on research and personal development was in the area of curriculum development. In 1982 he became a Non-stipendiary Priest in the Church of England, an interest he maintained throughout his retirement. Tony Hobson died in December 2002.

The set of teaching units for "Just the Maths" has been the result of a pruning, honing and computer-processing exercise (over some four or five years) of **many** years' personal teaching materials, into a form which may be easily accessible to students of Science and Engineerng in the future.

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"JUST THE MATHS"

UNIT NUMBER

1.1

ALGEBRA 1
(Introduction to algebra)

by

A.J. Hobson

UNIT 1.1 - ALGEBRA 1 - INTRODUCTION TO ALGEBRA

DEFINITION

An **"Algebra"** is any Mathematical system which uses the concepts of Equality (=), Addition (+), Subtraction (-), Multiplication (× or .) and Division (÷).

Note:
The Algebra of Numbers is what we normally call **"Arithmetic"** and, as far as this unit is concerned, it is only the algebra of numbers which we shall be concerned with.

1.1.1 THE LANGUAGE OF ALGEBRA

Suppose we use the symbols a, b and c to denote numbers of arithmetic; then

(a) $a + b$ is called the **"sum of a and b"**.

Note:
$a + a$ is usually abbreviated to $2a$,
$a + a + a$ is usually abbreviated to $3a$ and so on.

(b) $a - b$ is called the **"difference between a and b"**.

(c) $a \times b$, $a.b$ or even just ab is called the **"product"** of a and b.

Notes:
(i)
$a.a$ is usually abbreviated to a^2,
$a.a.a$ is usually abbreviated to a^3 and so on.

(ii) $-1 \times a$ is usually abbreviated to $-a$ and is called the **"negation"** of a.

(d) $a \div b$ or $\frac{a}{b}$ is called the **"quotient"** or **"ratio"** of a and b.

(e) $\frac{1}{a}$, [also written a^{-1}], is called the **"reciprocal"** of a.

(f) $| a |$ is called the **"modulus"**, **"absolute value"** or **"numerical value"** of a. It can be defined by the two statements

$| a | = a$ when a is positive or zero;
$| a | = -a$ when a is negative or zero.

Note:
Further work on fractions (ratios) will appear later, but we state here for reference the rules for combining fractions together:

Rules for combining fractions together

1.
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

2.
$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

3.
$$\frac{a}{b}.\frac{c}{d} = \frac{a.c}{b.d}$$

4.
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a.d}{b.c}$$

EXAMPLES

1. How much more than the difference of 127 and 59 is the sum of 127 and 59 ?

   Solution
   The difference of 127 and 59 is $127 - 59 = 68$ and the sum of 127 and 59 is $127 + 59 = 186$. The sum exceeds the difference by $186 - 68 = 118$.

2. What is the reciprocal of the number which is 5 multiplied by the difference of 8 and 2 ?

   Solution
   We require the reciprocal of $5.(8 - 2)$; that is, the reciprocal of 30. The answer is therefore $\frac{1}{30}$.

3. Calculate the value of $4\frac{2}{3} - 5\frac{1}{9}$ expressing the answer as a fraction.

   Solution
   Converting both numbers to a single fraction, we require
   $$\frac{14}{3} - \frac{46}{9} = \frac{126 - 138}{27} = -\frac{12}{27} = -\frac{4}{9}.$$
   We could also have observed that the 'lowest common multiple' (see later) of the two denominators, 3 and 9, is 9; hence we could write the alternative solution
   $$\frac{42}{9} - \frac{46}{9} = -\frac{4}{9}.$$

4. Remove the modulus signs from the expression $\mid a-2 \mid$ in the cases when (i) a is greater than (or equal to) 2 and (ii) a is less than 2.

Solution
(i) If a is greater than or equal to 2,

$$\mid a-2 \mid = a-2;$$

(ii) If a is less than 2,

$$\mid a-2 \mid = -(a-2) = 2-a.$$

1.1.2 THE LAWS OF ALGEBRA

If the symbols a, b and c denote numbers of arithmetic, then the following Laws are obeyed by them:

(a) The Commutative Law of Addition $\quad a+b=b+a$

(b) The Associative Law of Addition $\quad a+(b+c)=(a+b)+c$

(c) The Commutative Law of Multiplication $\quad a.b=b.a$

(d) The Associative Law of Multipication $\quad a.(b.c)=(a.b).c$

(e) The Distributive Laws $\quad a.(b+c)=a.b+a.c$ and $(a+b).c=a.c+b.c$

Notes:

(i) A consequence of the Distributive Laws is the rule for multiplying together a pair of bracketted expressions. It will be encountered more formally later, but we state it here for reference:

$$(a+b).(c+d)=a.c+b.c+a.d+b.d$$

(ii) The alphabetical letters so far used for numbers in arithmetic have been taken from the **beginning** of the alphabet. These tend to be reserved for fixed quantities called **constants**. Letters from the **end** of the alphabet, such as $w,\ x,\ y,\ z$ are normally used for quantities which may take many values, and are called **variables**.

1.1.3 PRIORITIES IN CALCULATIONS

Suppose that we encountered the expression $5\times 6-4$. It would seem to be ambiguous, meaning either $30-4=26$ or $5\times 2=10$.

However, we may remove the ambiguity by using brackets where necessary, together with a rule for precedence between the use of the brackets and the symbols $+$, $-$, $\times$ and $\div$.

The rule is summarised in the abbreviation

B.O.D.M.A.S.

which means that the order of precedence is

B	brackets	( )	First Priority
O	of	$\times$	Joint Second Priority
D	division	$\div$	Joint Second Priority
M	multiplication	$\times$	Joint Second Priority
A	addition	$+$	Joint Third Priority
S	subtraction	$-$	Joint Third Priority

Thus, $5 \times (6 - 4) = 5 \times 2 = 10$
but $5 \times 6 - 4 = 30 - 4 = 26$.

Similarly, $12 \div 3 - 1 = 4 - 1 = 3$
whereas $12 \div (3 - 1) = 12 \div 2 = 6$.

1.1.4 FACTORS

If a number can be expressed as a product of other numbers, each of those other numbers is called a **"factor"** of the original number.

EXAMPLES

1. We may observe that

$$70 = 2 \times 7 \times 5$$

   so that the number 70 has factors of 2, 7 and 5. These three cannot be broken down into factors themselves because they are what are known as **"prime"** numbers (numbers whose only factors are themselves and 1). Hence the only factors of 70, apart from 70 and 1, are 2, 7 and 5.

2. Show that the numbers 78 and 182 have two common factors which are prime numbers.

   The two factorisations are as follows:

$$78 = 2 \times 3 \times 13,$$

$$182 = 2 \times 7 \times 13.$$

The common factors are thus 2 and 13, both of which are prime numbers.

Notes:

(i) If two or more numbers have been expressed as a product of their prime factors, we may easily identify the prime factors which are common to all the numbers and hence obtain the **"highest common factor"**, h.c.f.

For example, $90 = 2 \times 3 \times 3 \times 5$ and $108 = 2 \times 2 \times 3 \times 3 \times 3$. Hence the h.c.f $= 2 \times 3 \times 3 = 18$

(ii) If two or more numbers have been expressed as a product of their prime factors, we may also identify the **"lowest common multiple"**, l.c.m.

For example, $15 = 3 \times 5$ and $20 = 2 \times 2 \times 5$. Hence the smallest number into which both 15 and 20 will divide requires two factors of 2 (for 20), one factor of 5 (for both 15 and 20) and one factor of 3 (for 15). The l.c.m. is thus $2 \times 2 \times 3 \times 5 = 60$.

(iii) If the numerator and denominator of a fraction have factors in common, then such factors may be cancelled to leave the fraction in its **"lowest terms"**.

For example $\frac{15}{105} = \frac{3\times5}{3\times5\times7} = \frac{1}{7}$.

1.1.5 EXERCISES

1. Find the sum and product of

   (a) 3 and 6; (b) 10 and 7; (c) 2, 3 and 6;

   (d) $\frac{3}{2}$ and $\frac{4}{11}$; (e) $1\frac{2}{5}$ and $7\frac{3}{4}$; (f) $2\frac{1}{7}$ and $5\frac{4}{21}$.

2. Find the difference between and quotient of

   (a) 18 and 9; (b) 20 and 5; (c) 100 and 20;

   (d) $\frac{3}{5}$ and $\frac{7}{10}$; (e) $3\frac{1}{4}$ and $2\frac{2}{9}$; (f) $1\frac{2}{3}$ and $5\frac{5}{6}$.

3. Evaluate the following expressions:

   (a) $6 - 2 \times 2$; (b) $(6 - 2) \times 2$;
   (c) $6 \div 2 - 2$; (d) $(6 \div 2) - 2$;
   (e) $6 - 2 + 3 \times 2$; (f) $6 - (2 + 3) \times 2$;
   (g) $(6 - 2) + 3 \times 2$; (h) $\frac{16}{-2}$; (i) $\frac{-24}{-3}$; (j) $(-6) \times (-2)$.

4. Place brackets in the following to make them correct:

   (a) $6 \times 12 - 3 + 1 = 55$; (b) $6 \times 12 - 3 + 1 = 68$;
   (c) $6 \times 12 - 3 + 1 = 60$; (d) $5 \times 4 - 3 + 2 = 7$;
   (e) $5 \times 4 - 3 + 2 = 15$; (f) $5 \times 4 - 3 + 2 = -5$.

5. Express the following as a product of prime factors:

   (a) 26; (b) 100; (c) 27; (d) 71;

   (e) 64; (f) 87; (g) 437; (h) 899.

6. Find the h.c.f of

   (a) 12, 15 and 21; (b) 16, 24 and 40; (c) 28, 70, 120 and 160;

   (d) 35, 38 and 42; (e) 96, 120 and 144.

7. Find the l.c.m of

   (a) 5, 6, and 8; (b) 20 and 30; (c) 7, 9 and 12;

   (d) 100, 150 and 235; (e) 96, 120 and 144.

1.1.6 ANSWERS TO EXERCISES

1. (a) 9, 18; (b) 17,70; (c) 11,36; (d) $\frac{41}{22}$, $\frac{6}{11}$; (e) $\frac{183}{20}$, $\frac{217}{20}$; (f) $\frac{154}{21}$, $\frac{545}{49}$.

2. (a) 9,2; (b) 15,4; (c) 80,5; (d) $-\frac{1}{10}$, $\frac{6}{7}$; (e) $\frac{37}{36}$, $\frac{117}{80}$; (f) $-\frac{25}{6}$, $\frac{2}{7}$.

3. (a) 2; (b) 8; (c) 1; (d) 1; (e) 10;
   (f) -4; (g) 10; (h) -8; (i) 8; (j) 12;

4. (a) $6 \times (12 - 3) + 1 = 55$; (b) $6 \times 12 - (3 + 1) = 68$;
   (c) $6 \times (12 - 3 + 1) = 60$; (d) $5 \times (4 - 3) + 2 = 7$;
   (e) $5 \times 4 - (3 + 2) = 15$; (f) $5 \times (4 - [3 + 2]) = -5$.

5. (a) 2×13; (b) $2 \times 2 \times 5 \times 5$; (c) $3 \times 3 \times 3$; (d) 71×1;

   (e) $2 \times 2 \times 2 \times 2 \times 2 \times 2$; (f) 3×29; (g) 19×23; (h) 29×31.

6. (a) 3; (b) 8; (c) 2; (d) 1; (e) 24.

7. (a) 120; (b) 60; (c) 252; (d) 14100; (e) 1440.

"JUST THE MATHS"

UNIT NUMBER

1.2

ALGEBRA 2
(Numberwork)

by

A.J. Hobson

UNIT 1.2 - - ALGEBRA 2 - NUMBERWORK

1.2.1 TYPES OF NUMBER

In this section (and elsewhere) the meaning of the following types of numerical quantity will need to be appreciated:

(a) NATURAL NUMBERS
These are the counting numbers 1, 2, 3, 4, ...

(b) INTEGERS
These are the positive and negative whole numbers and zero;
i.e. ......-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...

(c) RATIONALS
These are the numbers which can be expressed as the ratio of two integers but can also be written as a terminating or recurring decimal (see also next section)

For example

$$\frac{2}{5} = 0.4$$

and

$$\frac{3}{7} = 0.428714287142871....$$

(d) IRRATIONALS
These are the numbers which cannot be expressed as either the ratio of two integers or a recurring decimal (see also next section)

Typical examples are numbers like

$$\pi \simeq 3.1415926.....$$

$$e \simeq 2.71828.....$$

$$\sqrt{2} \simeq 1.4142135....$$

$$\sqrt{5} \simeq 2.2360679....$$

The above four types of number form the system of **"real numbers"**.

1.2.2 DECIMAL NUMBERS

(a) Rounding to a specified number of decimal places
Most decimal quantities used in scientific work need to be approximated by **"rounding"** them (up or down as appropriate) to a specified number of decimal places, depending on the accuracy required.

When rounding to n decimal places, the digit in the n-th place is left as it is when the one after it is below 5; otherwise it is taken up by one digit.

EXAMPLES

1. 362.5863 = 362.586 to 3 decimal places;
   362.5863 = 362.59 to 2 decimal places;
   362.5863 = 362.6 to 1 decimal place;
   362.5863 = 363 to the nearest whole number.

2. 0.02158 = 0.0216 to 4 decimal places;
   0.02158 = 0.022 to three decimal places;
   0.02158 = 0.02 to 2 decimal places.

(b) Rounding to a specified number of significant figures
The first significant figure of a decimal quantity is the first non-zero digit from the left, whether it be before or after the decimal point.

Hence when rounding to a specified number of significant figures, we use the same principle as in **(a)**, but starting from the first significant figure, then working to the right.

EXAMPLES

1. 362.5863 = 362.59 to 5 significant figures;
   362.5863 = 362.6 to 4 significant figures;
   362.5863 = 363 to 3 significant figures;
   362.5863 = 360 to 2 significant figures;
   362.5863 = 400 to 1 significant figure.

2. 0.02158 = 0.0216 to 3 significant figures; 0.02158 = 0.022 to 2 significant figures;
   0.02158 = 0.02 to 1 significant figure.

1.2.3 THE USE OF ELECTRONIC CALCULATORS

(a) B.O.D.M.A.S.

The student will normally need to work to the instruction manual for the particular calculator being used; but care must be taken to remember the B.O.D.M.A.S. rule for priorities in calculations when pressing the appropriate buttons.

For example, in working out $7.25 + 3.75 \times 8.32$, the multiplication should be carried out first, then the addition. The answer is 38.45, not 91.52.

Similarly, in working out $6.95 \div [2.43 - 1.62]$, it is best to evaluate $2.43 - 1.62$, then generate its reciprocal with the $\frac{1}{x}$ button, then multiply by 6.95. The answer is 8.58, not 1.24

(b) Other Useful Numerical Functions

Other useful functions to become familiar with for scientific work with numbers are those indicated by labels such as $\sqrt{x}$, x^2, x^y and $x^{\frac{1}{y}}$, using, where necessary, the **"shift"** control to bring the correct function into operation.

For example:

$$\sqrt{173} \simeq 13.153;$$
$$173^2 = 29929;$$
$$23^3 = 12167;$$
$$23^{\frac{1}{3}} \simeq 2.844$$

(c) The Calculator Memory

Familiarity with the calculator's memory facility will be essential for more complicated calculations in which various parts need to be stored temporarily while the different steps are being carried out.

For example, in order to evaluate

$$(1.4)^3 - 2(1.4)^2 + 5(1.4) - 3 \simeq 2.824$$

we need to store each of the four terms in the calculation (positively or negatively) then recall their total sum at the end.

1.2.4 SCIENTIFIC NOTATION

(a) Very large numbers, especially decimal numbers are customarily written in the form

$$a \times 10^n$$

where n is a positive integer and a lies between 1 and 10.

For instance,

$$521983677.103 = 5.21983677103 \times 10^8.$$

(b) Very small decimal numbers are customarily written in the form

$$a \times 10^{-n}$$

where n is a positive integer and a lies between 1 and 10.

For instance,

$$0.00045938 = 4.5938 \times 10^{-4}.$$

Note:
An electronic calculator will allow you to enter numbers in scientific notation by using the $\boxed{\textbf{EXP}}$ or $\boxed{\textbf{EE}}$ buttons.

EXAMPLES

1. Key in the number 3.90816×10^{57} on a calculator.

   Press **3.90816** $\boxed{\textbf{EXP}}$ **57**

   In the display there will now be 3.90816 57 or $3.90816^{\times 10^{57}}$.

2. Key in the number 1.5×10^{-27} on a calculator

   Press **1.5** $\boxed{\textbf{EXP}}$ **27** $\boxed{+/-}$

   In the display there will now be 1.5 - 27 or $1.5^{\times 10^{-27}}$.

Notes:

(i) On a calculator or computer, scientific notation is also called *floating point notation.*

(ii) When performing a calculation involving decimal numbers, it is always a good idea to check that the result is reasonable and that a major arithmetical error has not been made with the calculator.

For example,

$$69.845 \times 196.574 = 6.9845 \times 10^1 \times 1.96574 \times 10^3.$$

This product can be **estimated** for reasonableness as:

$$7 \times 2 \times 1000 = 14000.$$

The answer obtained by calculator is 13729.71 to two decimal places which is 14000 when rounded to the nearest 1000, indicating that the exact result could be reasonably expected.

(iii) If a set of measurements is made with an accuracy to a given number of significant figures, then it may be shown that any calculation involving those measurements will be accurate only to one significant figure more than the least number of significant figures in any measurement.

For example, the edges of a rectangular piece of cardboard are measured as 12.5cm and 33.43cm respectively and hence the area may be evaluated as

$$12.5 \times 33.43 = 417.875\text{cm}^2.$$

Since one of the edges is measured only to three significant figures, the area result is accurate only to four significant figures and hence must be stated as 417.9cm^2.

1.2.5 PERCENTAGES

Definition
A percentage is a fraction whose denominator is 100. We use the per-cent symbol, %, to represent a percentage.

For instance, the fraction $\frac{17}{100}$ may be written 17%

EXAMPLES

1. Express $\frac{2}{5}$ as a percentage.
   Solution
   $$\frac{2}{5} = \frac{2}{5} \times \frac{20}{20} = \frac{40}{100} = 40\%$$
2. Calculate 27% of 90.
   Solution
   $$27\% \text{ of } 90 = \frac{27}{100} \times 90 = \frac{27}{10} \times 9 = 24.3$$

3. Express 30% as a decimal.

   Solution

$$30\% = \frac{30}{100} = 0.3$$

1.2.6 RATIO

Sometimes, a more convenient way of expressing the ratio of two numbers is to use a colon (:) in place of either the standard division sign (÷) or the standard notation for fractions.

For instance, the expression 7:3 could be used instead of either $7 \div 3$ or $\frac{7}{3}$. It denotes that two quantities are "in the ratio 7 to 3" which implies that the first number is seven thirds times the second number or, alternatively, the second number is three sevenths times the first number. Although more cumbersome, the ratio 7:3 could also be written $\frac{7}{3}$:1 or 1:$\frac{3}{7}$.

EXAMPLES

1. Divide 170 in the ratio 3:2

   Solution

   We may consider that 170 is made up of $3 + 2 = 5$ parts, each of value $\frac{170}{5} = 34$.

   Three of these make up a value of $3 \times 34 = 102$ and two of them make up a value of $2 \times 34 = 68$.

   Thus 170 needs to be divided into 102 and 68.

2. Divide 250 in the ratio 1:3:4

   Solution

   This time, we consider that 250 is made up of $1 + 3 + 4 = 8$ parts, each of value $\frac{250}{8} = 31.25$. Three of these make up a value of $3 \times 31.25 = 93.75$ and four of them make up a value of $4 \times 31.25 = 125$.

   Thus 250 needs to be divided into 31.25, 93.75 and 125.

1.2.7 EXERCISES

1. Write to 3 s.f.

   (a) 6962; (b) 70.406; (c) 0.0123;
   (d) 0.010991; (e) 45.607; (f) 2345.

2. Write 65.999 to

   (a) 4 s.f. (b) 3 s.f. (c) 2 s.f.
   (d) 1 s.f. (e) 2 d.p. (f) 1 d.p.

3. Compute the following in scientific notation:

   (a) $(0.003)^2 \times (0.00004) \times (0.00006) \times 5,000,000,000$;

   (b) $800 \times (0.00001)^2 \div (200,000)^4$.

4. Assuming that the following contain numbers obtained by measurement, use a calculator to determine their value and state the expected level of accuracy:

   (a)
   $$\frac{(13.261)^{0.5}(1.2)}{(5.632)^3};$$

   (b)
   $$\frac{(8.342)(-9.456)^3}{(3.25)^4}.$$

5. Calculate 23% of 124.

6. Express the following as percentages:

   (a) $\frac{9}{11}$; (b) $\frac{15}{20}$; (c) $\frac{9}{10}$; (d) $\frac{45}{50}$; (e) $\frac{75}{90}$.

7. A worker earns £400 a week, then receives a 6% increase. Calculate the new weekly wage.

8. Express the following percentages as decimals:

   (a) 50% (b) 36% (c) 75% (d) 100% (e) 12.5%

9. Divide 180 in the ratio 8:1:3

10. Divide 930 in the ratio 1:1:3

11. Divide 6 in the ratio 2:3:4

12. Divide 1200 in the ratio 1:2:3:4

13. A sum of £2600 is to be divided in the ratio $2\frac{3}{4} : 1\frac{1}{2} : 2\frac{1}{4}$. Calculate the amount of money in each part of the division.

1.2.8. ANSWERS TO EXERCISES

1. (a) 6960; (b) 70.4; (c) 0.0123;
   (d) 0.0110; (e) 45.6; (f) 2350.
2. (a) 66.00; (b) 66.0; (c) 66;
   (d) 70; (e) 66.00; (f) 66.0
3. (a) 1.08 - 04 or $1.08^{\times 10^{-4}}$ (b) 5 - 29 or $5^{\times 10^{-29}}$;
4. (a) 0.0245, accurate to three sig. figs. (b) -63.22, accurate to four sig. figs.
5. 28.52
6. (a) 81.82% (b) 75% (c) 90% (d) 90% (e) 83.33%
7. £424.
8. (a) 0.5; (b) 0.36; (c) 0.75; (d) 1; (e) 0.125
9. 120, 15, 45.
10. 186, 186, 558.
11. 1.33, 2, 2.67
12. 120, 240, 360, 480
13. £1100, £600 £900.

“JUST THE MATHS”

UNIT NUMBER

1.3

ALGEBRA 3
(Indices and radicals (or surds))

by

A.J.Hobson

UNIT 1.3 - ALGEBRA 3 - INDICES AND RADICALS (or Surds)

1.3.1 INDICES

(a) Positive Integer Indices

It was seen earlier that, for any number a, a^2 denotes $a.a$, a^3 denotes a.a.a, a^4 denotes $a.a.a.a$ and so on.

Suppose now that a and b are arbitrary numbers and that m and n are natural numbers (i.e. positive whole numbers)

Then the following rules are the basic Laws of Indices:

Law No. 1

$$a^m \times a^n = a^{m+n}$$

Law No. 2

$$a^m \div a^n = a^{m-n}$$

assuming, for the moment, that m is greater than n.

Note:
It is natural to use this rule to give a definition to a^0 which would otherwise be meaningless.

Clearly $\frac{a^m}{a^m} = 1$ but the present rule for indices suggests that $\frac{a^m}{a^m} = a^{m-m} = a^0$.
Hence, we **define** a^0 to be equal to 1.

Law No. 3

$$(a^m)^n = a^{mn}$$

$$a^m b^m = (ab)^m$$

EXAMPLE

Simplify the expression,

$$\frac{x^2y^3}{z} \div \frac{xy}{z^5}.$$

Solution

The expression becomes

$$\frac{x^2y^3}{z} \times \frac{z^5}{xy} = xy^2z^4.$$

(b) Negative Integer Indices

Law No. 4

$$a^{-1} = \frac{1}{a}$$

Note:
It has already been mentioned that a^{-1} means the same as $\frac{1}{a}$; and the logic behind this statement is to maintain the basic Laws of Indices for negative indices as well as positive ones.

For example $\frac{a^m}{a^{m+1}}$ is clearly the same as $\frac{1}{a}$ but, using Law No. 2 above, it could also be thought of as $a^{m-[m+1]} = a^{-1}$.

Law No. 5

$$a^{-n} = \frac{1}{a^n}$$

Note:
This time, we may observe that $\frac{a^m}{a^{m+n}}$ is clearly the same as $\frac{1}{a^n}$; but we could also use Law No. 2 to interpret it as $a^{m-[m+n]} = a^{-n}$

Law No. 6

$$a^{-\infty} = 0$$

Note:
Strictly speaking, no power of a number can ever be equal to zero, but Law No. 6 asserts that a very large negative power of a number a gives a very small value; the larger the negative power, the smaller will be the value.

EXAMPLE

Simplify the expression,

$$\frac{x^5y^2z^{-3}}{x^{-1}y^4z^5} \div \frac{z^2x^2}{y^{-1}}.$$

Solution

The expression becomes

$$x^5y^2z^{-3}xy^{-4}z^{-5}y^{-1}z^{-2}x^{-2} = x^4y^{-3}z^{-10}.$$

(c) Rational Indices

(i) Indices of the form $\frac{1}{n}$ where n is a natural number.

In order to preserve Law No. 3, we interpret $a^{\frac{1}{n}}$ to mean a number which gives the value a when it is raised to the power n. It is called an **"n-th Root of a"** and, sometimes there is more than one value.

ILLUSTRATION

$$81^{\frac{1}{4}} = \pm 3 \text{ but } (-27)^{\frac{1}{3}} = -3 \text{ only.}$$

(ii) Indices of the form $\frac{m}{n}$ where m and n are natural numbers with no common factor.

The expression $y^{\frac{m}{n}}$ may be interpreted in two ways as either $(y^m)^{\frac{1}{n}}$ or $(y^{\frac{1}{n}})^m$. It may be shown that both interpretations give the same result but, sometimes, the arithmetic is shorter with one rather than the other.

ILLUSTRATION

$$27^{\frac{2}{3}} = 3^2 = 9 \text{ or } 27^{\frac{2}{3}} = 729^{\frac{1}{3}} = 9.$$

Note:
It may be shown that all of the standard laws of indices may be used for fractional indices.

1.3.2 RADICALS (or Surds)

The symbol "$\sqrt{\ }$" is called a **"radical"** (or **"surd"**). It is used to indicate the positive or **"principal"** square root of a number. Thus $\sqrt{16} = 4$ and $\sqrt{25} = 5$.

The number under the radical is called the **"radicand"**.

Most of our work on radicals will deal with square roots, but we may have occasion to use other roots of a number. For instance the **principal n-th root** of a number a is denoted by $\sqrt[n]{a}$, and is a number x such that $x^n = a$. The number n is called the **index** of the radical but, of course, when $n = 2$ we usually leave the index out.

ILLUSTRATIONS

1. $\sqrt[3]{64} = 4$ since $4^3 = 64$.
2. $\sqrt[3]{-64} = -4$ since $(-4)^3 = -64$.
3. $\sqrt[4]{81} = 3$ since $3^4 = 81$.
4. $\sqrt[5]{32} = 2$ since $2^5 = 32$.
5. $\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$.

Note:
If the index of the radical is an odd number, then the radicand may be positive or negative; but if the index of the radical is an even number, then the radicand may not be negative since no even power of a negative number will ever give a negative result.

(a) Rules for Square Roots

In preparation for work which will follow in the next section, we list here the standard rules for square roots:

(i) $(\sqrt{a})^2 = a$

(ii) $\sqrt{a^2} = |a|$

(iii) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(iv) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

assuming that all of the radicals can be evaluated.

ILLUSTRATIONS

1. $\sqrt{9 \times 4} = \sqrt{36} = 6$ and $\sqrt{9} \times \sqrt{4} = 3 \times 2 = 6$.
2. $\sqrt{\frac{144}{36}} = \sqrt{4} = 2$ and $\frac{\sqrt{144}}{\sqrt{36}} = \frac{12}{6} = 2$.

(b) Rationalisation of Radical (or Surd) Expressions.

It is often desirable to eliminate expressions containing radicals from the denominator of a quotient. This process is called

rationalising the denominator.

The process involves multiplying numerator and denominator of the quotient by the same amount - an amount which eliminates the radicals in the denominator (often using the fact that the square root of a number multiplied by itself gives just the number;
i.e. $\sqrt{a}.\sqrt{a} = a$). We illustrate with examples:

EXAMPLES

1. Rationalise the surd form $\frac{5}{4\sqrt{3}}$

   Solution

   We simply multiply numerator and denominator by $\sqrt{3}$ to give

$$\frac{5}{4\sqrt{3}} = \frac{5}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{12}.$$

2. Rationalise the surd form $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$

   Solution

   Here we observe that, if we can convert the denominator into the cube root of b^n, where n is a whole multiple of 3, then the square root sign will disappear.

   We have

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}} \times \frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}} = \frac{\sqrt[3]{ab^2}}{\sqrt[3]{b^3}} = \frac{\sqrt[3]{ab^2}}{b}.$$

   If the denominator is of the form $\sqrt{a} + \sqrt{b}$, we multiply the numerator and the denominator by the expression $\sqrt{a} - \sqrt{b}$ because

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

3. Rationalise the surd form $\frac{4}{\sqrt{5}+\sqrt{2}}$.

   Solution

   Multiplying numerator and denominator by $\sqrt{5} - \sqrt{2}$ gives

$$\frac{4}{\sqrt{5} + \sqrt{2}} \times \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \frac{4\sqrt{5} - 4\sqrt{2}}{3}.$$

4. Rationalise the surd form $\frac{1}{\sqrt{3}-1}$.

Solution

Multiplying numerator and denominator by $\sqrt{3}+1$ gives

$$\frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{2}.$$

(c) Changing numbers to and from radical form

The modulus of any number of the form $a^{\frac{m}{n}}$ can be regarded as the principal n-th root of a^m; i.e.

$$\mid a^{\frac{m}{n}} \mid = \sqrt[n]{a^m}.$$

If a number of the type shown on the left is converted to the type on the right, we are said to have expressed it in radical form.

If a number of the type on the right is converted to the type on the left, we are said to have expressed it in exponential form.

Note:
The word **"exponent"** is just another word for **"power"** or **"index"** and the standard rules of indices will need to be used in questions of the type discussed here.

EXAMPLES

1. Express the number $x^{\frac{2}{5}}$ in radical form.

   Solution

   The answer is just

$$\sqrt[5]{x^2}.$$

2. Express the number $\sqrt[3]{a^5b^4}$ in exponential form.

   Solution

   Here we have

$$\sqrt[3]{a^5b^4} = (a^5b^4)^{\frac{1}{3}} = a^{\frac{5}{3}}b^{\frac{4}{3}}.$$

1.3.3 EXERCISES

1. Simplify
   (a) $5^7 \times 5^{13}$; (b) $9^8 \times 9^5$; (c) $11^2 \times 11^3 \times 11^4$.
2. Simplify
   (a) $\frac{15^3}{15^2}$; (b) $\frac{4^{18}}{4^9}$; (c) $\frac{5^{20}}{5^{19}}$.
3. Simplify
   (a) a^7a^3; (b) a^4a^5;
   (c) $b^{11}b^{10}b$; (d) $3x^6 \times 5x^9$.
4. Simplify
   (a) $(7^3)^2$; (b) $(4^2)^8$; (c) $(7^9)^2$.
5. Simplify
   (a) $(x^2y^3)(x^3y^2)$; (b) $(2x^2)(3x^4)$;
   (c) $(a^2bc^2)(b^2ca)$; (d) $\frac{6c^2d^3}{3cd^2}$.
6. Simplify
   (a) $\left(4^{-3}\right)^2$ (b) $a^{13}a^{-2}$;
   (c) $x^{-9}x^{-7}$; (d) $x^{-21}x^2x$;
   (e) $\frac{x^2y^{-1}}{z^3} \div \frac{z^2}{x^{-1}y^3}$.
7. Without using a calculator, evaluate the following:
   (a) $\frac{4^{-8}}{4^{-6}}$; (b) $\frac{3^{-5}}{3^{-8}}$.
8. Evaluate the following:
   (a) $64^{\frac{1}{3}}$; (b) $144^{\frac{1}{2}}$;
   (c) $16^{-\frac{1}{4}}$; (d) $25^{-\frac{1}{2}}$;
   (e) $16^{\frac{3}{2}}$; (f) $125^{-\frac{2}{3}}$.
9. Simplify the following radicals:
   (a) $-\sqrt[3]{-8}$; (b) $\sqrt{36x^4}$; (c) $\sqrt{\frac{9a^2}{36b^2}}$.
10. Rationalise the following surd forms:
    (a) $\frac{\sqrt{2}}{\sqrt{3}}$; (b) $\frac{\sqrt[3]{18}}{\sqrt[3]{2}}$; (c) $\frac{2+\sqrt{5}}{\sqrt{3}-2}$; (d) $\frac{\sqrt{a}}{\sqrt{a}+3\sqrt{b}}$.
11. Change the following to exponential form:
    (a) $\sqrt[4]{7^2}$; (b) $\sqrt[5]{a^2b}$; (c) $\sqrt[3]{9^5}$.

12. Change the following to radical form:
    (a) $b^{\frac{3}{5}}$; (b) $r^{\frac{5}{3}}$; (c) $s^{\frac{7}{3}}$.

1.3.4 ANSWERS TO EXERCISES

1. (a) 5^{20}; (b) 9^{13}; (c) 11^9.
2. (a) 15; (b) 4^9; (c) 5.
3. (a) a^{10}; (b) a^9; (c) b^{22}; (d) $15x^{15}$.
4. (a) 7^6; (b) 4^{16}; (c) 7^{18}.
5. (a) x^5y^5; (b) $6x^6$; (c) $a^3b^3c^3$; (d) $2cd$.
6. (a) 4^{-6}; (b) a^{11}; (c) x^{-16}; (d) x^{-18}; (e) xy^2z^{-5}.
7. (a) $\frac{1}{16}$; (b) 27.
8. (a) 4; (b) ± 12; (c) $\pm\frac{1}{2}$;
   (d) $\pm\frac{1}{5}$; (e) ± 64; (f) $\frac{1}{25}$;
9. (a) 2; (b) $6x^2$; (c) $\left|\frac{a}{2b}\right|$.
10. (a) $\frac{\sqrt{6}}{3}$; (b) $\frac{\sqrt[3]{72}}{2} = \sqrt[3]{9}$; (c) $-(2+\sqrt{5})(2+\sqrt{3})$; (d) $\frac{a-3\sqrt{ab}}{a-9b}$.
11. (a) $\left|7^{\frac{1}{2}}\right|$; (b) $a^{\frac{2}{5}}b^{\frac{1}{5}}$; (c) $9^{\frac{5}{3}}$.
12. (a) $\sqrt[5]{b^3}$; (b) $\sqrt[3]{r^5}$; (c) $\sqrt[3]{s^7}$.

"JUST THE MATHS"

UNIT NUMBER

1.4

ALGEBRA 4
(Logarithms)

by

A.J.Hobson

UNIT 1.4 - ALGEBRA 4 - LOGARITHMS

1.4.1 COMMON LOGARITHMS

The system of numbers with which we normally count and calculate has a base of 10; this means that each of the successive digits of a particular number correspond to that digit multiplied by a certain power of 10.

For example

$$73,520 = 7 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 2 \times 10^1.$$

Note:
Other systems (not discussed here) are sometimes used - such as the binary system which uses successive powers of 2.

The question now arises as to whether a given number can be expressed as a single power of 10, not necessarily an integer power. It will certainly need to be a **positive** number since powers of 10 are not normally negative (or even zero).

It can easily be verified by calculator, for instance that

$$1.99526 \simeq 10^{0.3}$$

and

$$2 \simeq 10^{0.30103}.$$

DEFINITION
In general, when it occurs that

$$x = 10^y,$$

for some positive number x, we say that y is the **"logarithm to base 10"** of x
(or **" common logarithm"** of x) and we write

$$y = \log_{10} x.$$

EXAMPLES

1. $\log_{10} 1.99526 = 0.3$ from the illustrations above.
2. $\log_{10} 2 = 0.30103$ from the illustrations above.
3. $\log_{10} 1 = 0$ simply because $10^0 = 1$.

1.4.2 LOGARITHMS IN GENERAL

In practice, with scientific work, only two bases of logarithms are ever used; but it will be useful to include here a general discussion of the definition and properties of logarithms to **any** base so that unnecessary repetition may be avoided. We consider only positive bases of logarithms in the general discussion.

DEFINITION
If B is a fixed positive number and x is another positive number such that

$$x = B^y,$$

we say that y is the **"logarithm to base B"** of x and we write

$$y = \log_B x.$$

EXAMPLES

1. $\log_B 1 = 0$ simply because $B^0 = 1$.
2. $\log_B B = 1$ simply because $B^1 = B$.
3. $\log_B 0$ doesn't really exist because no power of B could ever be equal to zero. But, since a very large negative power of B will be a very small positive number, we usually write

$$\log_B 0 = -\infty.$$

1.4.3 USEFUL RESULTS

In preparation for the general properties of logarithms, we note the following two results which can be obtained directly from the definition of a logarithm:

(a) For any positive number x,

$$x = B^{\log_B x}.$$

In other words, any positive number can be expressed as a power of B without necessarily using a calculator.

We have simply replaced the y in the statement $x = B^y$ by $\log_B x$ in the equivalent statement $y = \log_B x$.

(b) For any number y,

$$y = \log_B B^y.$$

In other words, any number can be expressed in the form of a logarithm without necessarily using a calculator.

We have simply replaced x in the statement $y = \log_B x$ by B^y in the equivalent statement $x = B^y$.

1.4.4 PROPERTIES OF LOGARITHMS

The following properties were once necessary for performing numerical calculations before electronic calculators came into use. We do not use logarithms for this purpose nowadays; but we do need their properties for various topics in scientific mathematics.

(a) The Logarithm of Product.

$$\log_B p.q = \log_B p + \log_B q.$$

Proof:
We need to show that, when $p.q$ is expressed as a power of B, that power is the expression on the right hand side of the above formula.

From Result (a) of the previous section,

$$p.q = B^{\log_B p}.B^{\log_B q} = B^{\log_B p + \log_B q},$$

by elementary properties of indices.

The result therefore follows.

(b) The Logarithm of a Quotient

$$\log_B \frac{p}{q} = \log_B p - \log_B q.$$

Proof:
The proof is along similar lines to that in (i).

From Result (a) of the previous section,

$$\frac{p}{q} = \frac{B^{\log_B p}}{B^{\log_B q}} = B^{\log_B p - \log_B q},$$

by elementary properties of indices.

The result therefore follows.

(c) The Logarithm of an Exponential

$$\log_B p^n = n \log_B p,$$

where n need not be an integer.

Proof:
From Result (a) of the previous section,

$$p^n = \left(B^{\log_B p}\right)^n = B^{n \log_B p},$$

by elementary properties of indices.

(d) The Logarithm of a Reciprocal

$$\log_B \frac{1}{q} = -\log_B q.$$

Proof:
This property may be proved in two ways as follows:

Method 1.
The left-hand side $= \log_B 1 - \log_B q = 0 - \log_B q = -\log_B q$.

Method 2.
The left-hand side $= \log_B q^{-1} = -\log_B q$.

(e) Change of Base

$$\log_B x = \frac{\log_A x}{\log_A B}.$$

Proof:
Suppose $y = \log_B x$, then $x = B^y$ and hence

$$\log_A x = \log_A B^y = y \log_A B.$$

Thus,

$$y = \frac{\log_A x}{\log_A B}$$

and the result follows.

Note:
The result shows that the logarithms of any set of numbers to a given base will be directly

proportional to the logarithms of the same set of numbers to another given base. This is simply because the number $\log_A B$ is a constant.

1.4.5 NATURAL LOGARITHMS

It was mentioned earlier that, in scientific work, only two bases of logarithms are ever used. One of these is base 10 and the other is a base which arises **naturally** out of elementary calculus when discussing the simplest available result for the "derivative" (rate of change) of a logarithm.

This other base turns out to be a non-recurring, non-terminating decimal quantity (irrational number) which is equal to 2.71828.....and clearly this would be inconvenient to write into the logarithm notation.

We therefore denote it by e to give the **"natural logarithm"** of a number, x, in the form $\log_e x$, although most scientific books use the alternative notation $\ln x$.

Note:
From the earlier change of base formula we can say that

$$\log_{10} x = \frac{\log_e x}{\log_e 10} \qquad \text{and} \qquad \log_e x = \frac{\log_{10} x}{\log_{10} e}.$$

EXAMPLES

1. Solve for x the indicial equation

$$4^{3x-2} = 26^{x+1}.$$

Solution

The secret of solving an equation where an unknown quantity appears in a power (or index or exponent) is to take logarithms of both sides first.

Here we obtain

$$(3x-2)\log_{10} 4 = (x+1)\log_{10} 26;$$

$$(3x-2)0.6021 = (x+1)1.4150;$$

$$1.8063x - 1.2042 = 1.4150x + 1.4150;$$

$$(1.8603 - 1.4150)x = 1.4150 + 1.2042;$$

$$0.3913x = 2.6192;$$

$$x = \frac{2.6192}{0.3913} \simeq 6.6936$$

2. Rewrite the expression

$$4x + \log_{10}(x+1) - \log_{10} x - \frac{1}{2}\log_{10}(x^3 + 2x^2 - x)$$

as the common logarithm of a single mathematical expression.

Solution

The secret here is to make sure that every term in the given expression is converted, where necessary, to a logarithm with no multiple in front of it or behind it. In this case, we need first to write $4x = \log_{10} 10^{4x}$ and $\frac{1}{2}\log_{10}(x^3 + 2x^2 - x) = \log_{10}\left(x^3 + 2x^2 - x\right)^{\frac{1}{2}}$.
We can then use the results for the logarithms of a product and a quotient to give

$$\log_{10} \frac{10^{4x}(x+1)}{x\sqrt{(x^3 + 2x^2 - x)}}.$$

3. Rewrite without logarithms the equation

$$2x + \ln x = \ln(x - 7).$$

Solution

This time, we need to convert both sides to the natural logarithm of a single mathematical expression in order to remove the logarithms completely.

$$2x + \ln x = \ln e^{2x} + \ln x = \ln xe^{2x}.$$

Hence,

$$xe^{2x} = x - 7.$$

4. Solve for x the equation

$$6\ln 4 + \ln 2 = 3 + \ln x.$$

Solution

In view of the facts that $6\ln 4 = \ln 4^6$ and $3 = \ln e^3$, the equation can be written

$$\ln 2(4^6) = \ln xe^3.$$

Hence,

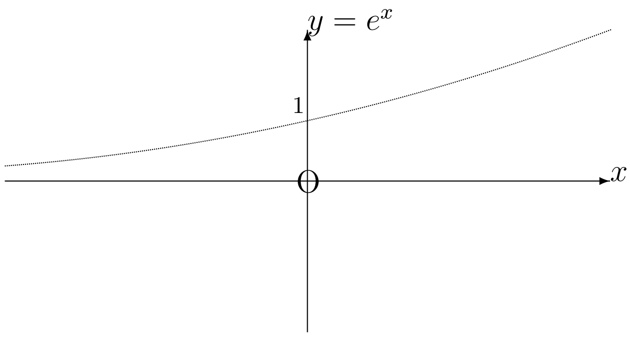
$$2(4^6) = xe^3,$$

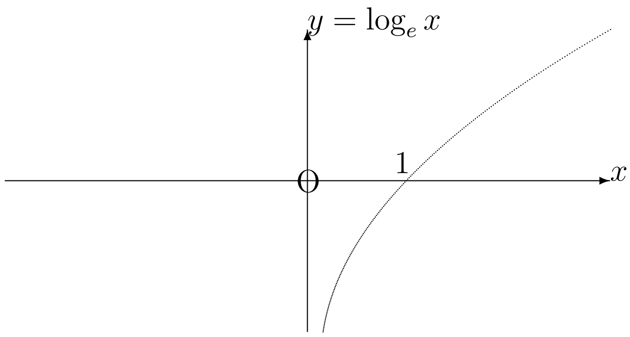
so that

$$x = \frac{2(4^6)}{e^3} \simeq 407.856$$

1.4.6 GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In the applications of mathematics to science and engineering, two commonly used "functions" are $y = e^x$ and $y = \log_e x$. Their graphs are as follows:





They are closely linked with each other by virtue of the two equivalent statements:

$$P = \log_e Q \quad \text{and} \quad Q = e^P$$

for any number P and any postive number Q.

Because of these statements, we would expect similarities in the graphs of the equations

$$y = \log_e x \quad \text{and} \quad y = e^x.$$

1.4.7 LOGARITHMIC SCALES

In a certain kind of graphical work (see Unit 5.3), some use is made of a linear scale along which numbers can be allocated according to their logarithmic distances from a chosen origin of measurement.

Considering firstly that 10 is the base of logarithms, the number 1 is always placed at the zero of measurement (since $\log_{10} 1 = 0$); 10 is placed at the first unit of measurement (since $\log_{10} 10 = 1$), 100 is placed at the second unit of measurement (since $\log_{10} 100 = 2$), and so on.

Negative powers of 10 such as $10^{-1} = 0.1$, $10^{-2} = 0.01$ etc. are placed at the points corresponding to -1 and -2 etc. respectively on an ordinary linear scale.

The logarithmic scale appears therefore in **"cycles"**, each cycle corresponding to a range of numbers between two consecutive powers of 10.

Intermediate numbers are placed at intervals which correspond to their logarithm values.

An extract from a typical logarithmic scale would be as follows:

0.1 0.2 0.3 0.4 1 2 3 4 10

Notes:
(i) A given set of numbers will determine how many cycles are required on the logarithmic scale. For example .3, .6, 5, 9, 23, 42, 166 will require **four** cycles.

(ii) Commercially printed logarithmic scales do not specify the base of logarithms; the change of base formula implies that logarithms to different bases are proportional to each other and hence their logarithmic scales will have the same relative shape.

1.4.8 EXERCISES

1. Without using tables or a calculator, evaluate

   (a) $\log_{10} 27 \div \log_{10} 3$;

   (b) $(\log_{10} 16 - \log_{10} 2) \div \log_{10} 2$.

2. Using properties of logarithms where possible, solve for x the following equations:

   (a) $\log_{10} \frac{7}{2} + 2\log_{10} \frac{4}{3} - \log_{10} \frac{7}{45} = 1 + \log_{10} x$;

   (b) $2\log_{10} 6 - (\log_{10} 4 + \log_{10} 9) = \log_{10} x$.

   (c) $10^x = 5(2^{10})$.

3. From the definition of a logarithm or the change of base formula, evaluate the following:

   (a) $\log_2 7$;

   (b) $\log_3 0.04$;

   (c) $\log_5 3$;

   (d) $3\log_3 2 - \log_3 4 + \log_3 \frac{1}{2}$.

4. Obtain y in terms of x for the following equations:

   (a) $2\ln y = 1 - x^2$;

   (b) $\ln x = 5 - 3\ln y$.

5. Rewrite the following statements without logarithms:

   (a) $\ln x = -\frac{1}{2}\ln(1 - 2v^3) + \ln C$;

   (b) $\ln(1 + y) = \frac{1}{2}x^2 + \ln 4$;

   (c) $\ln(4 + y^2) = 2\ln(x + 1) + \ln A$.

6. (a) If $\frac{I_0}{I} = 10^{ac}$, find c in terms of the other quantities in the formula.

   (b) If $y^p = Cx^q$, find q in terms of the other quantities in the formula.

1.4.9 ANSWERS TO EXERCISES

1. (a) 3; (b) 3.
2. (a) 4; (b) 1; (c) 3.70927
3. (a) 2.807; (b) −2.930; (c) 0.683; (d) 0
4. (a)
$$y = e^{\frac{1}{2}(1-x^2)};$$
(b)
$$y = \sqrt[3]{\frac{e^5}{x}}.$$
5. (a)
$$x = \frac{C}{\sqrt{1-2v^3}};$$
(b)
$$1 + y = 4e^{\frac{x^2}{2}};$$
(c)
$$4 + y^2 = A(x+1)^2.$$
6. (a)
$$c = \frac{1}{a}\log_{10}\frac{I_0}{I};$$
(b)
$$q = \frac{p\log y - \log C}{\log x},$$
using any base.

"JUST THE MATHS"

UNIT NUMBER

1.5

ALGEBRA 5
(Manipulation of algebraic expressions)

by

A.J.Hobson

UNIT 1.5 - ALGEBRA 5
MANIPULATION OF ALGEBRAIC EXPRESSIONS

1.5.1 SIMPLIFICATION OF EXPRESSIONS

An algebraic expression will, in general, contain a mixture of alphabetical symbols together with one or more numerical quantities; some of these symbols and numbers may be bracketted together.

Using the Language of Algebra and the Laws of Algebra discussed earlier, the method of simplification is to remove brackets and collect together any terms which have the same format

Some elementary illustrations are as follows:

1. $a + a + a + 3 + b + b + b + b + 8 \equiv 3a + 4b + 11.$
2. $11p^2 + 5q^7 - 8p^2 + q^7 \equiv 3p^2 + 6q^7.$
3. $a(2a - b) + b(a + 5b) - a^2 - 4b^2 \equiv 2a^2 - ab + ba + 5b^2 - a^2 - 4b^2 \equiv a^2 + b^2.$

More frequently, the expressions to be simplified will involve symbols which represent both the constants and variables encountered in scientific work. Typical examples in pure mathematics use symbols like x, y and z to represent the variable quantities.

Further illustrations use this kind of notation and, for simplicity, we shall omit the full-stop type of multiplication sign between symbols.

1. $x(2x + 5) + x^2(3 - x) \equiv 2x^2 + 5x + 3x^2 - x^3 \equiv 5x^2 + 5x - x^3.$
2. $x^{-1}(4x - x^2) - 6(1 - 3x) \equiv 4 - x - 6 + 18x \equiv 17x - 2.$

We need also to consider the kind of expression which involves two or more brackets multiplied together; but the routine is just an extension of what has already been discussed.

For example consider the expression

$$(a + b)(c + d).$$

Taking the first bracket as a single item for the moment, the Distributive Law gives

$$(a + b)c + (a + b)d.$$

Using the Distributive Law a second time gives

$$ac + bc + ad + bd.$$

In other words, each of the two terms in the first bracket are multiplied by each of the two terms in the second bracket, giving four terms in all.

Again, we illustrate with examples:

EXAMPLES

1. $(x + 3)(x - 5) \equiv x^2 + 3x - 5x - 15 \equiv x^2 - 2x - 15.$
2. $(x^3 - x)(x + 5) \equiv x^4 - x^2 + 5x^3 - 5x.$
3. $(x + a)^2 \equiv (x + a)(x + a) \equiv x^2 + ax + ax + a^2 \equiv x^2 + 2ax + a^2.$
4. $(x + a)(x - a) \equiv x^2 + ax - ax - a^2 \equiv x^2 - a^2.$

The last two illustrations above are significant for later work because they incorporate, respectively, the standard results for a **"perfect square"** and **"the difference between two squares"**.

1.5.2 FACTORISATION

Introduction
In an algebraic context, the word **"factor"** means the same as **"multiplier"**. Thus, to factorise an algebraic expression, is to write it as a product of separate multipliers or factors.

Some simple examples will serve to introduce the idea:

EXAMPLES

1.
$$3x + 12 \equiv 3(x + 4).$$

2.
$$8x^2 - 12x \equiv x(8x - 12) \equiv 4x(2x - 3).$$

3.
$$5x^2 + 15x^3 \equiv x^2(5 + 15x) \equiv 5x^2(1 + 3x).$$

4.

$$6x + 3x^2 + 9xy \equiv x(6 + 3x + 9y) \equiv 3x(2 + x + 3y).$$

Note:
When none of the factors can be broken down into simpler factors, the original expression is said to have been factorised into **"irreducible factors"**.

Factorisation of quadratic expressions

A **"quadratic expression"** is an expression of the form

$$ax^2 + bx + c,$$

where, usually, a, b and c are fixed numbers (constants) while x is a variable number. The numbers a and b are called, respectively, the **"coefficients"** of x^2 and x while c is called the **"constant term"**; but, for brevity, we often say that the quadratic expression has coefficients a, b and c.

Note:
It is important that the coefficient a does not have the value zero otherwise the expression is not quadratic but **"linear"**.

The method of factorisation is illustrated by examples:

(a) When the coefficient of x^2 is 1

EXAMPLES

1.

$$x^2 + 5x + 6 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn.$$

This implies that $5 = m + n$ and $6 = mn$ which, by inspection gives $m = 2$ and $n = 3$. Hence

$$x^2 + 5x + 6 \equiv (x + 2)(x + 3).$$

2.

$$x^2 + 4x - 21 \equiv (x + m)(x + n) \equiv x^2 + (m + n)x + mn.$$

This implies that $4 = m + n$ and $-21 = mn$ which, by inspection, gives $m = -3$ and $n = 7$. Hence

$$x^2 + 4x - 21 \equiv (x - 3)(x + 7).$$

Notes:
(i) In general, for simple cases, it is better to try to carry out the factorisation entirely by inspection. This avoids the cumbersome use of m and n in the above two examples as follows:

$$x^2 + 2x - 8 \equiv (x+?)(x+?).$$

The two missing numbers must be such that their sum is 2 and their product is -8. The required values are therefore -2 and 4. Hence

$$x^2 + 2x - 8 \equiv (x - 2)(x + 4).$$

(ii) It is necessary, when factorising a quadratic expression, to be aware that either a perfect square or the difference of two squares might be involved. In these cases, the factorisation is a little simpler. For instance:

$$x^2 + 10x + 25 \equiv (x + 5)^2$$

and

$$x^2 - 64 \equiv (x - 8)(x + 8).$$

(iii) Some quadratic expressions will not conveniently factorise at all. For example, in the expression

$$x^2 - 13x + 2,$$

we cannot find two whole numbers whose sum is -13 while, at the same time, their product is 2.

(b) When the coefficient of x^2 is not 1

Quadratic expressions of this kind are usually more difficult to factorise than those in the previous paragraph. We first need to determine the possible pairs of factors of the coefficient of x^2 and the possible pairs of factors of the constant term; then we need to consider the possible combinations of these which provide the correct factors of the quadratic expression.

EXAMPLES

1. To factorise the expression

$$2x^2 + 11x + 12,$$

we observe that 2 is the product of 2 and 1 only, while 12 is the product of either 12 and 1, 6 and 2 or 4 and 3. All terms of the quadratic expression are positive and hence we may try $(2x+1)(x+12)$, $(2x+12)(x+1)$, $(2x+6)(x+2)$, $(2x+2)(x+6)$, $(2x+4)(x+3)$ and $(2x+3)(x+4)$. Only the last of these is correct and so

$$2x^2 + 11x + 12 \equiv (2x + 3)(x + 4).$$

2. To factorise the expression

$$6x^2 + 7x - 3,$$

we observe that 6 is the product of either 6 and 1 or 3 and 2 while 3 is the product of 3 and 1 only. A negative constant term implies that, in the final result, its two factors must have opposite signs. Hence we may try $(6x + 3)(x - 1)$, $(6x - 3)(x + 1)$, $(6x + 1)(x - 3)$, $(6x - 1)(x + 3)$, $(3x + 3)(2x - 1)$, $(3x - 3)(2x + 1)$, $(3x + 1)(2x - 3)$ and $(3x - 1)(2x + 3)$. Again, only the last of these is correct and so

$$6x^2 + 7x - 3 \equiv (3x - 1)(2x + 3).$$

Note:
The more factors there are in the coefficients considered, the more possibilities there are to try of the final factorisation.

1.5.3 COMPLETING THE SQUARE IN A QUADRATIC EXPRESSION

The following work is based on the standard expansions

$$(x + a)^2 \equiv x^2 + 2ax + a^2$$

and

$$(x - a)^2 \equiv x^2 - 2ax + a^2.$$

Both of these last expressions are called **"complete squares"** (or **"perfect squares"**) in which we observe that one half of the coefficient of x, when multiplied by itself, gives the constant term. That is

$$(\frac{1}{2} \times 2a)^2 = a^2.$$

ILLUSTRATIONS

1.

$$x^2 + 6x + 9 \equiv (x + 3)^2.$$

2.

$$x^2 - 8x + 16 \equiv (x-4)^2.$$

3.

$$4x^2 - 4x + 1 \equiv 4\left[x^2 - x + \frac{1}{4}\right] \equiv 4\left(x - \frac{1}{2}\right)^2.$$

Of course it may happen that a given quadratic expression is NOT a complete square; but, by using one half of the coefficient of x, we may express it as the sum or difference of a complete square and a constant. This process is called **"completing the square"**, and the following examples illustrate it:

EXAMPLES

1.

$$x^2 + 6x + 11 \equiv (x+3)^2 + 2.$$

2.

$$x^2 - 8x + 7 \equiv (x-4)^2 - 9.$$

3.

$$\begin{aligned} 4x^2 - 4x + 5 &\equiv 4\left[x^2 - x + \frac{5}{4}\right] \\ &\equiv 4\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{5}{4}\right] \\ &\equiv 4\left[\left(x - \frac{1}{2}\right)^2 + 1\right] \\ &\equiv 4\left(x - \frac{1}{2}\right)^2 + 4. \end{aligned}$$

1.5.4 ALGEBRAIC FRACTIONS

We first recall the basic rules for combining fractions, namely

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

We also note that a single algebraic fraction may sometimes be simplified by the cancellation of common factors between the numerator and the denominator.

EXAMPLES

1.
$$\frac{5}{25+15x} \equiv \frac{1}{5+3x}, \quad \text{assuming that} \quad x \neq -\frac{5}{3}.$$

2.
$$\frac{4x}{3x^2+x} \equiv \frac{4}{3x+1}, \quad \text{assuming that} \quad x \neq 0 \quad \text{or} \quad -\frac{1}{3}.$$

3.
$$\frac{x+2}{x^2+3x+2} \equiv \frac{x+2}{(x+2)(x+1)} \equiv \frac{1}{x+1}, \quad \text{assuming that} \quad x \neq -1 \quad \text{or} \quad -2.$$

These elementary principles may now be used with more advanced combinations of algebraic fractions

EXAMPLES

1. Simplify the expression
$$\frac{3x+6}{x^2+3x+2} \times \frac{x+1}{2x+8}.$$

   Solution

   Using factorisation where possible, together with the rule for multiplying fractions, we obtain
$$\frac{3(x+2)(x+1)}{2(x+4)(x+1)(x+2)} \equiv \frac{3}{2(x+4)},$$
   assuming that $x \neq -1, -2$ or -4.

2. Simplify the expression
$$\frac{3}{x+2} \div \frac{x}{2x+4}.$$

   Solution

   Using factorisation where possible together with the rule for dividing fractions, we obtain
$$\frac{3}{x+2} \times \frac{2x+4}{x} \equiv \frac{3}{x+2} \times \frac{2(x+2)}{x} \equiv \frac{6}{x},$$
   assuming that $x \neq 0$ or -2.

3. Express
$$\frac{4}{x+y} - \frac{3}{y}$$
as a single fraction.

Solution

From the basic rule for adding and subtracting fractions, we obtain
$$\frac{4y - 3(x+y)}{(x+y)y} \equiv \frac{y - 3x}{(x+y)y},$$
assuming that $y \neq 0$ and $x \neq -y$.

4. Express
$$\frac{x}{x+1} + \frac{4 - x^2}{x^2 - x - 2}$$
as a single fraction.

Solution

This example could be tackled in the same way as the previous one but it is worth noticing that $x^2 - x - 2 \equiv (x+1)(x-2)$. Consequently, it is worth putting both fractions over the simplest common denominator, namely $(x+1)(x-2)$. Hence we obtain, if $x \neq 2$ or -1,
$$\frac{x(x-2)}{(x+1)(x-2)} + \frac{4 - x^2}{(x+1)(x-2)} \equiv \frac{x^2 - 2x + 4 - x^2}{(x+1)(x-2)} \equiv \frac{2(2-x)}{(x+1)(x-2)} \equiv -\frac{2}{x+1}.$$

1.5.5 EXERCISES

1. Write down in their simplest forms
   (a) $5a - 2b - 3a + 6b$; (b) $11p + 5q - 2q + p$.
2. Simplify the following expressions:
   (a) $3x^2 - 2x + 5 - x^2 + 7x - 2$; (b) $x^3 + 5x^2 - 2x + 1 + x - x^2$.
3. Expand and simplify the following expressions:
   (a) $x(x^2 - 3x) + x^2(4x + 7)$; (b) $(2x - 1)(2x + 1) - x^2 + 5x$;
   (c) $(x + 3)(2x^2 - 5)$; (d) $2(3x + 1)^2 + 5(x - 7)^2$.
4. Factorise the following expressions:
   (a) $xy + 4x^2y$; (b) $2abc - 6ab^2$;
   (c) $\pi r^2 + 2\pi rh$; (d) $2xy^2z + 4x^2z$.

5. Factorise the following quadratic expressions:
   (a) $x^2 + 8x + 12$; (b) $x^2 + 11x + 18$; (c) $x^2 + 13x - 30$;
   (d) $3x^2 + 11x + 6$; (e) $4x^2 - 12x + 9$; (f) $9x^2 - 64$.

6. Complete the square in the following quadratic expressions:
   (a) $x^2 - 10x - 26$; (b) $x^2 - 5x + 4$; (c) $7x^2 - 2x + 1$.

7. Simplify the following:
   (a) $\frac{x^2+4x+4}{x^2+5x+6}$; (b) $\frac{x^2-1}{x^2+2x+1}$,
   assuming the values of x to be such that no denominators are zero.

8. Express each of the following as a single fraction:
   (a) $\frac{3}{x} + \frac{4}{y}$; (b) $\frac{4}{x} - \frac{6}{2x}$;
   (c) $\frac{1}{x+1} + \frac{1}{x+2}$; (d) $\frac{5x}{x^2+5x+4} - \frac{3}{x+4}$,
   assuming that the values of x and y are such that no denominators are zero.

1.5.6 ANSWERS TO EXERCISES

1. (a) $2a + 4b$; (b) $12p + 3q$.

2. (a) $2x^2 + 5x + 3$; (b) $x^3 + 4x^2 - x + 1$.

3. (a) $5x^3 + 4x^2$; (b) $3x^2 + 5x - 1$.
   (c) $2x^3 + 6x^2 - 5x - 15$; (d) $23x^2 - 58x + 247$.

4. (a) $xy(1 + 4x)$; (b) $2ab(c - 3b)$;
   (c) $\pi r(r + 2h)$; (d) $2xz(y^2 + 2x)$.

5. (a) $(x + 2)(x + 6)$; (b) $(x + 2)(x + 9)$; (c) $(x - 2)(x + 15$;
   (d) $(3x + 2)(x + 3)$; (e) $(2x - 3)^2$; (f) $(3x - 8)(3x + 8)$.

6. (a) $(x - 5)^2 - 51$; (b) $\left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$; (c) $7\left[\left(x - \frac{1}{7}\right)^2 + \frac{6}{49}\right]$.

7. (a) $\frac{x+2}{x+3}$; (b) $\frac{x-1}{x+1}$.

8. (a) $\frac{3y+4x}{xy}$; (b) $\frac{1}{x}$;
   (c) $\frac{2x+3}{(x+1)(x+2)}$; (d) $\frac{2x-3}{x^2+5x+4}$.

"JUST THE MATHS"

UNIT NUMBER

1.6

ALGEBRA 6
(Formulae and algebraic equations)

by

A.J.Hobson

UNIT 1.6 - ALGEBRA 6 - FORMULAE AND ALGEBRAIC EQUATIONS

1.6.1 TRANSPOSITION OF FORMULAE

In dealing with technical formulae, it is often required to single out one of the quantities involved in terms of all the others. We are said to **"transpose the formula"** and make that quantity **"the subject of the equation"**.

In order to do this, steps of the following types may be carried out on both sides of a given formula:

(a) Addition or subtraction of the same value;

(b) Multiplication or division by the same value;

(c) The raising of both sides to equal powers;

(d) Taking logarithms of both sides.

EXAMPLES

1. Make x the subject of the formula

$$y = 3(x + 7).$$

Solution
Dividing both sides by 3 gives $\frac{y}{3} = x + 7$; then subtracting 7 gives $x = \frac{y}{3} - 7$.

2. Make y the subject of the formula

$$a = b + c\sqrt{x^2 - y^2}.$$

Solution
(i) Subtracting b gives $a - b = c\sqrt{x^2 - y^2}$;
(ii) Dividing by c gives $\frac{a-b}{c} = \sqrt{x^2 - y^2}$;
(iii) Squaring both sides gives $\left(\frac{a-b}{c}\right)^2 = x^2 - y^2$;
(iv) Subtracting x^2 gives $\left(\frac{a-b}{c}\right)^2 - x^2 = -y^2$;
(v) Multiplying throughout by -1 gives $x^2 - \left(\frac{a-b}{c}\right)^2 = y^2$;

(vi) Taking square roots of both sides gives

$$y = \pm\sqrt{x^2 - \left(\frac{a-b}{c}\right)^2}.$$

3. Make x the subject of the formula

$$e^{2x-1} = y^3.$$

Solution
Taking natural logarithms of both sides of the formula

$$2x - 1 = 3\ln y.$$

Hence

$$x = \frac{3\ln y + 1}{2}.$$

Note:
A genuine scientific formula will usually involve quantities which can assume only positive values; in which case we can ignore the negative value of a square root.

1.6.2 SOLUTION OF LINEAR EQUATIONS

A Linear Equation in a variable quantity x has the general form

$$ax + b = c.$$

Its solution is obtained by first subtracting b from both sides then dividing both sides by a. That is

$$x = \frac{c-b}{a}.$$

EXAMPLES

1. Solve the equation

$$5x + 11 = 20.$$

Solution
The solution is clearly $x = \frac{20-11}{5} = \frac{9}{5} = 1.8$

2. Solve the equation
$$3 - 7x = 12.$$

Solution
This time, the solution is $x = \frac{12-3}{-7} = \frac{9}{-7} \simeq -1.29$

1.6.3 SOLUTION OF QUADRATIC EQUATIONS

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0,$$

where a, b and c are constants and $a \neq 0$.

We shall discuss three methods of solving such an equation related very closely to the previous discussion on quadratic expressions. The first two methods can be illustrated by examples.

(a) By Factorisation
This method depends on the ability to determine the factors of the left hand side of the given quadratic equation. This will usually be by trial and error.

EXAMPLES

1. Solve the quadratic equation
$$6x^2 + x - 2 = 0.$$

   Solution
   In factorised form, the equation can be written
$$(3x + 2)(2x - 1) = 0.$$
   Hence, $x = -\frac{2}{3}$ or $x = \frac{1}{2}$.

2. Solve the quadratic equation
$$15x^2 - 17x - 4 = 0.$$

   Solution
   In factorised form, the equation can be written
$$(5x + 1)(3x - 4) = 0.$$
   Hence, $x = -\frac{1}{5}$ or $x = \frac{4}{3}$

(b) By Completing the square

By looking at some numerical examples of this method, we shall be led naturally to a third method involving a **universal** formula for solving any quadratic equation.

EXAMPLES

1. Solve the quadratic equation

$$x^2 - 4x - 1 = 0.$$

Solution
On completing the square, the equation can be written

$$(x-2)^2 - 5 = 0.$$

Thus,

$$x - 2 = \pm\sqrt{5}.$$

That is,

$$x = 2 \pm \sqrt{5}.$$

Left as it is, this is an answer in **"surd form"** but it could, of course, be expressed in decimals as 4.236 and -0.236.

2. Solve the quadratic equation

$$4x^2 + 4x - 2 = 0.$$

Solution
The equation may be written

$$4\left[x^2 + x - \frac{1}{2}\right] = 0$$

and, on completing the square,

$$4\left[\left(x + \frac{1}{2}\right)^2 - \frac{3}{4}\right] = 0.$$

Hence,

$$\left(x + \frac{1}{2}\right)^2 = \frac{3}{4},$$

giving

$$x + \frac{1}{2} = \pm\sqrt{\frac{3}{4}}.$$

That is,

$$x = -\frac{1}{2} \pm \sqrt{\frac{3}{4}}$$

or

$$x = \frac{-1 \pm \sqrt{3}}{2}.$$

(c) By the Quadratic Formula

Starting now with an arbitrary quadratic equation

$$ax^2 + bx + c = 0,$$

we shall use the method of completing the square in order to establish the **general** solution.

The sequence of steps is as follows:

$$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0;$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] = 0;$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a};$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}};$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}};$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:
The quantity $b^2 - 4ac$ is called the **"discriminant"** of the equation and gives either two solutions, one solution or no solutions according as its value is positive, zero or negative.

The single solution case is usually interpreted as a pair of coincident solutions while the no solution case really means no **real** solutions. A more complete discussion of this case arises in the subject of **"complex numbers"** (see Unit 6.1).

EXAMPLES

Use the quadratic formula to solve the following:

1.
$$x^2 + 2x - 35 = 0.$$

Solution

$$x = \frac{-2 \pm \sqrt{4 + 140}}{2} = \frac{-2 \pm 12}{2} = 5 \quad \text{or} \quad -7.$$

2.
$$2x^2 - 3x - 7 = 0.$$

Solution

$$x = \frac{3 \pm \sqrt{9 + 56}}{4} = \frac{3 \pm \sqrt{65}}{4} = \frac{3 \pm 8.062}{4} \simeq 2.766 \quad \text{or} \quad -1.266$$

3.
$$9x^2 - 6x + 1 = 0.$$

Solution

$$x = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6}{18} = \frac{1}{3} \quad \text{only}.$$

4.
$$5x^2 + x + 1 = 0.$$

Solution

$$x = \frac{-1 \pm \sqrt{1 - 20}}{10}.$$

Hence, there are no real solutions

1.6.4 EXERCISES

1. Make the given symbol the subject of the following formulae
   (a) x: $a(x-a) = b(x+b)$;
   (b) b: $a = \frac{2-7b}{3+5b}$;
   (c) r: $n = \frac{1}{2L}\sqrt{\frac{r}{p}}$;
   (d) x: $ye^{x^2+1} = 5$.
2. Solve, for x, the following equations
   (a) $14x = 35$;
   (b) $3x - 4.7 = 2.8$;
   (c) $4(2x-5) = 3(2x+8)$.
3. Solve the following quadratic equations by factorisation:
   (a) $x^2 + 5x - 14 = 0$;
   (b) $8x^2 + 2x - 3 = 0$.
4. Where possible, solve the following quadratic equations by the formula:
   (a) $2x^2 - 3x + 1 = 0$; (b) $4x = 45 - x^2$;
   (c) $16x^2 - 24x + 9 = 0$; (d) $3x^2 + 2x + 11 = 0$.

1.6.5 ANSWERS TO EXERCISES

1. (a) $x = \frac{b^2+a^2}{a-b}$;
   (b) $b = \frac{2-3a}{7+5a}$;
   (c) $r = 4n^2L^2p$;
   (d) $x = \pm\sqrt{\ln 5 - \ln y - 1}$.
2. (a) 2.5; (b) 2.5; (c) 22.
3. (a) $x = -7$, $x = 2$;
   (b) $x = -\frac{3}{4}$, $x = \frac{1}{2}$.
4. (a) $x = 1$, $x = \frac{1}{2}$;
   (b) $x = 5$, $x = -9$;
   (c) $x = \frac{3}{4}$ only;
   (d) No solutions.

"JUST THE MATHS"

UNIT NUMBER

1.7

ALGEBRA 7
(Simultaneous linear equations)

by

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UNIT 1.7 - ALGEBRA 7 - SIMULTANEOUS LINEAR EQUATIONS

Introduction

When Mathematics is applied to scientific work, it is often necessary to consider several statements involving several variables which are required to have a common solution for those variables. We illustrate here the case of two simultaneous linear equations in two variables x and y and three simultaneous linear equations in three unknowns x, y and z.

1.7.1 TWO SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the simultaneous linear equations:

$$\begin{aligned} ax + by &= p, \\ cx + dy &= q. \end{aligned}$$

To obtain the solution we first eliminate one of the variables in order to calculate the other. For instance, to eliminate x, we try to make coefficient of x the same in both equations so that, subtracting one statement from the other, x disappears. In this case, we multiply the first equation by c and the second equation by a to give

$$\begin{aligned} cax + cby &= cp, \\ acx + ady &= aq. \end{aligned}$$

Subtracting the second equation from the first, we obtain

$$y(cb - ad) = cp - aq$$

which, in turn, means that

$$y = \frac{cp - aq}{cb - ad} \text{ provided } cb - ad \neq 0.$$

Having found the value of y, we could then either substitute back into one of the original equations to find x or begin again by eliminating y.

However, it is better not to think of the above explanation as providing a **formula** for solving two simultaneous linear equations. Rather, a numerical example should be dealt with from first principles with the numbers provided.

Note:
$cb - ad = 0$ relates to a degenerate case in which the left hand sides of the two equations are proportional to each other. Such cases will not be dealt with at this stage.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 6x - 2y &= 1, \qquad (1) \\ 4x + 7y &= 9. \qquad (2) \end{aligned}$$

Solution

Multiplying the first equation by 4 and the second equation by 6,

$$\begin{aligned} 24x - 8y &= 4, \qquad (4) \\ 24x + 42y &= 54. \qquad (5) \end{aligned}$$

Subtracting the second of these from the first, we obtain $-50y = -50$ and hence $y = 1$.

Substituting back into equation (1), $6x - 2 = 1$, giving $6x = 3$, and, hence, $x = \frac{1}{2}$.

Alternative Method

Multiplying the first equation by 7 and the second equation by -2, we obtain

$$\begin{aligned} 42x - 14y &= 7, \qquad (5) \\ -8x - 14y &= -18. \qquad (6) \end{aligned}$$

Subtracting equation (6) from equation (5) gives $50x = 25$ and hence, $x = \frac{1}{2}$.

Substituting into equation (1) gives $3 - 2y = 1$ and hence, $-2y = -2$; that is $y = 1$.

1.7.2 THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

Here, we consider three simultaneous equations of the general form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3; \end{aligned}$$

but the method will be illustrated by a particular example.

The object of the method is to eliminate one of the variables from two different pairs of the three equations so that we are left with a pair of simultaneous equations from which to calculate the other two variables.

EXAMPLE

Solve, for x, y and z, the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 9, \qquad (1) \\ 2x + y - z &= 1, \qquad (2) \\ 3x - 2y + z &= 8. \qquad (3) \end{aligned}$$

Solution

Firstly, we may eliminate z from equations (2) and (3) by adding them together. We obtain

$$5x - y = 9. \qquad (4)$$

Secondly, we may eliminate z from equations (1) and (2) by doubling equation (2) and adding it to equation (1). We obtain

$$5x + y = 11. \qquad (5)$$

If we now add equation (4) to equation (5), y will be eliminated to give

$$10x = 20 \quad \text{or} \quad x = 2.$$

Similarly, if we subtract equation (4) from equation (5), x will be eliminated to give

$$2y = 2 \quad \text{or} \quad y = 1.$$

Finally, if we substitute our values of x and y into one of the original equations [say equation (3)] we obtain

$$z = 8 - 3x + 2y = 8 - 6 + 2 = 4.$$

Thus,

$$x = 2, \quad y = 1 \quad \text{and} \quad z = 4.$$

1.7.3 ILL-CONDITIONED EQUATIONS

In the simultaneous linear equations of genuine scientific problems, the coefficients will often be decimal quantitities that have already been subjected to rounding errors; and the solving process will tend to amplify these errors. The result may be that such errors swamp the values of the variables being solved for; and we have what is called an **"ill-conditioned"** set of equations. The opposite of this is a **"well-conditioned"** set of equations and all of those so far discussed have been well-conditioned. But let us consider, now, the following example:

EXAMPLE

The simultaneous linear equations

$$\begin{aligned} x + y &= 1, \\ 1.001x + y &= 2 \end{aligned}$$

have the common solution $x = 1000$, $y = -999$.

However, suppose that the coefficient of x in the second equation is altered to 1.000, which is a mere 0.1%. Then the equations have no solution at all since $x + y$ cannot be equal to 1 and 2 at the same time.

Secondly, suppose that the coefficient of x in the second equation is altered to 0.999 which is still only a 0.2% alteration.

The solutions obtained are now $x = -1000$, $y = 1001$ and so a change of about 200% has occurred in original values of x and y.

1.7.4 EXERCISES

1. Solve, for x and y, the following pairs of simultaneous linear equations:

   (a)
$$\begin{aligned} x - 2y &= 5, \\ 3x + y &= 1; \end{aligned}$$

   (b)
$$\begin{aligned} 2x + 3y &= 42, \\ 5x - y &= 20. \end{aligned}$$

2. Solve, for x, y and z, the following sets of simultaneous equations:

   (a)
$$\begin{aligned} x + y + z &= 0, \\ 2x - y - 3z &= 4, \\ 3x + 3y &= 7; \end{aligned}$$

   (b)
$$\begin{aligned} x + y - 10 &= 0, \\ y + z - 3 &= 0, \\ x + z + 1 &= 0; \end{aligned}$$

   (c)
$$\begin{aligned} 2x - y - z &= 6, \\ x + 3y + 2z &= 1, \\ 3x - y - 5z &= 1; \end{aligned}$$

(d)

$$\begin{aligned} 2x - 5y + 2z &= 14, \\ 9x + 3y - 4z &= 13, \\ 7x + 3y - 2z &= 3; \end{aligned}$$

(e)

$$\begin{aligned} 4x - 7y + 6z &= -18, \\ 5x + y - 4z &= -9, \\ 3x - 2y + 3z &= 12. \end{aligned}$$

3. Solve the simultaneous linear equations

$$\begin{aligned} 1.985x - 1.358y &= 2.212, \\ 0.953x - 0.652y &= 1.062, \end{aligned}$$

and compare with the solutions obtained by changing the constant term, 1.062, of the second equation to 1.061.

1.7.5 ANSWERS TO EXERCISES

1. (a) $x = 1$, $y = -2$; (b) $x = 6$, $y = 10$.
2. (a) $x = -\frac{2}{9}$, $y = \frac{23}{9}$, $z = -\frac{7}{3}$;
   (b) $x = 3$, $y = 7$, $z = -4$;
   (c) $x = 3$, $y = -2$, $z = 2$;
   (d) $x = 1$, $y = -4$, $z = -4$;
   (e) $x = 3$, $y = 12$, $z = 9$.
3.

$$x \simeq 0.6087, \quad y \simeq -0.7391$$

compared with

$$x \simeq 30.1304, \quad y \simeq 42.413$$

a change of 4850% in x and 5839% in y.

"JUST THE MATHS"

UNIT NUMBER

1.8

ALGEBRA 8
(Polynomials)

by

A.J.Hobson

UNIT 1.8 - ALGEBRA 8 - POLYNOMIALS

Introduction

The work already covered in earlier units has frequently been concerned with mathematical expressions involving constant quantities together positive integer powers of a variable quantity (usually x). The general form of such expressions is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n$$

which is called a
"polynomial of degree n in x", having **"coefficients"** $a_0, a_1, a_2, a_3, ...a_n$, usually constant.

Note:
Polynomials of degree 1, 2 and 3 are called respectively **"linear"**, **"quadratic"** and **"cubic"** polynomials.

1.8.1 THE FACTOR THEOREM

If $P(x)$ denotes an algebraic polynomial which has the value zero when $x = \alpha$, then $x - \alpha$ is a factor of the polynomial and
$P(x) \equiv (x - \alpha)\times$ another polynomial, $Q(x)$, of one degree lower.

Notes:
(i) The statement of this Theorem includes some functional notation (i.e. $P(x), Q(x)$) which will be discussed fully as an introduction to the subject of Calculus in Unit 10.1.

(ii) $x = \alpha$ is called a **"root"** of the polynomial.

1.8.2 APPLICATION TO QUADRATIC AND CUBIC EXPRESSIONS

(a) Quadratic Expressions

Suppose we are given a quadratic expression where we suspect that at least one of the values of x making it zero is a whole number. We may systematically try

$$x = 0, 1, -1, 2, -2, 3, -3, 4, -4, ...$$

until such a value of x (say $x = \alpha$) is located. Then one of the factors of the quadratic expression is $x - \alpha$ enabling us to determine the other factor(s) easily.

EXAMPLES

1. By trial and error, the quadratic expression

$$x^2 + 2x - 3$$

has the value zero when $x = 1$; hence, $x - 1$ is a factor.

By further trial and error, the complete factorisation is

$$(x - 1)(x + 3).$$

2. By trial and error, the quadratic expression

$$3x^2 + 20x - 7$$

has the value zero when $x = -7$; hence, $(x + 7)$ is a factor.

By further trial and error, the complete factorisation is

$$(x + 7)(3x - 1).$$

(b) Cubic Expressions

The method just used for the factorisation of quadratic expressions may also be used for polynomials having powers of x higher than two. In particular, it may be used for cubic expressions whose standard form is

$$ax^3 + bx^2 + cx + d,$$

where a, b, c and d are constants.

EXAMPLES

1. Factorise the cubic expression

$$x^3 + 3x^2 - x - 3$$

assuming that there is at least one whole number value of x for which the expression has the value zero.

Solution

By trial and error, the cubic expression has the value zero when $x = 1$. Hence, by the Factor Theorem, $(x - 1)$ is a factor.

Thus,

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)Q(x)$$

where $Q(x)$ is some quadratic expression to be found and, if possible, factorised further. In other words,

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(px^2 + qx + r)$$

for some constants p, q and r.

Usually, it is fairly easy to determine $Q(x)$ using trial and error by comparing, initially, the highest and lowest powers of x on both sides of the above identity; then comparing any intermediate powers as necessary. We obtain

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x^2 + 4x + 3).$$

In this example, the quadratic expression does factorise even further giving

$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x + 1)(x + 3).$$

2. Factorise the cubic expression

$$x^3 + 4x^2 + 4x + 1.$$

Solution

By trial and error, we discover that the cubic expression has value zero when $x = -1$, and so $x + 1$ must be a factor.
Hence,

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)Q(x),$$

where $Q(x)$ is a quadratic expression to be found. In other words,

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)(px^2 + qx + r)$$

for some constants p, q and r.

Comparing the relevant coefficients, we obtain

$$x^3 + 4x^2 + 4x + 1 \equiv (x + 1)(x^2 + 3x + 1)$$

but, this time, the quadratic part of the answer will not conveniently factorise further.

1.8.3 CUBIC EQUATIONS

There is no convenient general method of solving a cubic equation; hence, the discussion here will be limited to those equations where at least one of the solutions is known to be a fairly small whole number, that solution being obtainable by trial and error.

We illustrate with examples.

EXAMPLES

1. Solve the cubic equation
$$x^3 + 3x^2 - x - 3 = 0.$$

   Solution

   By trial and error, one solution is $x = 1$ and so $(x - 1)$ must be a factor of the left hand side. In fact, we obtain the new form of the equation as
$$(x - 1)(x^2 + 4x + 3) = 0.$$
   That is,
$$(x - 1)(x + 1)(x + 3) = 0.$$
   Hence, the solutions are $x = 1$, $x = -1$ and $x = -3$.

2. Solve the cubic equation
$$2x^3 - 7x^2 + 5x + 54 = 0.$$

   Solution

   By trial and error, one solution is $x = -2$ and so $(x + 2)$ must be a factor of the left hand side. In fact we obtain the new form of the equation as
$$(x + 2)(2x^2 - 11x + 27) = 0.$$
   The quadratic factor will not conveniently factorise, but we can use the quadratic formula to determine the values of x, if any, which make it equal to zero.They are
$$x = \frac{11 \pm \sqrt{121 - 216}}{4}.$$
   The discriminant here is negative so that the only (real) solution to the cubic equation is $x = -2$.

1.8.4 LONG DIVISION OF POLYNOMIALS

(a) Exact Division

Having used the Factor Theorem to find a linear factor of a polynomial expression, there will always be a remaining factor, $Q(x)$, which is some other polynomial whose degree is one lower than that of the original; but the determination of $Q(x)$ by trial and error is not the only method of doing so. An alternative method is to use the technique known as **"long division of polynomials"**; and the working is illustrated by the following examples:

EXAMPLES

1. Factorise the cubic expression
$$x^3 + 3x^2 - x - 3.$$

   Solution

   By trial and error, the cubic expression has the value zero when $x = 1$ so that $(x - 1)$ is a factor.

   Dividing the given cubic expression (called the **"dividend"**) by $(x - 1)$ (called the **"divisor"**), we have the following scheme:

$$\begin{array}{rl} & x^2 + 4x + 3 \\ x - 1) & \overline{x^3 + 3x^2 - x - 3} \\ & \underline{x^3 - x^2} \\ & \quad 4x^2 - x - 3 \\ & \quad \underline{4x^2 - 4x} \\ & \qquad 3x - 3 \\ & \qquad \underline{3x - 3} \\ & \qquad\quad 0 \end{array}$$

   Note:

   At each stage, using positive powers of x or constants only, we examine what the highest power of x in the divisor would have to be multiplied by in order to give the highest power of x in the dividend. The results are written in the top line as the **"quotient"** so that, on multiplying down, we can subtract to find each **"remainder"**.

   The process stops when, to continue would need negative powers of x; the final remainder will be zero when the divisor is a factor of the original expression.

   We conclude here that
$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x^2 + 4x + 3).$$

   Further factorisation leads to the complete result,
$$x^3 + 3x^2 - x - 3 \equiv (x - 1)(x + 1)(x + 3).$$

2. Solve, completely, the cubic equation
$$x^3 + 4x^2 + 4x + 1 = 0.$$

   Solution

   By trial and error, one solution is $x = -1$ so that $(x + 1)$ is a factor of the left hand side.

Dividing the left hand side of the equation by $(x+1)$, we have the following scheme:

$$
\begin{array}{r}
x^2+3x+1 \\
x+1\overline{)x^3+4x^2+4x+1} \\
\underline{x^3+x^2} \\
3x^2+4x+1 \\
\underline{3x^2+3x} \\
x+1 \\
\underline{x+1} \\
0
\end{array}
$$

Hence, the equation becomes

$$(x+1)(x^2+3x+1)=0,$$

giving $x=-1$ and $x=\frac{-3\pm\sqrt{9-4}}{2}\simeq -0.382 \quad \text{or} \quad -2.618$

(b) Non-exact Division

The technique for long division of polynomials may be used to divide any polynomial by another polynomial of lower or equal degree, even when this second polynomial is not a factor of the first.

The chief difference in method is that the remainder will not be zero; but otherwise we proceed as before.

EXAMPLES

1. Divide the polynomial $6x+5$ by the polynomial $3x-1$.

   Solution

$$
\begin{array}{r}
2 \\
3x-1\overline{)6x+5} \\
\underline{6x-2} \\
7
\end{array}
$$

Hence,

$$\frac{6x+5}{3x-1}\equiv 2+\frac{7}{3x-1}.$$

2. Divide $3x^2 + 2x$ by $x + 1$.

Solution

$$\begin{array}{r}
3x - 1 \\
x + 1 \overline{)3x^2 + 2x} \\
\underline{3x^2 + 3x} \\
-x \\
\underline{-x - 1} \\
1
\end{array}$$

Hence,

$$\frac{3x^2 + 2x}{x + 1} \equiv 3x - 1 + \frac{1}{x + 1}.$$

3. Divide $x^4 + 2x^3 - 2x^2 + 4x - 1$ by $x^2 + 2x - 3$.

Solution

$$\begin{array}{r}
x^2 \qquad\quad + 1 \\
x^2 + 2x - 3 \overline{)x^4 + 2x^3 - 2x^2 + 4x - 1} \\
\underline{x^4 + 2x^3 - 3x^2} \qquad\qquad \\
x^2 + 4x - 1 \\
\underline{x^2 + 2x - 3} \\
2x + 2
\end{array}$$

Hence,

$$\frac{x^4 + 2x^3 - 2x^2 + 4x - 1}{x^2 + 2x - 3} \equiv x^2 + 1 + \frac{2x + 2}{x^2 + 2x - 3}.$$

1.8.5 EXERCISES

1. Use the Factor Theorem to factorise the following quadratic expressions: assuming that at least one root is an integer:
   (a) $2x^2 + 7x + 3$;
   (b) $1 - 2x - 3x^2$.
2. Factorise the following cubic polynomials assuming that at least one root is an integer:
   (a) $x^3 + 2x^2 - x - 2$;
   (b) $x^3 - x^2 - 4$;
   (c) $x^3 - 4x^2 + 4x - 3$.

3. Solve completely the following cubic equations assuming that at least one solution is an integer:

   (a) $x^3 + 6x^2 + 11x + 6 = 0$;

   (b) $x^3 + 2x^2 - 31x + 28 = 0$.

4. (a) Divide $2x^3 - 11x^2 + 18x - 8$ by $x - 2$;

   (b) Divide $3x^3 + 12x^2 + 13x + 4$ by $x + 1$.

5. (a) Divide $x^2 - 2x + 1$ by $x^2 + 3x - 2$, expressing your answer in the form of a constant plus a ratio of two polynomials.

   (b) Divide x^5 by $x^2 - 2x - 8$, expressing your answer in the form of a polynomial plus a ratio of two other polynomials.

1.8.6 ANSWERS TO EXERCISES

1. (a) $(2x + 1)(x + 3)$; (b) $(x + 1)(1 - 3x)$.
2. (a) $(x - 1)(x + 1)(x + 2)$; (b) $(x - 2)(x^2 + x + 2)$; (c) $(x - 3)(x^2 - x + 1)$.
3. (a) $x = -1$, $x = -2$, $x = -3$; (b) $x = 1$, $x = 4$, $x = -7$.
4. (a) $2x^2 - 7x + 4$; (b) $3x^2 + 9x + 4$.
5. (a)

$$1 + \frac{3 - 5x}{x^2 + 3x - 2};$$

   (b)

$$x^3 + 2x^2 + 12x + 40 + \frac{176x + 320}{x^2 - 2x - 8}.$$

"JUST THE MATHS"

UNIT NUMBER

1.9

ALGEBRA 9
(The theory of partial fractions)

by

A.J.Hobson

UNIT 1.9 - ALGEBRA 9 - THE THEORY OF PARTIAL FRACTIONS

1.9.1 INTRODUCTION

The theory of partial fractions applies chiefly to the ratio of two polynomials in which the degree of the numerator is strictly less than that of the denominator. Such a ratio is called a **"proper rational function"**.

For a rational function which is not proper, it is necessary first to use long division of polynomials in order to express it as the sum of a polynomial and a proper rational function.

RESULT

A proper rational function whose denominator has been factorised into its irreducible factors can be expressed as a sum of proper rational functions, called **"partial fractions"**; the denominators of the partial fractions are the irreducible factors of the denominator in the original rational function.

ILLUSTRATION

From previous work on fractions, it can be verified that

$$\frac{1}{2x+3} + \frac{3}{x-1} \equiv \frac{7x+8}{(2x+3)(x-1)}$$

and the expression on the left hand side may be interpretted as the decomposition into partial fractions of the expression on the right hand side.

1.9.2 STANDARD TYPES OF PARTIAL FRACTION PROBLEM

(a) Denominator of the given rational function has all linear factors.

EXAMPLE

Express the rational function

$$\frac{7x+8}{(2x+3)(x-1)}$$

in partial fractions.

Solution

There will be two partial fractions each of whose numerator must be of lower degree than 1; i.e. it must be a **constant**

We write

$$\frac{7x+8}{(2x+3)(x-1)} \equiv \frac{A}{2x+3} + \frac{B}{x-1}.$$

Multiplying throughout by $(2x+3)(x-1)$, we obtain

$$7x + 8 \equiv A(x-1) + B(2x+3).$$

In order to determine A and B, any two suitable values of x may be substituted on both sides; and the most obvious values in this case are $x = 1$ and $x = -\frac{3}{2}$.

It may, however, be argued that these two values of x must be disallowed since they cause denominators in the first identity above to take the value zero.

Nevertheless, we shall use these values in the second identity above since the arithmetic involved is negligibly different from taking values infinitesimally close to $x = 1$ and $x = -\frac{3}{2}$.

Substituting $x = 1$ gives

$$7 + 8 = B(2+3).$$

Hence,

$$B = \frac{7+8}{2+3} = \frac{15}{5} = 3.$$

Substituting $x = -\frac{3}{2}$ gives

$$7 \times -\frac{3}{2} + 8 = A(-\frac{3}{2} - 1).$$

Hence,

$$A = \frac{7 \times -\frac{3}{2} + 8}{-\frac{3}{2} - 1} = \frac{-\frac{5}{2}}{-\frac{5}{2}} = 1.$$

We conclude that

$$\frac{7x+8}{(2x+3)(x-1)} = \frac{1}{2x+3} + \frac{3}{x-1}.$$

The "Cover-up" Rule
A useful time-saver when the factors in the denominator of the given rational function are linear is to use the following routine which is equivalent to the method described above:

To obtain the constant numerator of the partial fraction for a particular linear factor, $ax+b$, in the original denominator, cover up $ax + b$ in the original rational function and then substitute $x = -\frac{b}{a}$ into what remains.

ILLUSTRATION
In the above example, we may simply cover up $x - 1$, then substitute $x = 1$ into the fraction

$$\frac{7x+8}{2x+3}.$$

Then we may cover up $2x+3$ and substitute $x = -\frac{3}{2}$ into the fraction

$$\frac{7x+8}{x-1}.$$

Note:
We shall see later how the cover-up rule can also be brought into effective use when not all of the factors in the denominator of the given rational function are linear.

(b) Denominator of the given rational function contains one linear and one quadratic factor

EXAMPLE

Express the rational function

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)}$$

in partial fractions.

Solution

We should observe firstly that the quadratic factor will not reduce conveniently into two linear factors. If it did, the method would be as in the previous paragraph. Hence we may write

$$\frac{3x^2+9}{(x-5)(x^2+2x+7)} \equiv \frac{A}{x-5} + \frac{Bx+C}{x^2+2x+7},$$

noticing that the second partial fraction may contain an x term in its numerator, yet still be a proper rational function.

Multiplying throughout by $(x-5)(x^2+2x+7)$, we obtain

$$3x^2+9 \equiv A(x^2+2x+7) + (Bx+C)(x-5).$$

A convenient value of x to substitute on both sides is $x = 5$ which gives

$$3 \times 5^2 + 9 = A(5^2 + 2 \times 5 + 7).$$

That is, $84 = 42A$ or $A = 2$.

No other convenient values of x may be substituted; but two polynomial expressions can be identical only if their corresponding coefficients are the same in value. We therefore equate suitable coefficients to find B and C; usually, the coefficients of the highest and lowest powers of x.

Equating coefficients of x^2, $3 = A + B$ and hence $B = 1$.

Equating constant terms (the coefficients of x^0), $9 = 7A - 5C = 14 - 5C$ and hence $C = 1$.

The result is therefore

$$\frac{3x^2 + 9}{(x-5)(x^2 + 2x + 7)} \equiv \frac{2}{x-5} + \frac{x+1}{x^2 + 2x + 7}.$$

Observations
It is easily verified that the value of A may be calculated by means of the cover-up rule, as in paragraph **(a)**; and, having found A, the values of B and C could be found by cross multiplying the numerators and denominators in the expression

$$\frac{2}{x-5} + \frac{?x+?}{x^2 + 2x + 7}$$

in order to arrive at the numerator of the original rational function. This process essentially compares the coefficients of x^2 and x^0 as before.

(c) Denominator of the given rational function contains a repeated linear factor

In general, examples of this kind will not be more complicated than for a rational function with one repeated linear factor together with either a non-repeated linear factor or a quadratic factor.

EXAMPLE

Express the rational function

$$\frac{9}{(x+1)^2(x-2)}$$

in partial fractions.

Solution

First we observe that, from paragraph **(b)**, the partial fraction corresponding to the repeated linear factor would be of the form

$$\frac{Ax + B}{(x+1)^2};$$

but this may be written

$$\frac{A(x+1)+B-A}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B-A}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2}.$$

Thus, a better form of statement for the problem as a whole, is

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{A}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{x-2}.$$

Eliminating fractions, we obtain

$$9 \equiv A(x+1)(x-2) + C(x-2) + D(x+1)^2.$$

Putting $x = -1$ gives $9 = -3C$ so that $C = -3$.

Putting $x = 2$ gives $9 = 9D$ so that $D = 1$.

Equating coefficients of x^2 gives $0 = A + D$ so that $A = -1$.

Therefore,

$$\frac{9}{(x+1)^2(x-2)} \equiv -\frac{1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}.$$

Notes:
(i) Similar partial fractions may be developed for higher repeated powers so that, for a repeated linear factor of power of n, there will be n corresponding partial fractions, each with a constant numerator. The labels for these numerators in future will be taken as A, B, C, etc. in sequence.

(ii) We observe that the numerator above the repeated factor itself (D in this case) could actually have been obtained by the cover-up rule; covering up $(x+1)^2$ in the original rational function, then substituting $x = -1$ into the rest.

(d) Keily's Method

A useful method for repeated linear factors is to use these factors one at a time, keeping the rest outside the expression as a factor.

EXAMPLE

Express the rational function

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{9}{(x+1)(x-2)}\right]$$

in partial fractions.

Solution

Using the cover-up rule inside the square brackets,

$$\frac{9}{(x+1)^2(x-2)} \equiv \frac{1}{x+1}\left[\frac{-3}{x+1} + \frac{3}{x-2}\right]$$

$$\equiv -\frac{3}{(x+1)^2} + \frac{3}{(x+1)(x-2)};$$

and, again by cover-up rule,

$$\equiv -\frac{3}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x-2}$$

as before.

Warning
Care must be taken with Keily's method when, even though the original rational function is proper, the resulting expression inside the square brackets is improper. This would have occurred, for instance, if the problem given had been

$$\frac{9x^2}{(x+1)^2(x-2)},$$

leading to

$$\frac{1}{x+1}\left[\frac{9x^2}{(x+1)(x-2)}\right].$$

In this case, long division would have to be used inside the square brackets before proceeding with Keily's method.

For such examples, it is probably better to use the method of paragraph **(c)**.

1.9.3 EXERCISES

Express the following rational functions in partial fractions:

1.
$$\frac{3x+5}{(x+1)(x+2)}.$$

2.

$$\frac{17x+11}{(x+1)(x-2)(x+3)}.$$

3.

$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$

4.

$$\frac{2x+1}{(x+2)^2(x-3)}.$$

5.

$$\frac{9+11x-x^2}{(x+1)^2(x+2)}.$$

6.

$$\frac{x^5}{(x+2)(x-4)}.$$

1.9.4 ANSWERS TO EXERCISES

1.

$$\frac{2}{x+1}+\frac{1}{x+2}.$$

2.

$$\frac{1}{x+1}+\frac{3}{x-2}-\frac{4}{x+3}.$$

3.

$$\frac{1}{x-1}+\frac{2x+1}{x^2+x-7}.$$

4.

$$\frac{3}{5(x+2)^2}-\frac{7}{25(x+2)}+\frac{7}{25(x-3)}.$$

5.

$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}.$$

6.

$$x^3+2x^2+12x+40+\frac{16}{3(x+2)}+\frac{512}{3(x-4)}.$$

"JUST THE MATHS"

UNIT NUMBER

1.10

ALGEBRA 10
(Inequalities 1)

by

A.J.Hobson

UNIT 1.10 - ALGEBRA 10 - INEQUALITIES 1.

1.10.1 INTRODUCTION

If the symbols a and b denote numerical quantities, then the statement

$$a < b$$

is used to mean "a is less than b" while the statement

$$b > a$$

is used to mean "b is greater than a"

These are called **"strict inequalities"** because there is no allowance for the possibilty that a and b might be equal to each other. For example, if a is the number of days in a particular month and b is the number of hours in that month, then $b > a$.

Some inequalities do allow the possibility of a and b being equal to each other and are called **"weak inequalities"** written in one of the forms

$$a \leq b \quad b \leq a \quad a \geq b \quad b \geq a$$

For example, if a is the number of students who enrolled for a particular module in a university and b is the number of students who eventually passed that module, then $a \geq b$.

1.10.2 ALGEBRAIC RULES FOR INEQUALITIES

Given two different numbers, one of them must be strictly less than the other. Suppose a is the smaller of the two and b the larger; i.e.

$$a < b$$

Then
1. $a + c < b + c$ for any number c.

2. $ac < bc$ when c is positive but $ac > bc$ when c is negative.

3. $\frac{1}{a} > \frac{1}{b}$ provided a and b are **both positive.**

Note:
The only other situation in 3. which is consistent with $a < b$ will occur if a is negative and b is positive. In this case, $\frac{1}{a} < \frac{1}{b}$ because a negative number is always less than a positive number.

EXAMPLES

1. Simplify the inequality
$$2x + 3y > 5x - y + 7.$$
Solution
We simply deal with this in the same way as we would deal with an equation by adding appropriate quantities to both sides or subtracting appropriate quantities from both sides. We obtain
$$-3x + 4y > 7 \quad \text{or} \quad 3x - 4y + 7 < 0.$$

2. Solve the inequality
$$\frac{1}{x-1} < 2,$$
assuming that $x \neq 1$.
Solution
Here we must be careful in case $x-1$ is negative; the argument is therefore in two parts:
(a) If $x > 1$, i.e. $x - 1$ is positive, then the inequality can be rewritten as
$$1 < 2(x-1) \quad \text{or} \quad x - 1 > \frac{1}{2}.$$
Hence,
$$x > \frac{3}{2}.$$
(b) If $x < 1$, i.e. $x - 1$ is negative, then the inequality is automatically true since a negative number is bound to be less than a positive number.
Conclusion:
The given inequality is satisfied when $x < 1$ and when $x > \frac{3}{2}$.

3. Solve the inequality
$$2x - 7 \leq 3.$$
Solution
Adding 7 to both sides, then dividing both sides by 2 gives
$$x \leq 5.$$

4. Solve the inequality
$$\frac{x-1}{x-6} \geq 0.$$

Solution

We first observe that the fraction on the left of the inequality can equal zero only when $x = 1$.

Secondly, the only way in which a fraction can be positive is for both numerator and denominator to be positive or both numerator and denominator to be negative.

(a) Suppose $x - 1 > 0$ and $x - 6 > 0$; these two are covered by $x > 6$.

(b) Suppose $x - 1 < 0$ and $x - 6 < 0$; these two are covered by $x < 1$.

Note:
The value $x = 6$ is problematic because the given expression becomes infinite; in fact, as x passes through 6 from values below it to values above it, there is a sudden change from $-\infty$ to $+\infty$.

Conclusion:
The inequality is satisfied when either $x > 6$ or $x \leq 1$.

1.10.3 INTERVALS

In scientific calculations, a variable quantity x may be restricted to a certain range of values called an **"interval"** which may extend to ∞ or $-\infty$; but, in many cases, such intervals have an upper and a lower **"bound"**. The standard types of interval are as follows:

(a) $a < x < b$ denotes an **"open interval"** of all the values of x between a and b but excluding a and b themselves. The symbol (a, b) is also used to mean the same thing. For example, if x is a purely decimal quantity, it must lie in the open interval

$$-1 < x < 1.$$

(b) $a \leq x \leq b$ denotes a **"closed interval"** of all the values of x from a to b inclusive. The symbol $[a, b]$ is also used to mean the same thing. For example, the expression $\sqrt{1 - x^2}$ has real values only when

$$-1 \leq x \leq 1.$$

Note:
It is possible to encounter intervals which are closed at one end but open at the other; they may be called either **"half open"** or **"half closed"**. For example

$$a < x \leq b \quad \text{or} \quad a \leq x < b,$$

which can also be denoted respectively by $(a, b]$ and $[a, b)$.

(c) Intervals of the types

$$x > a \quad x \geq a \quad x < a \quad x \leq a$$

are called **"infinite intervals"**.

1.10.4 EXERCISES

1. Simplify the following inequalities:

   (a)

   $$x + y \leq 2x + y + 1;$$

   (b)

   $$2a - b > 1 + a - 2b - c.$$

2. Solve the following inequalities to find the range of values of x.

   (a)

   $$x + 3 < 6;$$

   (b)

   $$-2x \geq 10;$$

   (c)

   $$\frac{2}{x} > 18;$$

   (d)

   $$\frac{x+3}{2x-1} \leq 0.$$

3. Classify the following intervals as open, closed, or half-open/half-closed:

   (a)

   $$(5, 8);$$

   (b)

   $$(-3, -2);$$

(c)

$$[2,4);$$

(d)

$$[8,23];$$

(e)

$$(-\infty,\infty);$$

(f)

$$(0,\infty);$$

(g)

$$[0,\infty).$$

1.10.5 ANSWERS TO EXERCISES

1. (a) $x \geq -1$;
   (b) $a+b+c > 1$.
2. (a) $x < 3$;
   (b) $x \leq -5$;
   (c) $x < \frac{1}{9}$ since $x > 0$;
   (d) $-3 \leq x < \frac{1}{2}$.
3. (a) open;
   (b) open;
   (c) half-open/half-closed;
   (d) closed;
   (e) open;
   (f) open;
   (g) half-open/half-closed.

"JUST THE MATHS"

UNIT NUMBER

1.11

ALGEBRA 11
(Inequalities 2)

by

A.J.Hobson

UNIT 1.11 - ALGEBRA 11 - INEQUALITIES 2.

1.11.1 RECAP ON MODULUS, ABSOLUTE VALUE OR NUMERICAL VALUE

As seen in Unit 1.1, the Modulus of a numerical quantity ignores any negative signs if there are any. For example, the modulus of -3 is 3, but the modulus of 3 is also 3.

The modulus of an unspecified numerical quantity x is denoted by the symbol

$$\mid x \mid$$

and is defined by the two statements:

$$\mid x \mid = x \quad \text{if} \quad x \geq 0;$$

$$\mid x \mid = -x \quad \text{if} \quad x \leq 0.$$

Notes:
(i) An alternative, but less convenient formula for the modulus of x is

$$\mid x \mid = +\sqrt{x^2}.$$

(ii) It is possible to show that, for any two numbers a and b,

$$\mid a + b \mid \leq \mid a \mid + \mid b \mid .$$

This is called the **"triangle inequality"** and can be linked to the fact that the length of any side of a triangle is never greater than the sum of the lengths of the other two sides.

The proof is a little involved since it is necessary to consider all possible cases of a and b being positive, negative or zero together with a consideration of their relative sizes. It will not be included here.

1.11.2 INTERVAL INEQUALITIES

(a) Using the Modulus notation

In this section, we investigate the meaning of the inequality

$$\mid x - a \mid < k,$$

where a is any number and k is a positive number.

Case 1. $x - a > 0$**.**

The inequality can be rewritten as

$$x - a < k \quad \text{i.e.} \quad x < a + k.$$

Case 2. $x - a < 0$**.**

The inequality can be rewritten as

$$-(x - a) < k \quad \text{i.e.} \quad a - x < k \quad \text{i.e.} \quad x > a - k.$$

Combining the two cases, we conclude that

$$\mid x - a \mid < k \quad \text{means} \quad a - k < x < a + k,$$

which is an open interval having $x = a$ at the centre and extending to a distance of a either side of the centre. A similar interpretation could be given of $\mid x - a \mid \leq k$.

EXAMPLE

Obtain the closed interval represented by the statement

$$\mid x + 3 \mid \leq 10.$$

Solution

Using $a = -3$ and $k = 10$, we have

$$-3 - 10 \leq x \leq -3 + 10.$$

That is,

$$-13 \leq x \leq 7.$$

(b) Using Factorised Polynomials

Suppose a polynomial in x has been factorised into a number of linear factors corresponding to the degree of the polynomial. Then, if certain values of x are substituted in, the polynomial will be positive (or zero) as long as the number of individual factors which become negative is even. Similarly, the polynomial will be negative (or zero) as long as the number of individual factors which become negative is odd.

These observations enable us to find the ranges of values of x for which a factorised polynomial is positive or negative.

EXAMPLE

Find the range of values of x for which the polynomial

$$(x+3)(x-1)(x-2)$$

is strictly positive.

Solution

The first task is to find what are called the **"critical values"**. These are the values of x at which the polynomial becomes equal to zero. In our case, the critical values are $x = -3, 1, 2$.

Next, the critical values divide the x-line into separate intervals where, for the moment, we exclude the critical values themselves. In this case, we obtain

$$x < -3, \quad -3 < x < 1, \quad 1 < x < 2, \quad x > 2.$$

All that is now necessary is to select a value from each of these intervals and investigate how it affects the signs of the factors of the polynomial and hence the sign of the polynomial itself.

$x < -3$ gives (neg)(neg)(neg) and therefore < 0;

$-3 < x < 1$ gives (pos)(neg)(neg) and therefore > 0;

$1 < x < 2$ gives (pos)(pos)(neg) and therefore < 0;

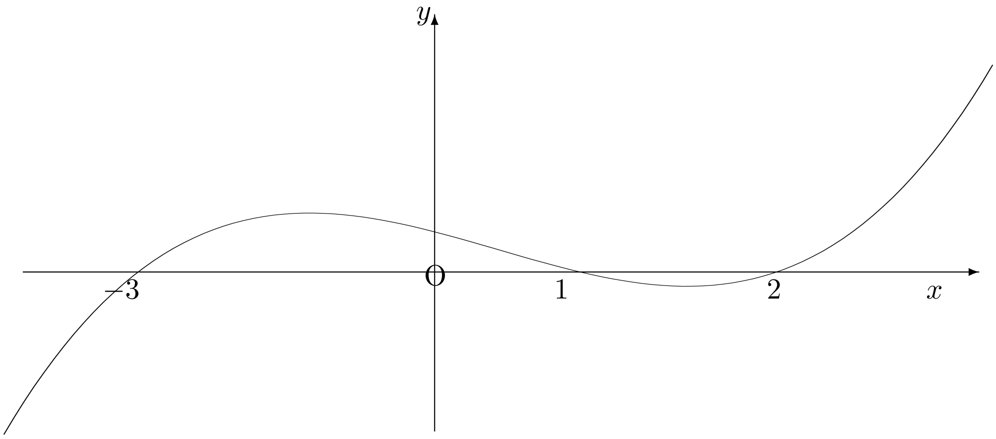
$x > 2$ gives (pos)(pos)(pos) and therefore > 0.

Clearly the critical values will not be included in the answer for this example because they cause the polynomial to have the value zero.

The required ranges are thus

$$-3 < x < 1 \quad \text{and} \quad x > 2.$$

Note:
An alternative method is to sketch the graph of the polynomial as a smooth curve passing through all the critical values on the x-axis. One point only between each critical value will make it clear whether the graph is on the positive side of the x-axis or the negative side. For the above example, the graph is as follows:



1.11.3 EXERCISES

1. Determine the **precise** ranges of values of x which satisfy the following inequalities:
   (a) $| x |< 2$;
   (b) $| x |> 3$;
   (c) $| x - 3 |< 1$;
   (d) $| x + 6 |< 4$;
   (e) $| x + 1 |\geq 2$;
   (f) $0 <| x + 3 |< 5$.

2. Determine the **precise** ranges of values of x which satisfy the following inequalities:
   (a) $(x - 4)(x + 2) \geq 0$;
   (b) $(x + 3)(x - 2)(x - 4) < 0$;
   (c) $(x + 1)^2(x - 3) > 0$;
   (d) $18x - 3x^2 > 0$.

3. By considering the expansion of $(a^2 - b^2)^2$, show that

$$a^4 + b^4 \geq 2a^2b^2.$$

1.11.4 ANSWERS TO EXERCISES

1. (a) $-2 < x < 2$;
   (b) $x < -3$ and $x > 3$;
   (c) $2 < x < 4$;
   (d) $-10 < x < -2$;
   (e) $x \geq 1$ and $x \leq -3$;
   (f) $-8 < x < -3$ and $-3 < x < 2$.
2. (a) $x \leq -2$ and $x \geq 4$;
   (b) $x < -3$ and $2 < x < 4$;
   (c) $x > 3$;
   (d) $0 < x < 6$.

"JUST THE MATHS"

UNIT NUMBER

2.1

SERIES 1
(Elementary progressions and series)

by

A.J.Hobson

UNIT 2.1 - SERIES 1 - ELEMENTARY PROGRESSIONS AND SERIES

2.1.1 ARITHMETIC PROGRESSIONS

The **"sequence"** of numbers,

$$a, a + d, a + 2d, a + 3d, ...$$

is said to form an **"arithmetic progression"**.

The symbol a represents the **"first term"**, the symbol d represents the **"common difference"** and the **"n-th term"** is given by the expression

$$a + (n - 1)d$$

EXAMPLES

1. Determine the n-th term of the arithmetic progression 15, 12, 9, 6,...

   Solution

   The n-th term is

   $$15 + (n - 1)(-3) = 18 - 3n.$$

2. Determine the n-th term of the arithmetic progression

   $$8, 8.125, 8.25, 8.375, 8.5, ...$$

   Solution

   The n-th term is

   $$8 + (n - 1)(0.125) = 7.875 + 0.125n.$$

3. The 13th term of an arithmetic progression is 10 and the 25th term is 20; calculate
   (a) the common difference;
   (b) the first term;
   (c) the 17th term.

   Solution

   Letting a be the first term and d be the common difference, we have

   $$a + 12d = 10$$

   and

   $$a + 24d = 20.$$

   (a) Subtracting the first of these from the second gives $12d = 10$, so that the common difference $d = \frac{10}{12} = \frac{5}{6} \simeq 0.83$

(b) Substituting into the first of the relationships between a and d gives

$$a + 12 \times \frac{5}{6} = 10.$$

That is,

$$a + 10 = 10.$$

Hence, $a = 0$.

(c) The 17th term is

$$0 + 16 \times \frac{5}{6} = \frac{80}{6} = \frac{40}{3} \simeq 13.3$$

2.1.2 ARITHMETIC SERIES

If the terms of an arithmetic progression are added together, we obtain what is called an **"arithmetic series"**. The total sum of the first n terms of such a series can be denoted by S_n so that

$$S_n = a + [a + d] + [a + 2d] + ... + [a + (n - 2)d] + [a + (n - 1)d];$$

but this is not the most practical way of evaluating the sum of the n terms, especially when n is a very large number.

A trick which provides us with a more convenient formula for S_n is to write down the existing formula **backwards**. That is,

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + ... + [a + 2d] + [a + d] + a.$$

Adding the two statements now gives

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + ... + [2a + (n - 1)d] + [2a + (n - 1)d],$$

where, on the right hand side, there are n repetitions of the same expression.

Hence,

$$2S_n = n[2a + (n - 1)d]$$

or

$$S_n = \frac{n}{2}[2a + (n - 1)d].$$

This version of the formula is suitable if we know the values of a, n and d; but an alternative version can be used if we know only the first term, the last term and the number of terms. In this case,

$$S_n = \frac{n}{2}[\text{FIRST} \quad + \quad \text{LAST}]$$

which is simply n times the average of the first and last terms.

EXAMPLES

1. Determine the sum of the natural numbers from 1 to 100.

   Solution

   The sum is given by
$$\frac{100}{2} \times [1 + 100] = 5050.$$

2. How many terms of the arithmetic series
$$10 + 12 + 14 + ...$$
   must be taken so that the sum of the series is 252 ?

   Solution

   The first term is clearly 10 and the common difference is 2.

   Hence, letting n be the number of terms, we require that
$$252 = \frac{n}{2}[20 + (n - 1) \times 2].$$
   That is,
$$252 = \frac{n}{2}[2n + 18] = n(n + 9).$$
   By trial and error, $n = 12$ will balance this equation; but it is more conclusive to obtain n as the solution to the quadratic equation
$$n^2 + 9n - 252 = 0$$
   or
$$(n - 12)(n + 21) = 0,$$
   which gives $n = 12$ only, since the negative value $n = -21$ may be ignored.

3. A contractor agrees to sink a well 250 metres deep at a cost of £2.70 for the first metre, £2.85 for the second metre and an extra 15p for each additional metre. Find the cost of the last metre and the total cost.

   Solution

   In this problem we are dealing with an arithmetic series of 250 terms whose first term is 2.70 and whose common difference is 0.15. The cost of the last metre is the 250-th term of the series and therefore

   £$[2.70 + 249 \times 0.15] =$ £40.05

   The total cost will be

   £$\frac{250}{2} \times [2.70 + 40.05] =$ £5343.75

2.1.3 GEOMETRIC PROGRESSIONS

The sequence of numbers

$$a, ar, ar^2, ar^3, ...$$

is said to form a "geometric progression".

The symbol a represents the **"first term"**, the symbol r represents the **"common ratio"** and the **"n-th term"** is given by the expression

$$ar^{n-1}.$$

EXAMPLES

1. Determine the n-th term of the geometric progression

$$3, -12, 48, -192, ...$$

   Solution

   The progression has n-th term

$$3(-4)^{n-1}$$

   which will always be positive when n is an odd number and negative when n is an even number.

2. Determine the seventh term of the geometric progression

$$3, 6, 12, 24, ...$$

   Solution

   The seventh term is

$$3(2^6) = 192.$$

3. The third term of a geometric progression is 4.5 and the ninth term is 16.2. Determine the common ratio.

   Solution

   Firstly, we have

$$ar^2 = 4.5$$

   and

$$ar^8 = 16.2$$

   Dividing the second of these by the first gives

$$\frac{ar^8}{ar^2} = \frac{16.2}{4.5}.$$

Therefore,

$$r^6 = 3.6$$

and so

$$r \simeq 1.238$$

4. The expenses of a company are £200,000 a year. It is decided that each year they shall be reduced by 5% of those for the preceding year.
   What will be the expenses during the fourth year, the first reduction taking place at the end of the first year ?

   Solution
   In this problem, we use a geometric progression with first term 200,000 and common ratio 0.95.

   The expenses during the fourth year will thus be the fourth term of the progression; that is, £200,000 $\times(0.95)^3$ = £171475.

2.1.4 GEOMETRIC SERIES

If the terms of a geometric progression are added together, we obtain what is called a **"geometric series"** . The total sum of a geometric series with n terms may be denoted by S_n so that

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

but, as with arithmetic series, this is not the most practical formula for evaluating S_n.

This time, a trick to establish a convenient formula for S_n is to write down both S_n and rS_n, the latter giving

$$rS_n = \qquad ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n.$$

Subtracting the second formula from the first gives

$$S_n - rS_n = a - ar^n,$$

so that

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

This is the version of the formula most commonly used since, in many practical applications, r will be less than one; but, for examples in which r is greater than one, it may be better to use the alternative version, namely

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

In fact, either version may be used whatever the value of r is.

EXAMPLES

1. Determine the sum of the geometric series

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

Solution

The sum is given by

$$S_6 = \frac{4(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = \frac{4(1 - 0.0156)}{0.5} \simeq 7.875$$

2. A sum of money £C is invested for n years at an interest of $100r\%$, compounded annually. What will be the total interest earned by the end of the n-th year ?

Solution

At the end of year 1, the interest earned will be Cr.
At the end of year 2, the interest earned will be $(C + Cr)r = Cr(1 + r)$.
At the end of year 3, the interest earned will be $C(1 + r)r + C(1 + r)r^2 = Cr(1 + r)^2$.

This pattern reveals that,
at the end of year n, the interest earned will be $Cr(1 + r)^{n-1}$.

Thus the total interest earned by the end of year n will be

$$Cr + Cr(1 + r) + Cr(1 + r)^2 + ... + Cr(1 + r)^{n-1},$$

which is a geometric series of n terms with first term Cr and common ratio $1 + r$. Its sum is therefore

$$\frac{Cr((1 + r)^n - 1)}{r} = C((1 + r)^n - 1).$$

Note:

The same result can be obtained using only a geometric progression as follows:

At the end of year 1, the total amount will be $C + Cr = C(1 + r)$.
At the end of year 2, the total amount will be $C(1 + r) + C(1 + r)r = C(1 + r)^2$.
At the end of year 3, the total amount will be $C(1 + r)^2 + C(1 + r)^2r = C(1 + r)^3$.
At the end of year n, the total amount will be $C(1 + r)^n$.

Thus the total interest earned will be $C(1 + r)^n - C = C((1 + r)^n - 1)$ as before.

The sum to infinity of a geometric series.

In a geometric series with n terms, suppose that the value of the common ratio, r, is numerically less than 1. Then the higher the value of n, the smaller the numerical value of r^n, to the extent that, as n approaches infinity, r^n approaches zero.

We conclude that, although it not possible to reach the end of a geometric series which has an infinite number of terms, its sum to infinity may be given by

$$S_\infty = \frac{a}{1-r}.$$

EXAMPLES

1. Determine the sum to infinity of the geometric series

$$5 - 1 + \frac{1}{5} - \ \ldots$$

Solution

The sum to infinity is

$$\frac{5}{1+\frac{1}{5}} = \frac{25}{6} \simeq 4.17$$

2. The yearly output of a silver mine is found to be decreasing by 25% of its previous year's output. If, in a certain year, its output was £25,000, what could be reckoned as its total future output ?

Solution

The total output, in pounds, for subsequent years will be given by

$$25000 \times 0.75 + 25000 \times (0.75)^2 + 25000 \times (0.75)^3 + ... = \frac{25000 \times 0.75}{1 - 0.75} = 75000.$$

2.1.5 MORE GENERAL PROGRESSIONS AND SERIES

Introduction

Not all progressions and series encountered in mathematics are either arithmetic or geometric. For instance:

$$1^2, 2^2, 3^2, 4^2, ..., n^2$$

has a clearly defined pattern but is not arithmetic or geometric.

An **arbitrary** progression of n numbers which conform to some regular pattern is often denoted by

$$u_1, u_2, u_3, u_4, ..., u_n,$$

and it may or may not be possible to find a simple formula for the sum S_n.

The Sigma Notation (Σ).

If the general term of a series with n terms is known, then the complete series can be written down in short notation as indicated by the following illustrations:

1.

$$a + (a + d) + (a + 2d) + ... + (a + [n - 1]d) = \sum_{r=1}^{n} (a + [r - 1]d).$$

2.

$$a + ar + ar^2 + ...ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}.$$

3.

$$1^2 + 2^2 + 3^2 + ...n^2 = \sum_{r=1}^{n} r^2.$$

4.

$$-1^3 + 2^3 - 3^3 + 4^3 + ...(-1)^n n^3 = \sum_{r=1}^{n} (-1)^r r^3.$$

Notes:

(i) It is sometimes more convenient to count the terms of a series from zero rather than 1. For example:

$$a + (a + d) + (a + 2d) + ...a + [n - 1]d = \sum_{r=0}^{n-1} (a + rd)$$

and

$$a + ar + ar^2 + ar^3 + ...ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

In general, for a series with n terms starting at u_0,

$$u_0 + u_1 + u_2 + u_3 + ... + u_{n-1} = \sum_{r=0}^{n-1} u_r.$$

(ii) We may also use the sigma notation for **"infinite series"** such as those we encountered in the sum to infinity of a geometric series. For example

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \sum_{r=1}^{\infty} \frac{1}{3^{r-1}} \text{ or } \sum_{r=0}^{\infty} \frac{1}{3^r}.$$

STANDARD RESULTS

It may be shown that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and

$$\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2.$$

Outline Proofs:

The first of these is simply the formula for the sum of an arithmetic series with first term 1 and last term n.

The second is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

The third is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1.$$

EXAMPLE

Determine the sum to n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + \ldots$$

Solution

The series is

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3r^2 + 2r = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r.$$

Using the three standard results, the summation becomes

$$\left[\frac{1}{2}n(n+1)\right]^2 + 3\left[\frac{1}{6}n(n+1)(2n+1)\right] + 2\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{4}n(n+1)[n(n+1) + 4n + 2 + 4] = \frac{1}{4}n(n+1)[n^2 + 5n + 6].$$

This simplifies to

$$\frac{1}{4}n(n+1)(n+2)(n+3).$$

2.1.6 EXERCISES

1. Write down the next two terms and also the n-th term of the following sequences of numbers which are either arithmetic progressions or geometric progressions:
   (a) 40, 29, 18, 7, ...;
   (b) $\frac{13}{3}$, $\frac{17}{3}$, 7, ...;
   (c) 5, 15, 45, ...;
   (d) 10, 9.2, 8.4, ...;
   (e) 81, -54, 36, ...;
   (f) $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{16}$, ...

2. The third term of an arithmetic series is 34 and the 17th term is -8. Find the sum of the first 20 terms.

3. For the geometric series 1 + 1.2 + 1.44 + ..., find the 6th term and the sum of the first 10 terms.

4. A parent places in a savings bank £25 on his son's first birthday, £50 on his second, £75 on his third and so on, increasing the amount by £25 on each birthday. How much will be saved up (apart from any accrued interest) when the boy reaches his 16th birthday if the final amount is added on this day ?

5. Every year, a gardner takes 4 runners from each of his one year old strawberry plants in order to form 4 additional plants. If he starts with 5 plants, how many new plants will he take at the end of the 6th year and what will then be his total number of plants ?

6. A superball is dropped from a height of 10m. At each rebound, it rises to a height which is 90% of the height from which it has just fallen. What is the total distance through which the ball will have moved before it finally comes to rest ?

7. Express the series
$$\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\frac{8}{9}+......n\text{terms}$$
in both of the forms
$$\sum_{r=1}^{n} u_r \quad \text{and} \quad \sum_{r=0}^{n-1} u_r.$$
Hint:
Find the pattern in the numerators and denominators separately

8. Determine the sum to n terms of the series

$$1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \ldots$$

2.1.7 ANSWERS TO EXERCISES

1. (a) -4, -15; $51 - 11n$;
   (b) $8\frac{1}{3}$, $9\frac{2}{3}$; $\frac{1}{3}(9 + 4n)$;
   (c) 135, 405; $5(3)^{n-1}$;

   (d) 7.6, 6.8; $10.8 - 0.8n$;
   (e) -24, 16; $(-1)^{n-1}2^{n-1}3^{5-n}$;
   (f) $\frac{9}{64}$, $\frac{27}{256}$; $\frac{3^{n-2}}{4^{n-1}}$.
2. $a = 40$, $d = -3$, $S_{20} = 230$.
3. 6th term $= (1.2)^5 \simeq 2.488$ and $S_{10} \simeq 25.96$
4. £3400.
5. Number at year 6 $= 5 \times 4^6 = 20480$; Total $= 27305$.
6. Total distance $= 10 + \frac{2\times 10\times 0.9}{1-0.9} = 190$m.
7. $$\sum_{r=1}^{n} \frac{2r}{2r+1} \quad \text{and} \quad \sum_{r=0}^{n-1} \frac{2(r+1)}{2r+3}.$$
8. $$\frac{1}{6}n(n+1)(4n+5).$$

"JUST THE MATHS"

UNIT NUMBER

2.2

SERIES 2
(Binomial series)

by

A.J.Hobson

UNIT 2.2 - SERIES 2 - BINOMIAL SERIES

INTRODUCTION

In this section, we shall be concerned with the methods of expanding (multiplying out) an expression of the form

$$(A + B)^n,$$

where A and B are either mathematical expressions or numerical values, and n is a given number which need not be a positive integer. However, we shall deal first with the case when n *is* a positive integer, since there is a useful aid to memory for obtaining the result.

2.2.1 PASCAL'S TRIANGLE

Initially, we consider some simple illustrations obtainable from very elementary algebraic techniques in earlier work:

1. $(A + B)^1 \equiv$
$$A + B.$$

2. $(A + B)^2 \equiv$
$$A^2 + 2AB + B^2.$$

3. $(A + B)^3 \equiv$
$$A^3 + 3A^2B + 3AB^2 + B^3.$$

4. $(A + B)^4 \equiv$
$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4.$$

OBSERVATIONS

(i) We notice that, in each result, the expansion begins with the maximum possible power of A and ends with the maximum possible power of B.

(ii) In the sequence of terms from beginning to end, the powers of A **decrease** in steps of 1 while the powers of B **increase** in steps of 1.

(iii) The coefficients in the illustrated expansions follow the diagramatic pattern called **PASCAL'S TRIANGLE**:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

and this suggests a general pattern where each line begins and ends with the number 1 and each of the other numbers is the sum of the two numbers above it in the previous line. For example, the next line would be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

giving the result

5. $(A + B)^5 \equiv$

$$A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$

(iv) The only difference which occurs in an expansion of the form

$$(A - B)^n$$

is that the terms are alternately positive and negative. For instance,

6. $(A - B)^6 \equiv$

$$A^6 - 6A^5B + 15A^4B^2 - 20A^3B^3 + 15A^2B^4 - 6AB^5 + B^6.$$

2.2.2 BINOMIAL FORMULAE

In $(A + B)^n$, if n is a large positive integer, then the method of Pascal's Triangle can become very tedious. If n is *not* a positive integer, then Pascal's Triangle cannot be used anyway.

A more general method which can be applied to any value of n is the binomial formula whose proof is best obtained as an application of differential calculus and hence will not be included here.

Before stating appropriate versions of the binomial formula, we need to introduce a standard notation called a **"factorial"** by means of the following definition:

DEFINITION
If n is a positive integer, the product

$$1.2.3.4.5..........n$$

is denoted by the symbol $n!$ and is called **"n factorial"**.

Note:
This definition could not be applied to the case when $n = 0$, but it is convenient to give a meaning to 0! We define it separately by the statement

$$0! = 1$$

and the logic behind this separate definition can be made plain in the applications of calculus. There is no meaning to $n!$ when n is a negative integer.

(a) Binomial formula for $(A + B)^n$ when n is a positive integer.

It can be shown that

$$(A + B)^n \equiv A^n + na^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2 + \frac{n(n-1)(n-2)}{3!}A^{n-3}B^3 + ... + B^n.$$

Notes:

(i) This is the same as the result which would be given by Pascal's Triangle.

(ii) The last term in the expansion is really

$$\frac{n(n-1)(n-2)(n-3).......3.2.1}{n!}A^{n-n}B^n = A^0B^n = B^n.$$

(iii) The coefficient of $A^{n-r}B^r$ in the expansion is

$$\frac{n(n-1)(n-2)(n-3).......(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

and this is sometimes denoted by the symbol $\binom{n}{r}$.

(iv) A commonly used version of the result is given by

$$(1 + x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + x^n.$$

EXAMPLES

1. Expand fully the expression $(1 + 2x)^3$.

   Solution

   We first note that
   $$(A + B)^3 \equiv A^3 + 3A^2B + \frac{3.2}{2!}AB^2 + B^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3.$$
   If we now replace A by 1 and B by $2x$, we obtain
   $$(1 + 2x)^3 \equiv 1 + 3(2x) + 3(2x)^2 + (2x)^3 \equiv 1 + 6x + 12x^2 + 8x^3.$$

2. Expand fully the expression $(2 - x)^5$.

   Solution

   We first note that
   $$(A + B)^5 \equiv A^5 + 5A^4B + \frac{5.4}{2!}A^3B^2 + \frac{5.4.3}{3!}A^2B^3 + \frac{5.4.3.2}{4!}AB^4 + B^5.$$
   That is,
   $$(A + B)^5 \equiv A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$
   We now replace A by 2 and B by $-x$ to obtain
   $$(2 - x)^5 \equiv 2^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)(-x)^4 + (-x)^5.$$
   That is,
   $$(2 - x)^5 \equiv 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5.$$

(b) Binomial formula for $(A + B)^n$ when n is negative or a fraction.

It turns out that the binomial formula for a positive integer index may still be used when the index is negative or a fraction, except that the series of terms will be an **infinite** series. That is, it will not terminate.

In order to state the most commonly used version of the more general result, we use the simplified form of the binomial formula in Note (iii) of the previous section:

RESULT

If n is negative or a fraction and x lies strictly between $x = -1$ and $x = 1$, it can be shown that
$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

EXAMPLES

1. Expand $(1+x)^{\frac{1}{2}}$ as far as the term in x^3.

   Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + ...$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - ......$$

   provided $-1 < x < 1$.

2. Expand $(2-x)^{-3}$ as far as the term in x^3 stating the values of x for which the series is valid.

   Solution

   We first convert the expression $(2-x)^{-3}$ to one in which the leading term in the bracket is 1. That is,

$$(2-x)^{-3} \equiv \left[2\left(1-\frac{x}{2}\right)\right]^{-3}$$

$$\equiv \frac{1}{8}\left(1+\left[-\frac{x}{2}\right]\right)^{-3}.$$

   The required binomial expansion is thus:

$$\frac{1}{8}\left[1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{x}{2}\right)^3+......\right].$$

   That is,

$$\frac{1}{8}\left[1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{5x^3}{4}+......\right].$$

   The expansion is valid provided that $-x/2$ lies strictly between -1 and 1. This will be so when x itself lies strictly between -2 and 2.

(c) Approximate Values

The Binomial Series may be used to calculate simple approximations, as illustrated by the following example:

EXAMPLE

Evaluate $\sqrt{1.02}$ correct to five places of decimals.

Solution

Using $1.02 = 1 + 0.02$, we may say that

$$\sqrt{1.02} = (1 + 0.02)^{\frac{1}{2}}.$$

That is,

$$\sqrt{1.02} = 1 + \frac{1}{2}(0.02) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1.2}(0.02)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2.3}(0.02)^3 + \ \ldots$$

$$= 1 + 0.01 - \frac{1}{8}(0.0004) + \frac{1}{16}(0.000008) - \ldots$$

$$= 1 + 0.01 - 0.00005 + 0.0000005 - \ \ldots$$

$$\simeq 1.010001 - 0.000050 = 1.009951$$

Hence $\sqrt{1.02} \simeq 1.00995$

2.2.3 EXERCISES

1. Expand the following, using Pascal's Triangle:

   (a)
   $$(1+x)^5;$$
   (b)
   $$(x+y)^6;$$
   (c)
   $$(x-y)^7;$$
   (d)
   $$(x-1)^8.$$

2. Use the result of question 1(a) to evaluate
   $$(1.01)^5$$
   without using a calculator.

3. Expand fully the following expressions:

   (a)
   $$(2x-1)^5;$$
   (b)
   $$\left(3+\frac{x}{2}\right)^4;$$
   (c)
   $$\left(x-\frac{2}{x}\right)^3.$$

4. Expand the following as far as the term in x^3, stating the values of x for which the expansions are valid:

   (a)
   $$(3+x)^{-1};$$

(b)

$$(1-2x)^{\frac{1}{2}};$$

(c)

$$(2+x)^{-4}.$$

5. Using the first four terms of the expansion for $(1+x)^n$, calculate an approximate value of $\sqrt{1.1}$, stating the result correct to five significant figures.

6. If x is small, show that

$$(1+x)^{-1}-(1-2x)^{\frac{1}{2}} \simeq \frac{3x^2}{2}.$$

2.2.4 ANSWERS TO EXERCISES

1. (a)

$$1+5x+10x^2+10x^3+5x^4+x^5;$$

(b)

$$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6;$$

(c)

$$x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7;$$

(d)

$$x^8-8x^7+28x^6-56x^5+70x^4-56x^3+28x^2-8x+1.$$

2. 1.0510100501 to ten places of decimals.

3. (a)

$$32x^5-80x^4+80x^3-40x^2+10x-1;$$

(b)

$$81+54x+\frac{27}{2}x^2+\frac{3}{2}x^3+\frac{1}{16}x^4;$$

(c)

$$x^3-6x+\frac{12}{x}-\frac{8}{x^3}.$$

4. (a)

$$\frac{1}{3}\left[1-\frac{x}{3}+\frac{x^2}{9}-\frac{x^3}{27}+......\right],$$

provided $-3 < x < 3$.

(b)

$$1-x-\frac{x^2}{2}-\frac{x^3}{2}-......,$$

provided $-\frac{1}{2} < x < \frac{1}{2}$.

(c)

$$\frac{1}{16}\left[1-2x+\frac{5x^2}{2}-\frac{5x^3}{2}+......\right],$$

provided $-2 < x < 2$.

5. 1.0488

6. Expand each bracket as far as the term in x^2.

"JUST THE MATHS"

UNIT NUMBER

2.3

SERIES 3
(Elementary convergence and divergence)

by

A.J.Hobson

UNIT 2.3 - SERIES 3 - ELEMENTARY CONVERGENCE AND DIVERGENCE

Introduction

In the examination of geometric series in Unit 2.1 and of binomial series in Unit 2.2, the idea was introduced of series which have a first term but no last term; that is, there are an infinite number of terms.

The general format of an infinite series may be specified by either

$$u_1 + u_2 + u_3 + \ldots = \sum_{r=1}^{\infty} u_r$$

or

$$u_0 + u_1 + u_2 + \ldots = \sum_{r=0}^{\infty} u_r.$$

In the first of these two forms, u_r is the r-th term while, in the second, u_r is the $(r+1)$-th term.

ILLUSTRATIONS

1.

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r} = \sum_{r=0}^{\infty} \frac{1}{r+1}.$$

2.

$$2 + 4 + 6 + 8 + \ldots = \sum_{r=1}^{\infty} 2r = \sum_{r=0}^{\infty} 2(r+1).$$

3.

$$1 + 3 + 5 + 7 + \ldots = \sum_{r=1}^{\infty} (2r-1) = \sum_{r=0}^{\infty} (2r+1).$$

2.3.1 THE DEFINITIONS OF CONVERGENCE AND DIVERGENCE

It has already been shown in Unit 2.1 (for geometric series) that an infinite series may have a **"sum to infinity"** even though it is not possible to reach the end of the series.

For example, the infinite geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ =\sum_{r=1}^{\infty}\frac{1}{2^r}$$

is such that the sum, S_n, of the first n terms is given by

$$S_n=\frac{\frac{1}{2}(1-\frac{1}{2}^n)}{1-\frac{1}{2}}=1-\frac{1}{2^n}.$$

As n becomes larger and larger, S_n approaches ever closer to the fixed value, 1.

We say that the **"limiting value"** of S_n as n **"tends to infinity"** is 1; and we write

$$\lim_{n\to\infty}\ S_n=1.$$

Since this limiting value is a **finite** number, we say that the series **"converges"** to 1.

DEFINITION (A)
For the infinite series

$$\sum_{r=1}^{\infty}\ u_r,$$

the expression

$$u_1+u_2+u_3+\ldots\ +u_n$$

is called its **"n-th partial sum"**.

DEFINITION (B)
If the n-th partial sum of an infinite series tends to a finite limit as n tends to infinity, the series is said to **"converge"**. In **all** other cases, the series is said to **"diverge"**.

ILLUSTRATIONS

1.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\;=\sum_{r=1}^{\infty}\frac{1}{2^r}\quad\text{converges.}$$

2.

$$1+2+3+4+\ldots.=\sum_{r=1}^{\infty}r\quad\text{diverges.}$$

3.

$$1-1+1-1+\ldots.=\sum_{r=1}^{\infty}(-1)^{n-1}\quad\text{diverges.}$$

Notes:

(i) The third illustration above shows that a series which diverges does not necessarily diverge to infinity.

(ii) Whether a series converges or diverges depends less on the starting terms than it does on the later terms. For example

$$7-15+2+39+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\;..$$

converges to $7-15+2+39+1=33+1=34$.

(iii) It will be seen, in the next section, that it is sometimes possible to test an infinite series for convergence or divergence without having to try and determine its sum to infinity.

2.3.2 TESTS FOR CONVERGENCE AND DIVERGENCE

In this section, the emphasis will be on the **use** of certain standard tests, rather than on their rigorous formal **proofs**. Only **outline** proofs will be suggested.

To begin with, we shall consider series of **positive** terms only.

TEST 1 - The r-th Term Test

An infinite series,

$$\sum_{r=1}^{\infty} u_r,$$

cannot converge unless its terms ultimately tend to zero; that is,

$$\lim_{r\to\infty} u_r = 0.$$

Outline Proof:

The series will converge only if the r-th partial sums, S_r, tend to a finite limit, L (say), as r tends to infinity; hence, if we observe that $u_r = S_r - S_{r-1}$, then u_r must tend to zero as r tends to infinity since S_r and S_{r-1} each tend to L.

ILLUSTRATIONS

1. The convergent series

   $$\sum_{r=1}^{\infty} \frac{1}{2^r},$$

   discussed earlier, is such that

   $$\lim_{r\to\infty} \frac{1}{2^r} = 0.$$

2. The divergent series

   $$\sum_{r=1}^{\infty} r,$$

   discussed earlier, is such that

   $$\lim_{r\to\infty} r \neq 0.$$

3. The series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is such that

$$\lim_{r \to \infty} \frac{1}{r} = 0,$$

but it will be shown later that this series is **divergent**.

That is, the converse of the r-th Term Test is not true. It does not imply that a series is convergent when its terms **do** tend to zero; merely that it is divergent when its terms **do not** tend to zero.

TEST 2 - The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\sum_{r=1}^{\infty} v_r$$

is a second series which is known to **converge**.

Then the first series converges provided that $u_r \leq v_r$.

Similarly, if

$$\sum_{r=1}^{\infty} w_r$$

is a series which is known to **diverge**, then the first series diverges provided that $u_r \geq w_r$.

Note:
It may be necessary to ignore the first few values of r.

Outline Proof:

Suppose we think of u_r and v_r as the heights of two sets of rectangles, all with a common base-length of one unit.

If the series

$$\sum_{r=1}^{\infty} v_r$$

is **convergent** it represents a **finite** total area of an infinite number of rectangles.

The series

$$\sum_{r=1}^{\infty} u_r$$

represents a **smaller** area and, hence, is also finite.

A similar argument holds when

$$\sum_{r=1}^{\infty} w_r$$

is a **divergent** series and $u_r \geq w_r$.

A divergent series of **positive** terms can diverge only to $+\infty$ so that the set of rectangles determined by u_r generates an area that is greater than an area which is already infinite.

EXAMPLES

1. Show that the series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \ . \ .$$

   is divergent.

   Solution

   The given series may be written as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ \ldots,$$

a series whose terms are all greater than (or, for the second term, equal to) $\frac{1}{2}$. But the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ \ldots$$

is a divergent series and, hence, the series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent.

2. Given that

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is a convergent series, show that

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

is also a convergent series.

Solution

First, we observe that, for $r = 1, 2, 3, 4, \ldots,$

$$\frac{1}{r(r+1)} < \frac{1}{r.r} = \frac{1}{r^2}.$$

Hence, the terms of the series

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

are smaller in value than those of a known convergent series. It therefore converges also.

Note:
It may be shown that the series

$$\sum_{r=1}^{\infty} \frac{1}{r^p}$$

is convergent whenever $p > 1$ and divergent whenever $p \leq 1$. This result provides a useful standard tool to use with the Comparison Test.

TEST 3 - D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = L;$$

Then the series converges if $L < 1$ and diverges if $L > 1$.

There is no conclusion if $L = 1$.

Outline Proof:

(i) If $L > 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be greater than 1 and so $u_{r+1} > u_r$ for a large enough value of r.

Hence, the terms cannot ultimately be decreasing; so Test 1 shows that the series diverges.

(ii) If $L < 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be less than 1 and so $u_{r+1} < u_r$ for a large enough value or r.

We will consider that this first occurs when $r = s$; and, from this value onwards, the terms steadily decrease in value.

Furthermore, we can certainly find a positive number, h, between L and 1 such that

$$\frac{u_{s+1}}{u_s} < h, \frac{u_{s+2}}{u_{s+1}} < h, \frac{u_{s+3}}{u_{s+2}} < h, \ \ldots$$

That is,

$$u_{s+1} < hu_s, \ u_{s+2} < hu_{s+1}, \ u_{s+3} < hu_{s+2}, \ \ldots,$$

which gives

$$u_{s+1} < hu_s, \ u_{s+2} < h^2u_s, \ u_{s+3} < h^3u_s, \ \ldots$$

But, since $L < h < 1$,

$$hu_s + h^2u_s + h^3u_s + \ \ldots$$

is a convergent geometric series; therefore, by the Comparison Test,

$$u_{s+1} + u_{s+2} + u_{s+3} + \ \ldots = \sum_{r=1}^{\infty} u_{s+r} \ \ \text{converges,}$$

implying that the original series converges also.

(iii) If $L = 1$, there will be no conclusion since we have already encountered examples of both a convergent series **and** a divergent series which give $L = 1$.

In particular,

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent and gives

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r}{r+1} = \lim_{r\to\infty} \frac{1}{1+\frac{1}{r}} = 1.$$

Also,

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent and gives

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r^2}{(r+1)^2} = \lim_{r\to\infty} \left(\frac{r}{r+1}\right)^2 = \lim_{r\to\infty} \left(\frac{1}{1+\frac{1}{r}}\right)^2 = 1.$$

Note:
A convenient way to calculate the limit as r tends to infinity of any ratio of two polynomials in r is first to divide the numerator and the denominator by the highest power of r.

For example,

$$\lim_{r\to\infty} \frac{3r^3+1}{2r^3+1} = \lim_{r\to\infty} \frac{3+\frac{1}{r^3}}{2+\frac{1}{r^3}} = \frac{3}{2}.$$

ILLUSTRATIONS

1. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} \frac{r}{2^r},$$

$$\frac{u_{r+1}}{u_r} = \frac{r+1}{2^{r+1}}.\frac{2^r}{r} = \frac{r+1}{2r}.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r+1}{2r} = \lim_{r\to\infty} \frac{1+\frac{1}{r}}{2} = \frac{1}{2}.$$

The limiting value is less than 1 so that the series converges.

2. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} 2^r,$$

$$\frac{u_{r+1}}{u_r} = \frac{2^{r+1}}{2^r} = 2.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} 2 = 2.$$

The limiting value is greater than 1 so that the series diverges.

2.3.3 EXERCISES

1. Use the "r-th Term Test" to show that the following series are divergent:

(a)

$$\sum_{r=1}^{\infty} \frac{r}{r+2};$$

(b)

$$\sum_{r=1}^{\infty} \frac{1+2r^2}{1+r^2}.$$

2. Use the "Comparison Test" to determine whether the following series are convergent or divergent:

(a)

$$\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)};$$

(b)

$$\sum_{r=1}^{\infty} \frac{r}{\sqrt{r^6+1}};$$

(c)

$$\sum_{r=1}^{\infty} \frac{r}{r^2+1}.$$

3. Use D'Alembert's Ratio Test to determine whether the following series are convergent or divergent:

   (a)

$$\sum_{r=1}^{\infty} \frac{2^r}{r^2};$$

   (b)

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)!};$$

   (c)

$$\sum_{r=1}^{\infty} \frac{r+1}{r!}.$$

4. Obtain an expression for the r-th term of the following infinite series and, hence, investigate them for convergence or divergence:

   (a)

$$\frac{1}{1\times 2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{4\times 2^4} + \ .\ .\ ;$$

   (b)

$$1 + \frac{3}{2\times 4} + \frac{7}{4\times 9} + \frac{15}{8\times 16} + \frac{31}{16\times 25} + \ .\ .\ .;$$

   (c)

$$\frac{1}{\sqrt{3}-1} + \frac{1}{2-\sqrt{2}} + \frac{1}{\sqrt{5}-\sqrt{3}} + \frac{1}{\sqrt{6}-2} + \frac{1}{\sqrt{7}-\sqrt{5}} + \ .\ .$$

2.3.4 ANSWERS TO EXERCISES

1. (a)

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = 1 \neq 0;$$

   (b)

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = 2 \neq 0.$$

2. (a) Convergent;
   (b) Convergent;
   (c) Divergent.

3. (a) Divergent;
   (b) Convergent;
   (c) Convergent.

4. (a)

$$u_r = \frac{1}{r \times 2^r};$$

   The series is convergent by D'Alembert's Ratio Test;

   (b)

$$u_r = \frac{2^r - 1}{2^{r-1} r^2};$$

   The series is convergent by Comparison Test;

   (c)

$$u_r = \frac{1}{\sqrt{r+2} - \sqrt{r}};$$

   The series is divergent by r-th Term Test.

Note:
For further discussion of limiting values, see Unit 10.1

"JUST THE MATHS"

UNIT NUMBER

2.4

SERIES 4
(Further convergence and divergence)

by

A.J.Hobson

UNIT 2.4 - SERIES 4- FURTHER CONVERGENCE AND DIVERGENCE

2.4.1 SERIES OF POSITIVE AND NEGATIVE TERMS

Introduction

In Units 2.2 and 2.3, most of the series considered have included only positive terms. But now we shall examine the concepts of convergence and divergence in cases where negative terms are present.

We note here, for example, that the r-th Term Test encountered in Unit 2.3 may be used for series whose terms are not necessarily all positive. This is because the formula

$$u_r = S_r - S_{r-1}$$

is valid for any series.

The series cannot converge unless the partial sums S_r and S_{r-1} both tend to the same finite limit as r tends to infinity which implies that u_r tends to zero as r tends to infinity.

A particularly simple kind of series with both positive and negative terms is one whose terms are alternately positive and negative. The following test is applicable to such series:

Test 4 - The Alternating Series Test

If

$$u_1 - u_2 + u_3 - u_4 + \ \ldots, \quad \text{where} \quad u_r > 0,$$

is such that

$$u_r > u_{r+1} \ \ \textbf{and} \ \ u_r \to 0 \ \text{ as } \ r \to \infty,$$

then the series converges.

Outline Proof:

(a) Suppose we re-group the series as

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ \ldots;$$

then, it may be considered in the form

$$\sum_{r=1}^{\infty} v_r,$$

where $v_1 = u_1 - u_2, v_2 = u_3 - u_4, v_3 = u_5 - u_6, \; \ldots .$

This means that v_r is positive, so that the corresponding r-th partial sums, $S_r = v_1 + v_2 + v_3 + \; \ldots \; + v_r$, steadily increase as r increases.

(b) Alternatively, suppose we re-group the series as

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots ;$$

then, it may be considered in the form

$$u_1 - \sum_{r=1}^{\infty} w_r,$$

where $w_1 = u_2 - u_3, w_2 = u_4 - u_5, w_3 = u_6 - u_7, \; \ldots .$

In this case, each partial sum, $S_r = u_1 - (w_1 + w_2 + w_3 + \; \ldots + w_r)$ is less than u_1 since positive quantities are being subtracted from it.

(c) We conclude that the partial sums of the original series are steadily increasing but are never greater than u_1. They must therefore tend to a finite limit as r tends to infinity; that is, the series converges.

ILUSTRATION

The series

$$\sum_{r=1}^{\infty} (-1)^{r-1}\frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \; \ldots$$

is convergent since

$$\frac{1}{r} > \frac{1}{r+1} \;\; \textbf{and} \;\; \frac{1}{r} \rightarrow 0 \;\; \text{as} \;\; r \rightarrow \infty.$$

2.4.2 ABSOLUTE AND CONDITIONAL CONVERGENCE

In this section, a link is made between a series having both positive and negative terms and the corresponding series for which all of the terms are positive.

By making this link, we shall be able to make use of earlier tests for convergence and divergence.

DEFINITION (A)
If

$$\sum_{r=1}^{\infty} u_r$$

is a series with both positive and negative terms, it is said to be **"absolutely convergent"** if

$$\sum_{r=1}^{\infty} |u_r|$$

is convergent.

DEFINITION (B)
If

$$\sum_{r=1}^{\infty} u_r$$

is a convergent series of positive and negative terms, but

$$\sum_{r=1}^{\infty} |u_r|$$

is a divergent series, then the first of these two series is said to be **"conditionally convergent"**.

ILLUSTRATIONS

1. The series

   $$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$

   converges absolutely since the series

   $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ \ldots$$

   converges.

2. The series
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ \ldots$$
is conditionally convergent since, although it converges (by the Alternating Series Test), the series
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \ldots$$
is divergent.

Notes:

(i) It may be shown that any series of positive and negative terms which is **absolutely** convergent will also be convergent.

(ii) Any test for the convergence of a series of positive terms may be used as a test for the absolute convergence of a series of both positive and negative terms.

2.4.3 TESTS FOR ABSOLUTE CONVERGENCE

The Comparison Test

Given the series
$$\sum_{r=1}^{\infty} u_r,$$
suppose that $|u_r| \leq v_r$ where
$$\sum_{r=1}^{\infty} v_r$$
is a convergent series of positive terms. Then, the given series is absolutely convergent.

D'Alembert's Ratio Test

Given the series
$$\sum_{r=1}^{\infty} u_r,$$
suppose that
$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = L.$$

Then the given series is absolutely convergent if $L < 1$.

Note:

If $L > 1$, then $|u_{r+1}| > |u_r|$ for large enough values of r showing that the **numerical** values of the terms steadily increase. This implies that u_r does **not** tend to zero as r tends to infinity and, hence, by the r-th Term Test, the series diverges.

If $L = 1$, there is no conclusion.

EXAMPLES

1. Show that the series
$$\frac{1}{1\times 2} - \frac{1}{2\times 3} - \frac{1}{3\times 4} + \frac{1}{4\times 5} - \frac{1}{5\times 6} - \frac{1}{6\times 7} + \ . \ .$$
is absolutely convergent.
Solution
The r-th term of the series is numerically equal to
$$\frac{1}{r(r+1)},$$
which is always less than $\frac{1}{r^2}$, the r-th term of a known convergent series.

2. Show that the series
$$\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \ . \ .$$
is conditionally convergent.
Solution
The r-th term of the series is numerically equal to
$$\frac{r}{r^2+1},$$
which tends to zero as r tends to infinity.
Also,
$$\frac{r}{r^2+1} > \frac{r+1}{(r+1)^2+1}$$

since this may be reduced to the true statement $r^2 + r > 1$.
Hence, by the Alternating Series Test, the series converges.
However, it is also true that

$$\frac{r}{r^2+1} > \frac{r}{r^2+r} = \frac{1}{r+1};$$

and, hence, by the Comparison Test, the series of absolute values is divergent, since

$$\sum_{r=1}^{\infty} \frac{1}{r+1}$$

is divergent.

2.4.4 POWER SERIES

A series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ . \ . \ = \sum_{r=0}^{\infty} a_r x^r \ \text{ or } \ \sum_{r=1}^{\infty} a_{r-1}x^{r-1},$$

where x is usually a variable quantity, is called a **"power Series in x with coefficients $a_0, \ a_1, \ a_2, \ a_3, \ . \ . \ .$"**.

Notes:

(i) In this kind of series, it is particularly useful to sum the series from $r = 0$ to infinity rather than from $r = 1$ to infinity so that the constant term at the beginning (if there is one) can be considered as the term in x^0.

But the various tests for convergence and divergence still apply in this alternative notation.

(ii) A power series will not necessarily be convergent (or divergent) for **all** values of x and it is usually required to determine the specific **range** of values of x for which the series converges. This can most frequently be done using D'Alembert's Ratio Test.

ILLUSTRATION

For the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ . \ . \ = \sum_{r=1}^{\infty} (-1)^{r-1}\frac{x^r}{r},$$

we have

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{(-1)^r x^{r+1}}{r+1}.\frac{r}{(-1)^{r-1}x^r}\right| = \left|\frac{r}{r+1}x\right|,$$

which tends to $|x|$ as r tends to infinity.

Thus, the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

If $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots,$$

which converges by the Alternating Series Test; while, if $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ \ldots,$$

which diverges.

The **precise** range of convergence for the given series is therefore $-1 < x \leq 1$.

2.4.5 EXERCISES

1. Show that the following alternating series are convergent:

   (a)

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ \ldots;$$

   (b)

$$\frac{1}{3^2} - \frac{2}{3^3} + \frac{3}{3^4} - \frac{4}{3^5} + \ \ldots$$

2. Show that the following series are conditionally convergent:

   (a)

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ \ldots;$$

(b)

$$\frac{2}{1\times 3}-\frac{3}{2\times 4}+\frac{4}{3\times 5}-\frac{5}{4\times 6}+\ \ldots$$

3. Show that the following series are absolutely convergent:

   (a)

$$\frac{3}{2}+\frac{4}{3}\times\frac{1}{2}-\frac{5}{4}\times\frac{1}{2^2}-\frac{6}{5}\times\frac{1}{2^3}+\ \ldots;$$

   (b)

$$\frac{1}{3}+\frac{1\times 2}{3\times 5}-\frac{1\times 2\times 3}{3\times 5\times 7}-\frac{1\times 2\times 3\times 4}{3\times 5\times 7\times 9}+\ ..$$

4. Obtain the precise range of values of x for which each of the following power series is convergent:

   (a)

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ \ldots;$$

   (b)

$$\frac{x}{1\times 2}+\frac{x^2}{2\times 3}+\frac{x^3}{3\times 4}+\frac{x^4}{4\times 5}+\ \ldots;$$

   (c)

$$2x+\frac{3x^2}{2^3}+\frac{4x^3}{3^3}+\frac{5x^4}{4^3}+\ \ldots;$$

   (d)

$$1+\frac{2x}{5}+\frac{3x^2}{25}+\frac{4x^3}{125}+\ \ldots$$

2.4.6 ANSWERS TO EXERCISES

1. (a) Use $u_r = \frac{1}{2^{r-1}}$ (numerically);

   (b) Use $u_r = \frac{r}{3^{r+1}}$ (numerically).

2. (a) Use $u_r = \frac{1}{\sqrt{r}}$ (numerically);

   (b) Use $u_r = \frac{r+1}{r(r+2)}$ (numerically).

3. (a) Use $u_r = \frac{r+2}{r+1} \times \frac{1}{2^{r-1}}$ - (numerically);

   (b) Use $u_r = \frac{2^r (r!)^2}{(2r+1)!}$ (numerically).

4. (a) The power series converges for all values of x;

   (b) $-1 \leq x \leq 1$;

   (c) $-1 \leq x \leq 1$;

   (d) $-5 < x < 5$.

Note:
For further discussion of limiting values, see Unit 10.1

"JUST THE MATHS"

UNIT NUMBER

3.1

TRIGONOMETRY 1
(Angles & trigonometric functions)

by

A.J.Hobson

UNIT 3.1 - TRIGONOMETRY 1
ANGLES AND TRIGONOMETRIC FUNCTIONS

3.1.1 INTRODUCTION

The following results will be assumed without proof:

(i) The Circumference, C, and Diameter, D, of a circle are directly proportional to each other through the formula

$$C = \pi D$$

or, if the radius is r,

$$C = 2\pi r.$$

(ii) The area, A, of a circle is related to the radius, r, by means of the formula

$$A = \pi r^2.$$

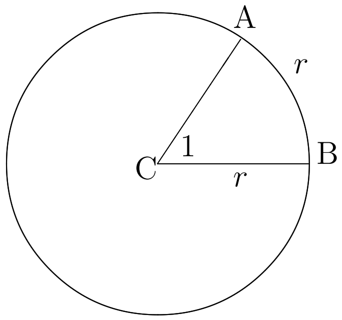
3.1.2 ANGULAR MEASURE

(a) Astronomical Units

The **"degree"** is a $\frac{1}{360}$th part of one complete revolution. It is based on the study of planetary motion where 360 is approximately the number of days in a year.

(b) Radian Measure

A **"radian"** is the angle subtended at the centre of a circle by an arc which is equal in length to the radius.



RESULTS

(i) Using the definition of a radian, together with the second formula for circumference on the previous page, we conclude that there are 2π radians in one complete revolution. That is, 2π radians is equivalent to $360°$ or, in other words π radians is equivalent to $180°$.

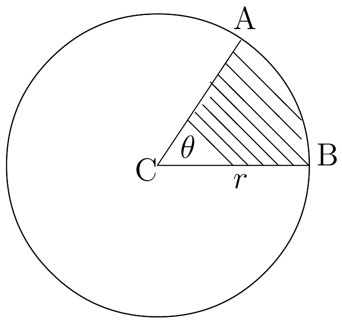
(ii) In the diagram overleaf, the arclength from A to B will be given by

$$\frac{\theta}{2\pi} \times 2\pi r = r\theta,$$

assuming that θ is measured in radians.

(iii) In the diagram below, the area of the sector ABC is given by

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta.$$



(c) Standard Angles

The scaling factor for converting degrees to radians is

$$\frac{\pi}{180}$$

and the scaling factor for converting from radians to degrees is

$$\frac{180}{\pi}.$$

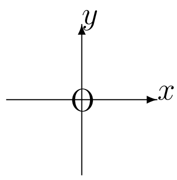
These scaling factors enable us to deal with any angle, but it is useful to list the expression, in radians, of some of the more well-known angles.

ILLUSTRATIONS

1. 15° is equivalent to $\frac{\pi}{180} \times 15 = \frac{\pi}{12}$.
2. 30° is equivalent to $\frac{\pi}{180} \times 30 = \frac{\pi}{6}$.
3. 45° is equivalent to $\frac{\pi}{180} \times 45 = \frac{\pi}{4}$.
4. 60° is equivalent to $\frac{\pi}{180} \times 60 = \frac{\pi}{3}$.
5. 75° is equivalent to $\frac{\pi}{180} \times 75 = \frac{5\pi}{12}$.
6. 90° is equivalent to $\frac{\pi}{180} \times 90 = \frac{\pi}{2}$.

(d) Positive and Negative Angles

For the measurement of angles in general, we consider the plane of the page to be divided into four quadrants by means of a cartesian reference system with axes Ox and Oy. The **"first quadrant"** is that for which x and y are both positive, and the other three quadrants are numbered from the first in an anticlockwise sense.



From the positive x-direction, we measure angles positively in the anticlockwise sense and negatively in the clockwise sense. Special names are given to the type of angles obtained as follows:

1. Angles in the range between $0°$ and $90°$ are called **"positive acute"** angles.

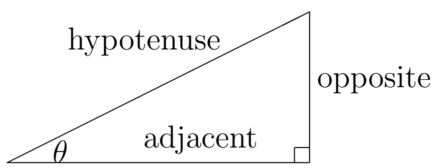
2. Angles in the range between $90°$ and $180°$ are called **"positive obtuse"** angles.

3. Angles in the range between $180°$ and $360°$ are called **"positive reflex"** angles.

4. Angles measured in the clockwise sense have similar names but preceded by the word **"negative"**.

3.1.3 TRIGONOMETRIC FUNCTIONS

We first consider a right-angled triangle in one corner of which is an angle θ other than the right-angle itself. The sides of the triangle are labelled in relation to this angle , θ, as **"opposite"**, **"adjacent"** and **"hypotenuse"** (see diagram below).



For future reference, we shall assume, without proof, the result known as **"Pythagoras' Theorem"**. This states that the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

DEFINITIONS

(a) The **"sine"** of the angle θ, denoted by $\sin\theta$, is defined by

$$\sin\theta \equiv \frac{\text{opposite}}{\text{hypotenuse}};$$

(b) The **"cosine"** of the angle θ, denoted by $\cos\theta$, is defined by

$$\cos\theta \equiv \frac{\text{adjacent}}{\text{hypotenuse}};$$

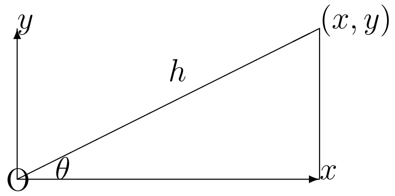
(c) The **"tangent"** of the angle θ, denoted by $\tan\theta$, is defined by

$$\tan\theta \equiv \frac{\text{opposite}}{\text{adjacent}}.$$

Notes:
(i) The traditional aid to remembering the above definitions is the abbreviation

S.O.H.C.A.H.T.O.A.

(ii) The definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be extended to angles of any size by regarding the end-points of the hypotenuse, with length h, to be, respectively, the origin and the point (x, y) in a cartesian system of reference.



For any values of x and y, positive, negative or zero, the three basic trigonometric functions are defined in general by the formulae

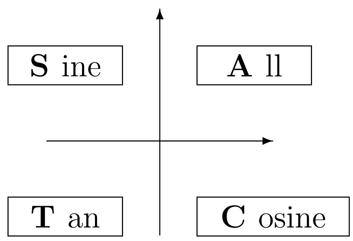
$$\sin\theta \equiv \frac{y}{h};$$

$$\cos\theta \equiv \frac{x}{h};$$

$$\tan\theta \equiv \frac{y}{x} \equiv \frac{\sin\theta}{\cos\theta}.$$

Clearly these reduce to the original definitions in the case when θ is a positive acute angle. Trigonometric functions can also be called **"trigonometric ratios"**.

(iii) It is useful to indicate diagramatically which of the three basic trigonometric functions have positive values in the various quadrants.



(iv) Three other trigonometric functions are sometimes used and are defined as the reciprocals of the three basic functions as follows:

"Secant"

$$\text{sec}\theta \equiv \frac{1}{\cos\theta};$$

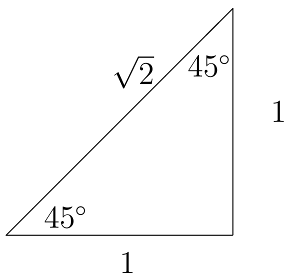
"Cosecant"

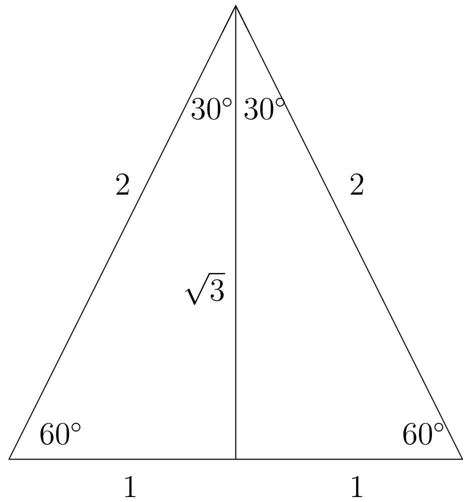
$$\text{cosec}\theta \equiv \frac{1}{\sin\theta};$$

"Cotangent"

$$\cot\theta \equiv \frac{1}{\tan\theta}.$$

(v) The values of the functions $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the particluar angles 30°, 45° and 60° are easily obtained without calculator from the following diagrams:





The diagrams show that

(a) $\sin 45° = \frac{1}{\sqrt{2}}$; (b) $\cos 45° = \frac{1}{\sqrt{2}}$; (c) $\tan 45° = 1$;

(d) $\sin 30° = \frac{1}{2}$; (e) $\cos 30° = \frac{\sqrt{3}}{2}$ (f) $\tan 30° = \frac{1}{\sqrt{3}}$;

(g) $\sin 60° = \frac{\sqrt{3}}{2}$; (h) $\cos 60° = \frac{1}{2}$; (i) $\tan 60° = \sqrt{3}$.

3.1.4 EXERCISES

1. Express each of the following angles as a multiple of π
   (a) 65°; (b) 105°; (c) 72°; (d) 252°;
   (e) 20°; (f) −160°; (g) 9°; (h) 279°.
2. On a circle of radius 24cms., find the length of arc which subtends an angle at the centre of
   (a) $\frac{2}{3}$ radians.; (b) $\frac{3\pi}{5}$ radians.;
   (c) 75°; (d) 130°.
3. A wheel is turning at the rate of 48 revolutions per minute. Express this angular speed in
   (a) revolutions per second; (b) radians per minute; (c) radians per second.

4. A wheel, 4 metres in diameter, is rotating at 80 revolutions per minute. Determine the distance, in metres, travelled in one second by a point on the rim.
5. A chord AB of a circle, radius 5cms., subtends a right-angle at the centre of the circle. Calculate, correct to two places of decimals, the areas of the two segments into which AB divides the circle.
6. If $\tan\theta$ is positive and $\cos\theta = -\frac{4}{5}$, what is the value of $\sin\theta$?
7. Determine the length of the chord of a circle, radius 20cms., subtending an angle of $150°$ at the centre.
8. A ladder leans against the side of a vertical building with its foot 4 metres from the building. If the ladder is inclined at $70°$ to the ground, how far from the ground is the top of the ladder and how long is the ladder ?

3.1.5 ANSWERS TO EXERCISES

1. (a) $\frac{13\pi}{36}$; (b) $\frac{7\pi}{12}$; (c) $\frac{2\pi}{5}$; (d) $\frac{7\pi}{5}$; (e) $\frac{\pi}{9}$; (f) -$\frac{8\pi}{9}$; (g) $\frac{\pi}{20}$; (h) $\frac{31\pi}{20}$.
2. (a) 16 cms.; (b) $\frac{72\pi}{5}$ cms.; (c) 10π cms.; (d) $\frac{52\pi}{3}$ cms.
3. (a) $\frac{4}{5}$ revs. per sec.; (b) 96π rads. per min.; (c) $\frac{8\pi}{5}$ rads. per sec.
4. $\frac{16\pi}{3}$ metres.
5. 7.13 square cms. and 71.41 square cms.
6. $\sin\theta = -\frac{3}{5}$.
7. The chord has a length of 38.6cms. approximately.
8. The top of ladder is 11 metres from the ground and the length of the ladder is 11.7 metres.

"JUST THE MATHS"

UNIT NUMBER

3.2

TRIGONOMETRY 2
(Graphs of trigonometric functions)

by

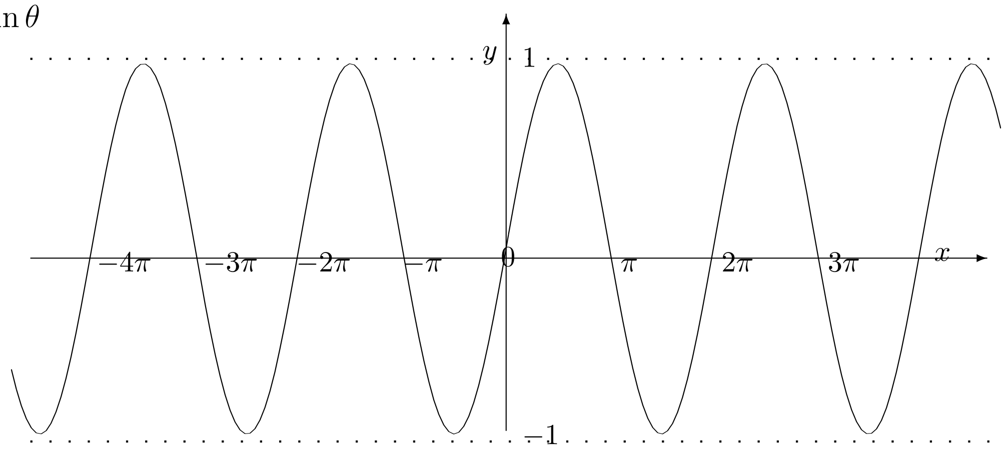
A.J.Hobson

UNIT 3.2 - TRIGONOMETRY 2.
GRAPHS OF TRIGONOMETRIC FUNCTIONS

3.2.1 GRAPHS OF ELEMENTARY TRIGONOMETRIC FUNCTIONS

The following diagrams illustrate the graphs of the basic trigonometric functions sinθ, cosθ and tanθ,

1. $y = \sin\theta$



The graph illustrates that

$$\sin(\theta + 2\pi) \equiv \sin\theta$$

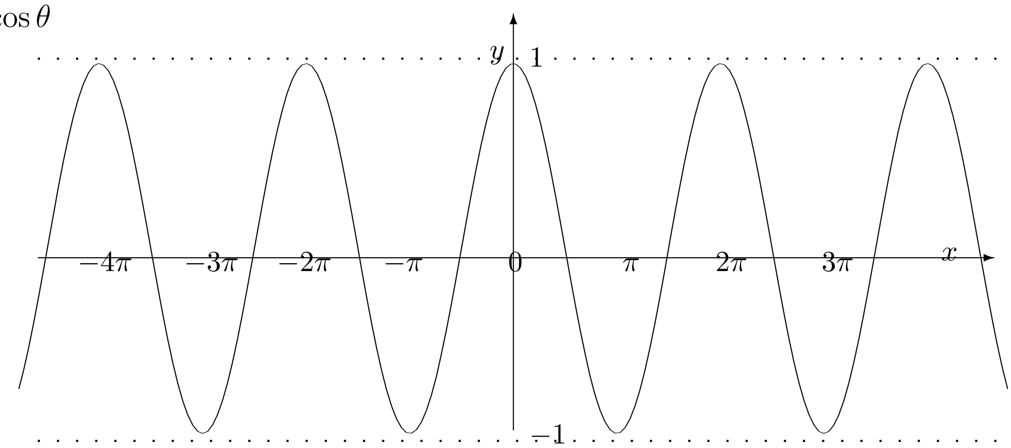
and we say that sinθ is a **"periodic function with period** 2π**"**.
Other numbers which can act as a period are $\pm 2n\pi$ where n is any integer; but 2π itself is the smallest positive period and, as such, is called the **"primitive period"** or sometimes the **"wavelength"**.

We may also observe that

$$\sin(-\theta) \equiv -\sin\theta$$

which makes sinθ what is called an **"odd function"**.

2. $y = \cos\theta$



The graph illustrates that

$$\cos(\theta + 2\pi) \equiv \cos\theta$$

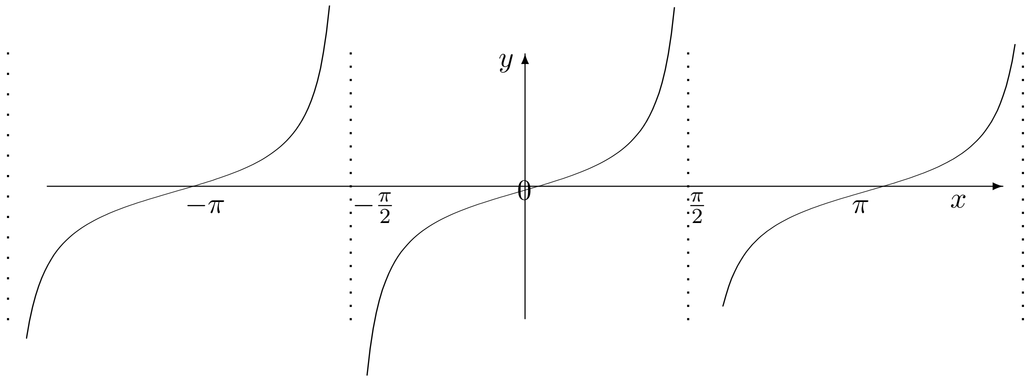
and so cosθ, like sinθ, is a periodic function with primitive period 2π

We may also observe that

$$\cos(-\theta) \equiv \cos\theta$$

which makes cosθ what is called an **"even function"**.

3. $y = \tan\theta$



This time, the graph illustrates that

$$\tan(\theta + \pi) \equiv \tan\theta$$

which implies that tanθ is a periodic function with primitive period π.

We may also observe that

$$\tan(-\theta) \equiv -\tan\theta$$

which makes tanθ an **"odd function"**.

3.2.2 GRAPHS OF MORE GENERAL TRIGONOMETRIC FUNCTIONS

In scientific work, it is possible to encounter functions of the form

$$\boxed{\text{A}\sin(\omega\theta + \alpha)} \text{ and } \boxed{\text{A}\cos(\omega\theta + \alpha)}$$

where ω and α are constants.

We may sketch their graphs by using the information in the previous examples 1. and 2.

EXAMPLES

1. Sketch the graph of

$$y = 5\cos(\theta - \pi).$$

Solution

The important observations to make first are that

(a) the graph will have the same shape as the basic cosine wave but will lie between $y=-5$ and $y=5$ instead of between $y=-1$ and $y=1$; we say that the graph has an **"amplitude"** of 5.

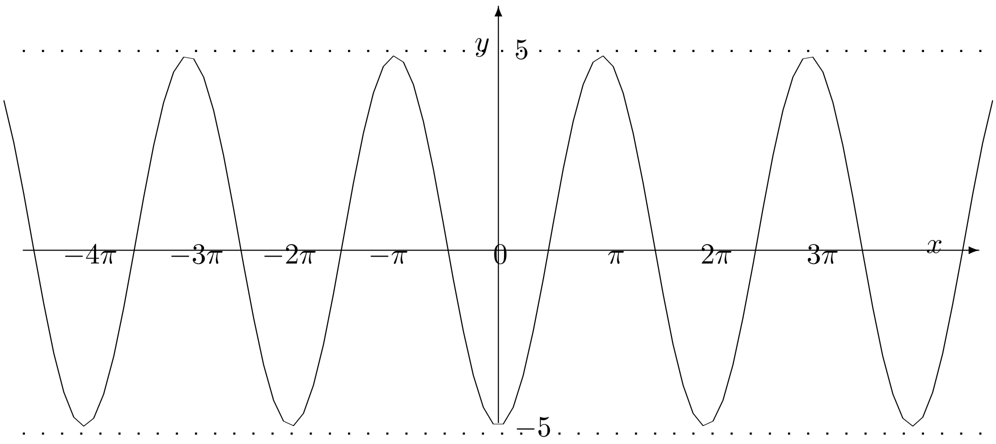
(b) the graph will cross the θ-axis at the points for which

$$\theta-\pi=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},......$$

that is

$$\theta=-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},......$$

(c) The y-axis must be placed between the smallest **negative** intersection with the θ-axis and the smallest **positive** intersection with the θ - axis (in proportion to their values). In this case, the y-axis must be placed half way between $\theta=-\frac{\pi}{2}$ and $\theta=\frac{\pi}{2}$.



Of course, in this example, from earlier trigonometry results, we could have noticed that

$$5\cos(\theta-\pi)\equiv-5\cos\theta$$

so that graph consists of an "upsidedown" cosine wave with an amplitude of 5. However, not all examples can be solved in this way.

2. Sketch the graph of

$$y=3\sin(2\theta+1).$$

Solution

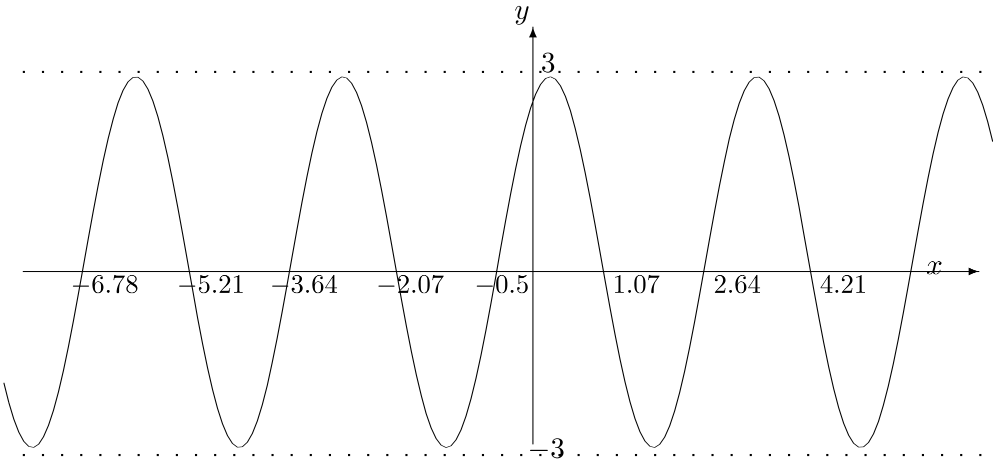
This time, the graph will have the same shape as the basic sine wave, but will have an amplitude of 3. It will cross the θ-axis at the points for which

$$2\theta+1=0,\pm\pi,\pm2\pi,\pm3\pi,\pm4\pi,......$$

and by solving for θ in each case, we obtain

$$\theta=...-6.78,-5.21,-3.64,-2.07,-0.5,1.07,2.64,4.21,5.78...$$

Hence, the y-axis must be placed between $\theta=-0.5$ and $\theta=1.07$ but at about one third of the way from $\theta=-0.5$



3.2.3 EXERCISES

1. Make a table of values of θ and y, with θ in the range from 0 to 2π in steps of $\frac{\pi}{12}$, and hence, sketch the graphs of

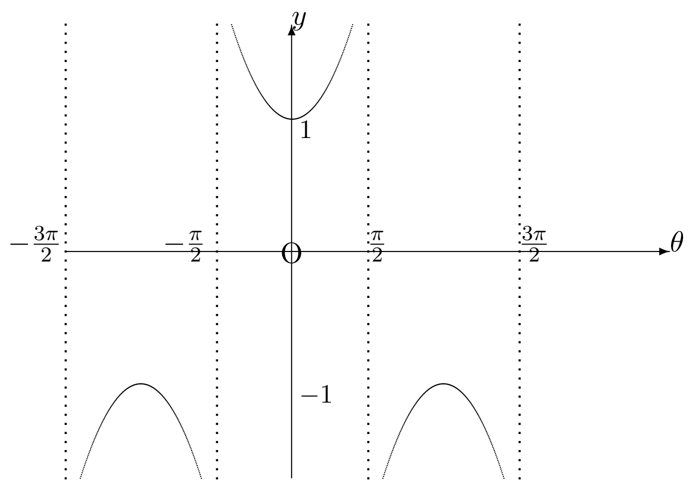
   (a)
   $$y = \sec\,\theta;$$
   (b)
   $$y = \text{cosec}\,\theta;$$
   (c)
   $$y = \cot\,\theta.$$

2. Sketch the graphs of the following functions:

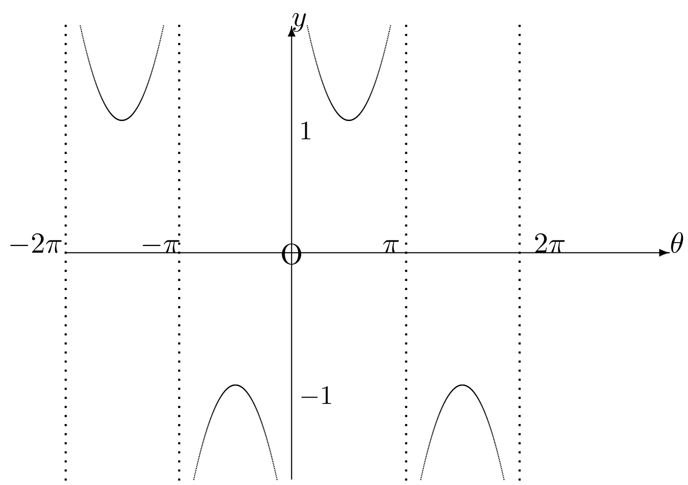
   (a)
   $$y = 2\sin\left(\theta + \frac{\pi}{4}\right);$$
   (b)
   $$y = 2\cos(3\theta - 1).$$
   (c)
   $$y = 5\sin(7\theta + 2).$$
   (d)
   $$y = -\cos\left(\theta - \frac{\pi}{3}\right).$$

3.2.4 ANSWERS TO EXERCISES

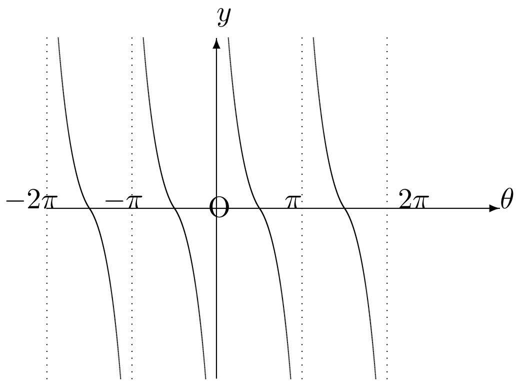
1. (a) The graph is



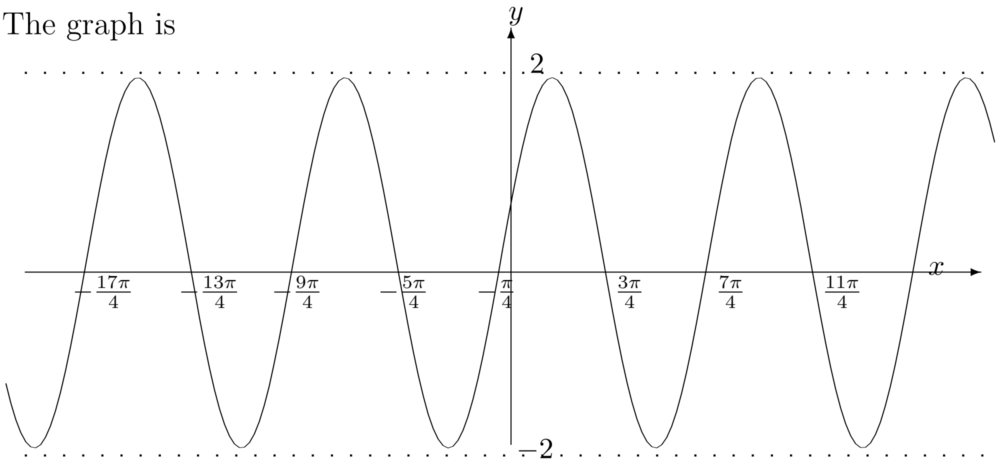
(b) The graph is



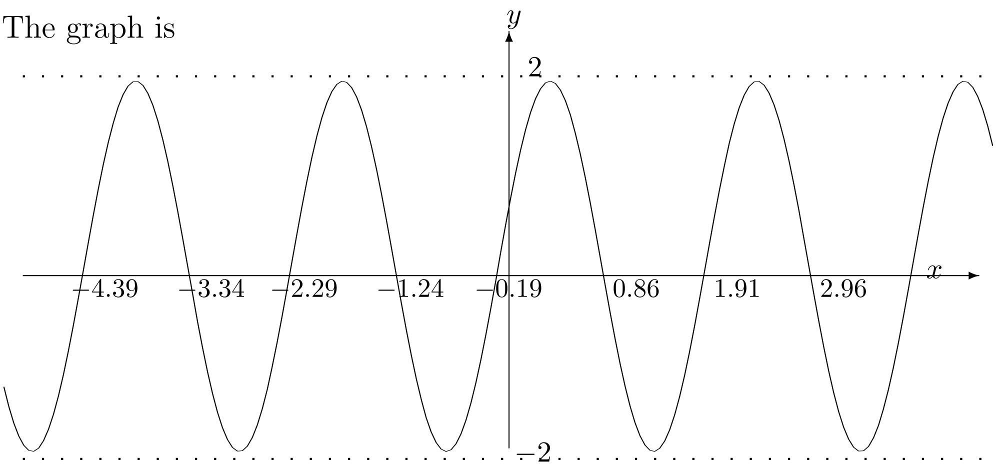
(c) The graph is



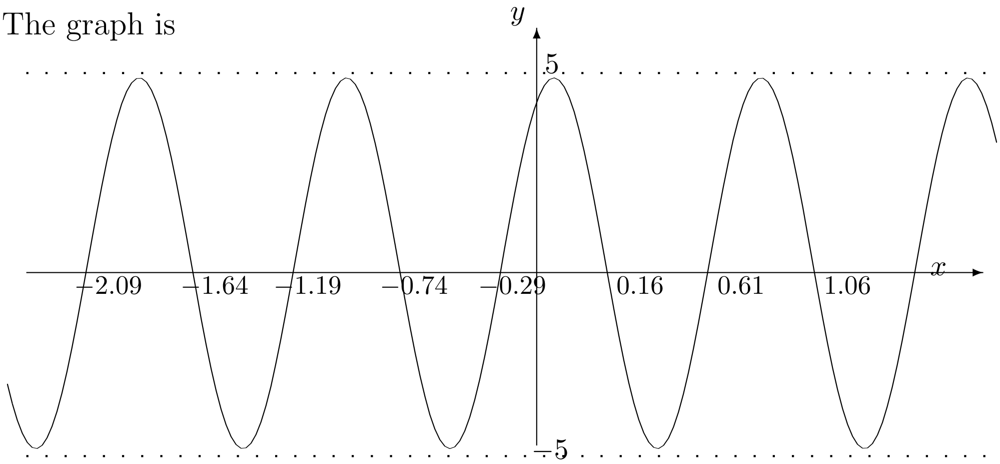
2. (a) The graph is



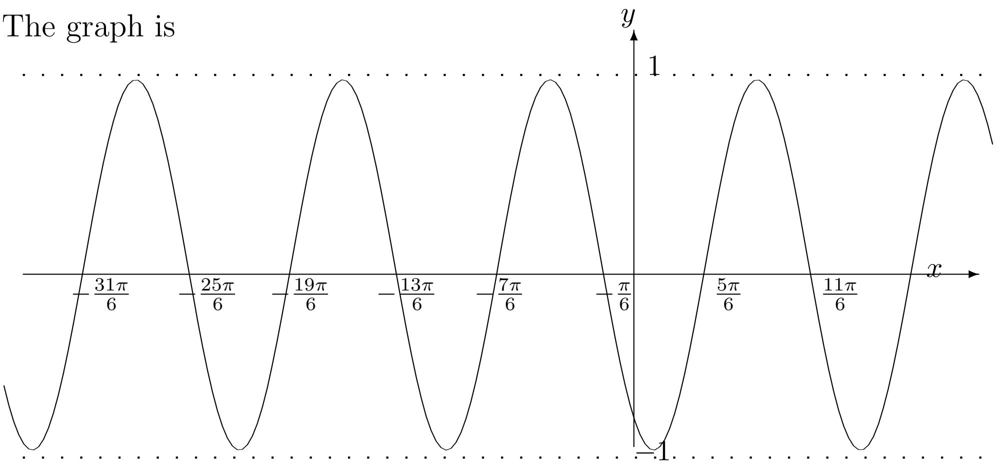
(b) The graph is



(c) The graph is



(d) The graph is



"JUST THE MATHS"

UNIT NUMBER

3.3

TRIGONOMETRY 3
(Approximations & inverse functions)

by

A.J.Hobson

UNIT 3.3 - TRIGONOMETRY APPROXIMATIONS AND INVERSE FUNCTIONS

3.3.1 APPROXIMATIONS FOR TRIGONOMETRIC FUNCTIONS

Three standard approximations for the functions $\sin\theta, \cos\theta$ and $\tan\theta$ respectively can be obtained from a set of results taken from the applications of Calculus. These are stated without proof as follows:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}.....$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}......$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + ...$$

These results apply **only if θ is in radians** but, if θ is small enough for θ^2 and higher powers of θ to be neglected, we conclude that

$$\sin\theta \simeq \theta,$$

$$\cos\theta \simeq 1,$$

$$\tan\theta \simeq \theta.$$

Better approximations are obtainable if more terms of the infinite series are used.

EXAMPLE

Approximate the function

$$5 + 2\cos\theta - 7\sin\theta$$

to a quartic polynomial in θ.

Solution

Using terms of the appropriate series up to and including the fourth power of θ, we deduce that

$$5 + 2\cos\theta - 7\sin\theta \simeq 5 + 2 - \theta^2 + \frac{\theta^4}{12} - 7\theta + 7\frac{\theta^3}{6}$$

$$= \frac{1}{12}\left[\theta^4 + 14\theta^3 - 12\theta^2 - 84\theta + 84\right].$$

3.3.2 INVERSE TRIGONOMETRIC FUNCTIONS

It is frequently necessary to determine possible angles for which the value of their sine, cosine or tangent is already specified. This is carried out using inverse trigonometric functions defined as follows:

(a) The symbol

$$\mathrm{Sin}^{-1}x$$

denotes any angle whose sine value is the number x. It is necessary that $-1 \leq x \leq 1$ since the sine of an angle is always in this range.

(b) The symbol

$$\mathrm{Cos}^{-1}x$$

denotes any angle whose cosine value is the number x. Again, $-1 \leq x \leq 1$.

(c) The symbol

$$\mathrm{Tan}^{-1}x$$

denotes any angle whose tangent value is x. This time, x may be any value because the tangent function covers the range from $-\infty$ to ∞.

We note that because of the $\boxed{\textbf{A}\text{ ll}}$, $\boxed{\textbf{S}\text{ ine}}$, $\boxed{\textbf{T}\text{ angent}}$, $\boxed{\textbf{C}\text{ osine}}$ diagram, (see Unit 3.1), there will be two **basic** values of an inverse function from two different quadrants. But either of these two values may be increased or decreased by a whole multiple of $360°$ (2π) to yield other acceptable answers and hence an infinite number of possible answers.

EXAMPLES

1. Evaluate $\mathrm{Sin}^{-1}(\frac{1}{2})$.

   Solution

   $\mathrm{Sin}^{-1}(\frac{1}{2}) = 30° \pm n360°$ or $150° \pm n360°$.

2. Evaluate $\mathrm{Tan}^{-1}(\sqrt{3})$.

   Solution

   $\mathrm{Tan}^{-1}(\sqrt{3}) = 60° \pm n360°$ or $240° \pm n360°$.

   This result is in fact better written in the combined form
   $\mathrm{Tan}^{-1}(\sqrt{3}) = 60° \pm n180°$
   THat is, angles in opposite quadrants have the same tangent.

   Another Type of Question

3. Obtain all of the solutions to the equation

$$\cos 3x = -0.432$$

   which lie in the interval $-180° \leq x \leq 180°$.

   Solution

   This type of question is of a slightly different nature since we are asked for a specified **selection** of values rather than the general solution of the equation.

We require that $3x$ be any one of the angles (within an interval $-540° \leq 3x \leq 540°$) whose cosine is equal to -0.432. Using a calculator, the simplest angle which satisfies this condition is $115.59°$; but the complete set is

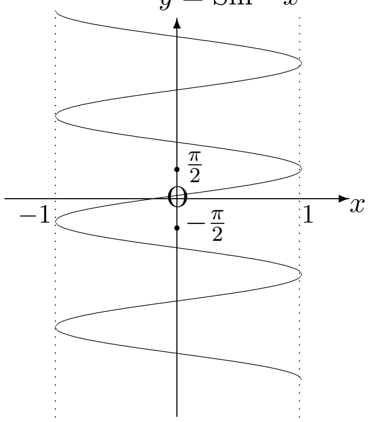
$$\pm 115.59° \quad \pm 244.41° \quad \pm 475.59°$$

Thus, on dividing by 3, the possibilities for x are

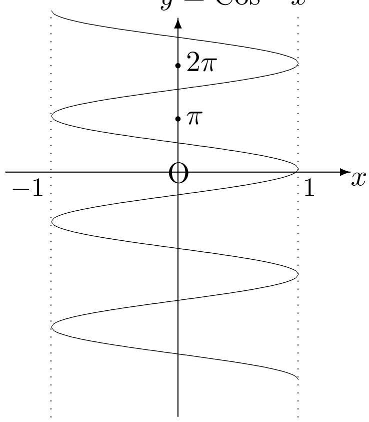
$$\pm 38.5° \quad \pm 81.5° \quad \pm 158.5°$$

Note: The graphs of inverse trigonometric functions are discussed fully in Unit 10.6, but we include them here for the sake of completeness

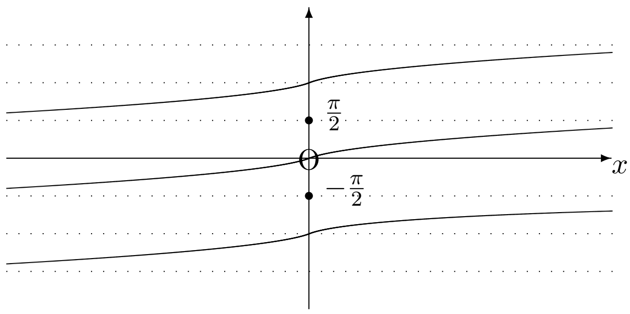
$y = \mathrm{Sin}^{-1}x$



$y = \mathrm{Cos}^{-1}x$



$y = \mathrm{Tan}^{-1}x$



Of all the possible values obtained for an inverse trigonometric function, one particular one is called the **"Principal Value"**. It is the unique value which lies in a specified range described below, the explanation of which is best dealt with in connection with differential calculus.

To indicate such a principal value, we use the lower-case initial letter of each inverse function.

(a) $\theta = \sin^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(b) $\theta = \cos^{-1}x$ lies in the range $0 \leq \theta \leq \pi$.

(c) $\theta = \tan^{-1}x$ lies in the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

EXAMPLES

1. Evaluate $\sin^{-1}(\frac{1}{2})$.

   Solution

   $\sin^{-1}(\frac{1}{2}) = 30^\circ$ or $\frac{\pi}{6}$.

2. Evaluate $\tan^{-1}(-\sqrt{3})$.

   Solution

   $\tan^{-1}(-\sqrt{3}) = -60^\circ$ or $-\frac{\pi}{3}$.

3. Write down a formula for u in terms of v in the case when

$$v = 5\cos(1 - 7u).$$

   Solution

   Dividing by 5 gives

$$\frac{v}{5} = \cos(1 - 7u).$$

   Taking the inverse cosine gives

$$\text{Cos}^{-1}\left(\frac{v}{5}\right) = 1 - 7u.$$

   Subtracting 1 from both sides gives

$$\text{Cos}^{-1}\left(\frac{v}{5}\right) - 1 = -7u.$$

   Dividing both sides by -7 gives

$$u = -\frac{1}{7}\left[\text{Cos}^{-1}\left(\frac{v}{5}\right) - 1\right].$$

3.3.3 EXERCISES

1. If powers of θ higher then three can be neglected, find an approximation for the function

$$6\sin\theta + 2\cos\theta + 10\tan\theta$$

   in the form of a polynomial in θ.

2. If powers of θ higher than five can be neglected, find an approximation for the function

$$2\sin\theta - \theta\cos\theta$$

   in the form of a polynomial in θ.

3. If powers of θ higher than two can be neglected, show that the function

$$\frac{\theta \sin \theta}{1 - \cos \theta}$$

is approximately equal to 2.

4. Write down the principal values of the following:

   (a) $\mathrm{Sin}^{-1} 1$;
   (b) $\mathrm{Sin}^{-1} \left(-\frac{1}{2}\right)$;
   (c) $\mathrm{Cos}^{-1} \left(-\frac{\sqrt{3}}{2}\right)$;
   (d) $\mathrm{Tan}^{-1} 5$;
   (e) $\mathrm{Tan}^{-1}(-\sqrt{3})$;
   (f) $\mathrm{Cos}^{-1} \left(-\frac{1}{\sqrt{2}}\right)$.

5. Solve the following equations for x in the interval $0 \leq x \leq 360°$:

   (a)
   $$\tan x = 2.46$$
   (b)
   $$\cos x = 0.241$$
   (c)
   $$\sin x = -0.786$$
   (d)
   $$\tan x = -1.42$$
   (e)
   $$\cos x = -0.3478$$
   (f)
   $$\sin x = 0.987$$

   Give your answers correct to one decimal place.

6. Solve the following equations for the range given, stating your final answers in degrees correct to one decimal place:

   (a) $\sin 2x = -0.346$ for $0 \leq x \leq 360°$;
   (b) $\tan 3x = 1.86$ for $0 \leq x \leq 180°$;
   (c) $\cos 2x = -0.57$ for $-180° \leq x \leq 180°$;
   (d) $\cos 5x = 0.21$ for $0 \leq x \leq 45°$;
   (e) $\sin 4x = 0.78$ for $0 \leq x \leq 180°$.

7. Write down a formula for u in terms of v for the following:

   (a) $v = \sin u$;
   (b) $v = \cos 2u$;
   (c) $v = \tan(u + 1)$.

8. If x is positive, show diagramatically that

(a)

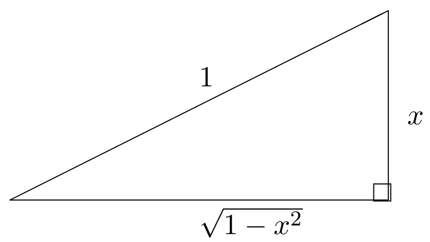
$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2};$$

(b)

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}.$$

3.3.4 ANSWERS TO EXERCISES

1. $\frac{7\theta^3}{3} - \theta^2 + 16\theta + 2$.
2. $\theta + \frac{\theta^3}{6} - \frac{\theta^5}{40}$.
3. Substitute approximations for $\sin\theta$ and $\cos\theta$.
4. (a) $\frac{\pi}{2}$; (b) $-\frac{\pi}{6}$; (c) $\frac{5\pi}{6}$; (d) 1.373; (e) $-\frac{\pi}{3}$; (f) $\frac{3\pi}{4}$.
5. (a) 67.9° or 247.9°;
   (b) 76.1° or 283.9°;
   (c) 231.8° or 308.2°;
   (d) 125.2° or 305.32°;
   (e) 110.4° or 249.6°;
   (f) 80.8° or 99.2°
6. (a) 100.1°, 169.9°, 280.1°, 349.9°
   (b) 20.6°, 80.6°, 140.6°
   (c) ±62.4°, ±117.6°
   (d) 15.6°
   (e) 32.2°, 102.8°, 122.2°
7. (a) $u = \mathrm{Sin}^{-1}v$; (b) $u = \frac{1}{2}\mathrm{Cos}^{-1}v$; (c) $u = \mathrm{Tan}^{-1}v - 1$.
8. A suitable diagram is



“JUST THE MATHS”

UNIT NUMBER

3.4

TRIGONOMETRY 4
(Solution of triangles)

by

A.J.Hobson

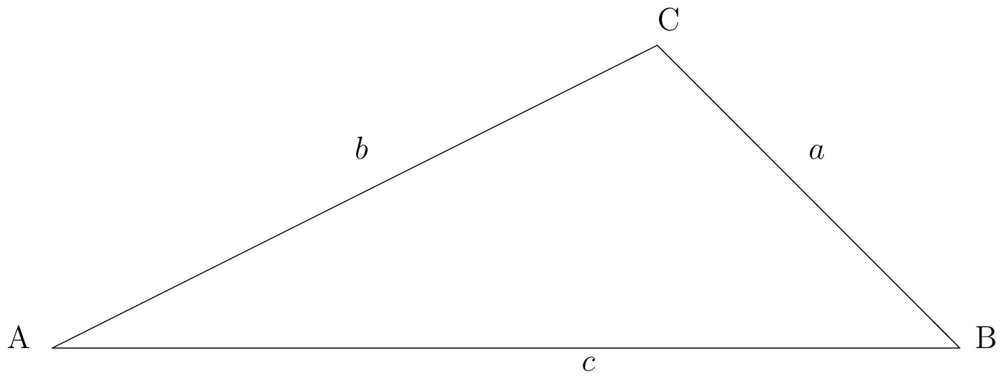
UNIT 3.4 - TRIGONOMETRY 4
SOLUTION OF TRIANGLES

3.4.1 INTRODUCTION

The **"solution of a triangle"** is defined to mean the complete set of data relating to the lengths of its three sides and the values of its three interior angles. It can be shown that these angles always add up to 180°.

If a sufficient amount of information is provided about **some** of this data, then it is usually possible to determine the remaining data.

We shall use a standardised type of diagram for an arbitrary triangle whose **"vertices"** (i.e. corners) are A,B and C and whose sides have lengths a, b and c. It is as follows:



The angles at A,B and C will be denoted by $\widehat{A}, \widehat{B}$ and $\widehat{C}$.

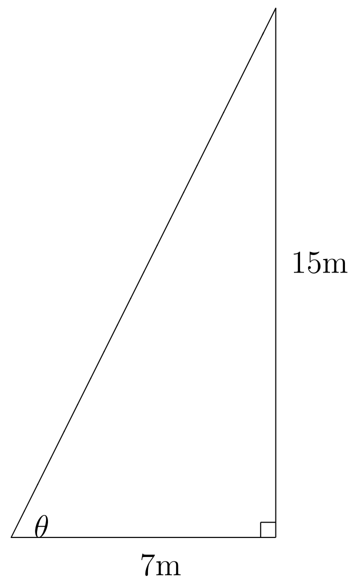
3.4.2 RIGHT-ANGLED TRIANGLES

Right-angled triangles are easier to solve than the more general kinds of triangle because all we need to use are the relationships between the lengths of the sides and the trigonometric ratios sine, cosine and tangent. An example will serve to illustrate the technique:

EXAMPLE

From the top of a vertical pylon, 15 meters high, a guide cable is to be secured into the (horizontal) ground at a distance of 7 meters from the base of the pylon.
What will be the length of the cable and what will be its inclination (in degrees) to the horizontal ?

Solution



From Pythagoras' Theorem, the length of the cable will be

$$\sqrt{7^2 + 15^2} \simeq 16.55\text{m}.$$

The angle of inclination to the horizontal will be θ, where

$$\tan\theta = \frac{15}{7}.$$

$$\text{Hence,} \quad \theta \simeq 65^\circ.$$

3.4.3 THE SINE AND COSINE RULES

Two powerful tools for the solution of triangles in general may be stated in relation to the earlier diagram as follows:

(a) The Sine Rule

$$\frac{a}{\sin \widehat{A}} = \frac{b}{\sin \widehat{B}} = \frac{c}{\sin \widehat{C}}.$$

(b) The Cosine Rule

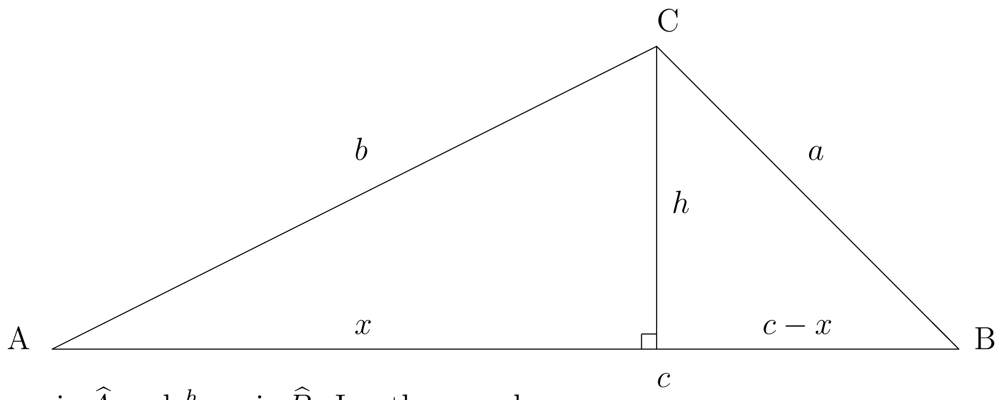
$$a^2 = b^2 + c^2 - 2bc \cos \widehat{A};$$
$$b^2 = c^2 + a^2 - 2ca \cos \widehat{B};$$
$$c^2 = a^2 + b^2 - 2ab \cos \widehat{C}.$$

Clearly, the last two of these are variations of the first.

We also observe that, whenever the angle on the right-hand-side is a right-angle, the Cosine Rule reduces to Pythagoras' Theorem.

The Proof of the Sine Rule

In the diagram encountered earlier, suppose we draw the perpendicular (of length h) from the vertex C onto the side AB.



Then $\frac{h}{b} = \sin \widehat{A}$ and $\frac{h}{a} = \sin \widehat{B}$. In other words,

$$b \sin \widehat{A} = a \sin \widehat{B}$$

or

$$\frac{b}{\sin \widehat{B}} = \frac{a}{\sin \widehat{A}}.$$

Clearly, the remainder of the Sine Rule can be obtained by considering the perpendicular drawn from a different vertex.

The Proof of the Cosine Rule

Using the same diagram as for the Sine Rule, we can assume that the side AB has lengths x and $c - x$ either side of the foot of the perpendicular drawn from C. Hence

$$h^2 = b^2 - x^2$$

and, at the same time,

$$h^2 = a^2 - (c - x)^2.$$

Expanding and equating the two expressions for h^2, we obtain

$$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$$

that is

$$a^2 = b^2 + c^2 - 2xc.$$

But $x = b \cos \widehat{A}$, and so

$$a^2 = b^2 + c^2 - 2bc \cos \widehat{A}.$$

EXAMPLES

1. Solve the triangle ABC in the case when $\widehat{A} = 20°, \widehat{B} = 30°$ and $c = 10$cm.

   Solution

   Firstly, the angle $\widehat{C} = 130°$ since the interior angles must add up to 180°.

   Thus, by the Sine Rule, we have

   $$\frac{a}{\sin 20°} = \frac{b}{\sin 30°} = \frac{10}{\sin 130°}.$$

   That is,

   $$\frac{a}{0.342} = \frac{b}{0.5} = \frac{10}{0.766}$$

   These give the results

   $$a = \frac{10 \times 0.342}{0.766} \cong 4.47\text{cm}$$
   $$b = \frac{10 \times 0.5}{0.766} \cong 6.53\text{cm}$$

2. Solve the triangle ABC in the case when $b = 9\text{cm}, c = 5\text{cm}$ and $\widehat{A} = 70°$.

   Solution

   In this case, the information prevents us from using the Sine Rule immediately, but the Cosine Rule **can** be applied as follows:

   $$a^2 = 25 + 81 - 90\cos 70°$$

   giving

   $$a^2 = 106 - 30.782 = 75.218$$

   Hence

   $$a \simeq 8.673\text{cm} \simeq 8.67\text{cm}$$

   Now we can use the Sine Rule to complete the solution

   $$\frac{8.673}{\sin 70°} = \frac{9}{\sin \widehat{B}} = \frac{5}{\sin \widehat{C}}.$$

   Thus,

   $$\sin \widehat{B} = \frac{9 \times \sin 70°}{8.673} = \frac{9 \times 0.940}{8.673} \simeq 0.975$$

   This suggests that $\widehat{B} \simeq 77.19°$ in which case $\widehat{C} \simeq 180° - 70° - 77.19° \simeq 32.81°$ but, for the moment, we must also allow the possibility that $\widehat{B} \simeq 102.81°$ which would give $\widehat{C} \simeq 7.19°$

   However, we can show that the alternative solution is unacceptable because it is not consistent with the whole of the Sine Rule statement for this example. Thus the only solution is the one for which

   $$a \simeq 8.67\text{cm}, \quad \widehat{B} \simeq 77.19°, \quad \widehat{C} \simeq 32.81°$$

Note: It is possible to encounter examples for which more than one solution **does** exist.

3.4.4 EXERCISES

Solve the triangle ABC in the following cases:

1. $c = 25\text{cm}$, $\widehat{A} = 35°$, $\widehat{B} = 68°$.
2. $c = 23\text{cm}$, $a = 30\text{cm}$, $\widehat{C} = 40°$.
3. $b = 4\text{cm}$, $c = 5\text{cm}$, $\widehat{A} = 60°$.
4. $a = 21\text{cm}$, $b = 23\text{cm}$, $c = 16\text{cm}$.

3.4.5 ANSWERS TO EXERCISES

1. $a \simeq 14.72\text{cm}$, $b \simeq 23.79\text{cm}$, $\widehat{C} \simeq 77°$.
2. $\widehat{A} \simeq 56.97°$, $\widehat{B} \simeq 83.03°$, $b = 35.52\text{cm}$;
   OR
   $\widehat{A} \simeq 123.03°$, $\widehat{B} \simeq 16.97°$, $b \simeq 10.44\text{cm}$.
3. $a \simeq 4.58\text{cm}$, $\widehat{B} \simeq 49.11°$, $\widehat{C} \simeq 70.89°$.
4. $\widehat{A} \simeq 62.13°$, $\widehat{B} \simeq 75.52°$, $\widehat{C} \simeq 42.35°$.

"JUST THE MATHS"

UNIT NUMBER

3.5

TRIGONOMETRY 5
(Trigonometric identities & wave-forms)

by

A.J.Hobson

UNIT 3.5 - TRIGONOMETRY 5
TRIGONOMETRIC IDENTITIES AND WAVE FORMS

3.5.1 TRIGONOMETRIC IDENTITIES

The standard trigonometric functions can be shown to satisfy a certain group of relationships for any value of the angle θ. They are called **"trigonometric identities"**.

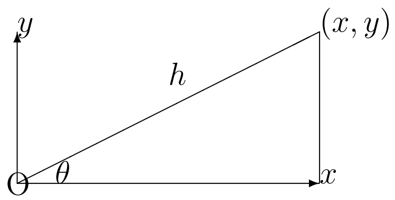
ILLUSTRATION

Prove that

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

Proof:

The following diagram was first encountered in Unit 3.1



From the diagram,

$$\cos\theta = \frac{x}{h} \quad \text{and} \quad \sin\theta = \frac{y}{h}.$$

But, by Pythagoras' Theorem,

$$x^2 + y^2 = h^2.$$

In other words,

$$\left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 = 1.$$

That is,

$$\cos^2\theta + \sin^2\theta \equiv 1.$$

It is also worth noting various consequences of this identity:

(a) $\cos^2\theta \equiv 1 - \sin^2\theta$; (rearrangement).

(b) $\sin^2\theta \equiv 1 - \cos^2\theta$; (rearrangement).

(c) $\sec^2\theta \equiv 1 + \tan^2\theta$; (divide by $\cos^2\theta$).

(d) $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$; (divide by $\sin^2\theta$).

Other Trigonometric Identities in common use will not be **proved** here, but they are listed for reference. However, a booklet of Mathematical Formulae should be obtained.

$$\sec\theta \equiv \frac{1}{\cos\theta} \quad \mathrm{cosec}\theta \equiv \frac{1}{\sin\theta} \quad \cot\theta \equiv \frac{1}{\tan\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \quad 1 + \tan^2\theta \equiv \sec^2\theta \quad 1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$$\sin A \equiv 2\sin\frac{1}{2}A\cos\frac{1}{2}A$$

$$\cos A \equiv \cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A \equiv 1 - 2\sin^2\frac{1}{2}A \equiv 2\cos^2\frac{1}{2}A - 1$$

$$\tan A \equiv \frac{2\tan\frac{1}{2}A}{1 - \tan^2\frac{1}{2}A}$$

$$\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A\cos B \equiv \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B \equiv \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A\cos B \equiv \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A\sin B \equiv \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

EXAMPLES

1. Show that
$$\sin^2 2x \equiv \frac{1}{2}(1 - \cos 4x).$$

   Solution
   From the standard trigonometric identities, we have
$$\cos 4x \equiv 1 - 2\sin^2 2x$$
   on replacing A by $2x$.
   Rearranging this new identity, gives the required result.

2. Show that
$$\sin\left(\theta + \frac{\pi}{2}\right) \equiv \cos\theta.$$

   Solution
   The left hand side can be expanded as
$$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2};$$
   and the result follows, because $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

3. Simplify the expression
$$\frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha}.$$

   Solution
   Using separate trigonometric identities in the numerator and denominator, the expression becomes
$$\frac{2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\cos\left(\frac{2\alpha-3\alpha}{2}\right)}{-2\sin\left(\frac{2\alpha+3\alpha}{2}\right).\sin\left(\frac{2\alpha-3\alpha}{2}\right)}$$
$$\equiv \frac{2\sin\left(\frac{5\alpha}{2}\right).\cos\left(\frac{-\alpha}{2}\right)}{-2\sin\left(\frac{5\alpha}{2}\right).\sin\left(\frac{-\alpha}{2}\right)}$$
$$\equiv \frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$
$$\equiv \cot\left(\frac{\alpha}{2}\right).$$

4. Express $2\sin 3x\cos 7x$ as the difference of two sines.
   Solution
$$2\sin 3x\cos 7x \equiv \sin(3x + 7x) + \sin(3x - 7x).$$
   Hence,
$$2\sin 3x\cos 7x \equiv \sin 10x - \sin 4x.$$

3.5.2 AMPLITUDE, WAVE-LENGTH, FREQUENCY AND PHASE ANGLE

In the scientific applications of Mathematics, importance is attached to trigonometric functions of the form

$$A\sin(\omega t+\alpha) \quad \text{and} \quad A\cos(\omega t+\alpha),$$

where A, ω and α are constants and t is usually a time variable.

It is useful to note, from trigonometric identities, that the expanded forms of the above two functions are given by

$$A\sin(\omega t+\alpha) \equiv A\sin\omega t\cos\alpha + A\cos\omega t\sin\alpha$$

and

$$A\cos(\omega t+\alpha) \equiv A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha.$$

(a) The Amplitude

In view of the fact that the sine and the cosine of any angle always lies within the closed interval from -1 to $+1$ inclusive, the constant, A, represents the maximum value (numerically) which can be attained by each of the above trigonometric functions.

A is called the **"amplitude"** of each of the functions.

(b) The Wave Length (Or Period)

If the value, t, increases or decreases by a whole multiple of $\frac{2\pi}{\omega}$, then the value, $(\omega t+\alpha)$, increases or decreases by a whole multiple of 2π; and, hence, the functions remain unchanged in value.

A graph, against t, of either $A\sin(\omega t+\alpha)$ or $A\cos(\omega t+\alpha)$ would be repeated in shape at regular intervals of length $\frac{2\pi}{\omega}$.

The repeated shape of the graph is called the **"wave profile"** and $\frac{2\pi}{\omega}$ is called the **"wave-length"**, or **"period"** of each of the functions.

(c) The Frequency

If t is indeed a time variable, then the wave length (or period) represents the time taken to complete a single wave-profile. Consequently, the number of wave-profiles completed in one unit of time is given by $\frac{\omega}{2\pi}$.

$\frac{\omega}{2\pi}$ is called the **"frequency"** of each of the functions.

Note:
The constant ω itself is called the **"angular frequency"**; it represents the change in the quantity $(\omega t+\alpha)$ for every unit of change in the value of t.

(d) The Phase Angle

The constant, α, affects the starting value, at $t=0$, of the trigonometric functions $A\sin(\omega t+\alpha)$ and $A\cos(\omega t+\alpha)$. Each of these is said to be **"out of phase"**, by an amount, α, with the trigonometric functions $A\sin\omega t$ and $A\cos\omega t$ respectively.

α is called the **"phase angle"** of each of the two original trigonometric functions; but it can take infinitely many values differing only by a whole multiple of 360° (if working in degrees) or 2π (if working in radians).

EXAMPLES

1. Express $\sin t + \sqrt{3}\cos t$ in the form $A\sin(t+\alpha)$, with α in degrees, and hence solve the equation,

$$\sin t + \sqrt{3}\cos t = 1$$

for t in the range $0° \leq t \leq 360°$.

Solution

We require that

$$\sin t + \sqrt{3}\cos t \equiv A\sin t\cos\alpha + A\cos t\sin\alpha$$

Hence,

$$A\cos\alpha = 1 \quad \text{and} \quad A\sin\alpha = \sqrt{3},$$

which gives $A^2 = 4$ (using $\cos^2\alpha + \sin^2\alpha \equiv 1$) and also $\tan\alpha = \sqrt{3}$.

Thus,

$$A = 2 \quad \text{and} \quad \alpha = 60° \text{ (principal value)}.$$

To solve the given equation, we may now use

$$2\sin(t + 60°) = 1,$$

so that

$$t + 60° = \text{Sin}^{-1}\frac{1}{2} = 30° + k360° \quad \text{or} \quad 150° + k360°,$$

where k may be any integer.

For the range $0° \leq t \leq 360°$, we conclude that

$$t = 330° \quad \text{or} \quad 90°.$$

2. Determine the amplitude and phase-angle which will express the trigonometric function $a\sin\omega t + b\cos\omega t$ in the form $A\sin(\omega t + \alpha)$.

Apply the result to the expression $3\sin 5t - 4\cos 5t$ stating α in degrees, correct to one decimal place, and lying in the interval from $-180°$ to $180°$.

Solution

We require that

$$A\sin(\omega t + \alpha) \equiv a\sin\omega t + b\cos\omega t;$$

and, hence, from trigonometric identities,

$$A \sin \alpha = b \quad \text{and} \quad A \cos \alpha = a.$$

Squaring each of these and adding the results together gives

$$A^2 = a^2 + b^2 \quad \text{that is} \quad A = \sqrt{a^2 + b^2}.$$

Also,

$$\frac{A \sin \alpha}{A \cos \alpha} = \frac{b}{a},$$

which gives

$$\alpha = \tan^{-1} \frac{b}{a};$$

but the particular angle chosen must ensure that $\sin \alpha = \frac{b}{A}$ and $\cos \alpha = \frac{a}{A}$ have the correct sign.

Applying the results to the expression $3 \sin 5t - 4 \cos 5t$, we have

$$A = \sqrt{3^2 + 4^2}$$

and

$$\alpha = \tan^{-1} \left(-\frac{4}{3} \right).$$

But $\sin \alpha \left(= -\frac{4}{5}\right)$ is negative and $\cos \alpha \left(= \frac{3}{5}\right)$ is positive so that α may be taken as an angle between zero and $-90°$; that is $\alpha = -53.1°$.

We conclude that

$$3 \sin 5t - 4 \cos 5t \equiv 5 \sin(5t - 53.1°).$$

3. Solve the equation

$$4 \sin 2t + 3 \cos 2t = 1$$

for t in the interval from $-180°$ to $180°$.

Solution

Expressing the left hand side of the equation in the form $A \sin(2t + \alpha)$, we require

$$A = \sqrt{4^2 + 3^2} = 5 \quad \text{and} \quad \alpha = \tan^{-1} \frac{3}{4}.$$

Also $\sin \alpha \left(= \frac{3}{5}\right)$ is positive and $\cos \alpha \left(= \frac{4}{5}\right)$ is positive so that α may be taken as an angle in the interval from zero to $90°$.

Hence, $\alpha = 36.87°$ and the equation to be solved becomes

$$5 \sin(2t + 36.87°) = 1.$$

Its solutions are obtained by making t the **"subject"** of the equation to give

$$t = \frac{1}{2} \left[\text{Sin}^{-1} \frac{1}{5} - 36.87° \right].$$

The possible values of $\text{Sin}^{-1} \frac{1}{5}$ are $11.53° + k360°$ and $168.46° + k360°$, where k may be any integer. But, to give values of t which are numerically less than $180°$, we may use only $k = 0$ and $k = 1$ in the first of these and $k = 0$ and $k = -1$ in the second.

The results obtained are

$$t = -12.67°, \ 65.80°, \ 167.33° \quad \text{and} \quad -114.21°$$

3.5.3 EXERCISES

1. Simplify the following expressions:

   (a)
$$(1 + \cos x)(1 - \cos x);$$

   (b)
$$(1 + \sin x)^2 - 2 \sin x(1 + \sin x).$$

2. Show that
$$\cos\left(\theta - \frac{\pi}{2}\right) \equiv \sin\theta$$

3. Express $2 \sin 4x \sin 5x$ as the difference of two cosines.

4. Use the table of trigonometric identities to show that

   (a)
$$\frac{\sin 5x + \sin x}{\cos 5x + \cos x} \equiv \tan 3x;$$

   (b)
$$\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x;$$

   (c)
$$\tan x.\tan 2x + 2 \equiv \tan 2x.\cot x;$$

   (d)
$$\cot(x + y) \equiv \frac{\cot x.\cot y - 1}{\cot y + \cot x}.$$

5. Solve the following equations by writing the trigonometric expression on the left-hand-side in the form suggested, being careful to see whether the phase angle is required in degrees or radians and ensuring that your final answers are in the range given:

   (a) $\cos t + 7 \sin t = 5, \quad 0° \leq t \leq 360°$, (transposed to the form $A\cos(t - \alpha)$, with α in degrees.
   (b) $\sqrt{2} \cos t - \sin t = 1, \quad 0° \leq t \leq 360°$, (transposed to the form $A\cos(t + \alpha)$, with α in degrees.
   (c) $2 \sin t - \cos t = 1, \quad 0 \leq t \leq 2\pi$, (transposed to the form $A \sin(t - \alpha)$, with α in radians.
   (d) $3 \sin t - 4 \cos t = 0.6, \quad 0° \leq t \leq 360°$, (transposed to the form $A\sin(t - \alpha)$, with α in degrees.

6. Determine the amplitude and phase-angle which will express the trigonometric function $a \cos \omega t + b \sin \omega t$ in the form $A\cos(\omega t + \alpha)$.

   Apply the result to the expression $4 \cos 5t - 4\sqrt{3} \sin 5t$ stating α in degrees and lying in the interval from $-180°$ to $180°$.

7. Solve the equation

$$2\cos 3t + 5\sin 3t = 4$$

for t in the interval from zero to $360°$, expressing t in decimals correct to two decimal places.

3.5.4 ANSWERS TO EXERCISES

1. (a) $\sin^2 x$; (b) $\cos^2 x$.
2. Use the cos(A - B) formula to expand left hand side.
3. $\cos x - \cos 9x$.
4. (a) Use the formulae for sinA + sinB and cosA + cosB;
   (b) Use the formulae for $\cos 2x$ to make the 1's cancel;
   (c) Both sides are identically equal to $\frac{2}{1-\tan^2 x}$;
   (d) Invert the formula for $\tan(x+y)$.
5. (a) $36.9°$, $126.9°$;
   (b) $19.5°$, $270°$;
   (c) 0, 3.14;
   (d) $226.24°$
6. $$A = \sqrt{a^2 + b^2}, \quad \text{and} \quad \alpha = \tan^{-1}\left(-\frac{b}{a}\right);$$

   $$4\cos 5t - 4\sqrt{3}\sin 5t \equiv 8\cos(5t + 60°).$$
7. $$\sqrt{29}\cos(3t - 68.20°) = 4 \quad \text{or} \quad \sqrt{29}\sin(3t + 21.80°) = 4$$

   give

   $$t = 8.72°,\ 36.74°,\ 156.74° \quad \text{and} \quad 276.74°$$

“JUST THE MATHS”

UNIT NUMBER

4.1

HYPERBOLIC FUNCTIONS 1
(Definitions, graphs and identities)

by

A.J.Hobson

UNIT 4.1 - HYPERBOLIC FUNCTIONS DEFINITIONS, GRAPHS AND IDENTITIES

4.1.1 INTRODUCTION

In this section, we introduce a new group of mathematical functions, based on the functions

$$e^x \quad \text{and} \quad e^{-x}$$

whose properties resemble, very closely, those of the standard trigonometric functions. But, whereas trigonometric functions can be related to the geometry of a circle (and are sometimes called the **"circular functions"**), it can be shown that the new group of functions are related to the geometry of a hyperbola (see unit 5.7). Because of this, they are called **"hyperbolic functions"**.

4.1.2 DEFINITIONS

(a) Hyperbolic Cosine

The hyperbolic cosine of a number, x, is denoted by $\cosh x$ and is defined by

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

Note:
The name of the function is pronounced **"cosh"**.

(b) Hyperbolic Sine

The hyperbolic sine of a number, x, is denoted by $\sinh x$ and is defined by

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}.$$

Note:
The name of the function is pronounced **"shine"**

(c) Hyperbolic Tangent

The hyperbolic tangent of a number, x, is denoted by $\tanh x$ and is defined by

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

Notes:
(i) The name of the function is pronounced **"than"**.

(ii) In terms of exponentials, it is easily shown that

$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}.$$

(d) Other Hyperbolic Functions

Other, less commonly used, hyperbolic functions are defined as follows:

(i) Hyperbolic secant, pronounced **"shek"**, is defined by

$$\mathrm{sech} x \equiv \frac{1}{\cosh x}.$$

(ii) Hyperbolic cosecant, pronounced **"coshek"** is defined by

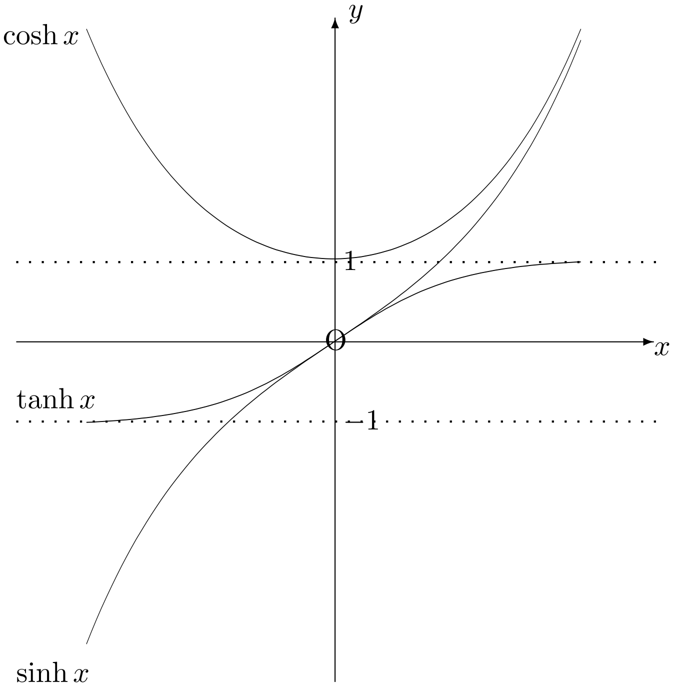
$$\mathrm{cosech} x \equiv \frac{1}{\sinh x}.$$

(iii) Hyperbolic cotangent, pronounced **"coth"** is defined by

$$\mathrm{coth} x \equiv \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x}.$$

4.1.3 GRAPHS OF HYPERBOLIC FUNCTIONS

It is useful to see the graphs of the functions $\cosh x$, $\sinh x$ and $\tanh x$ drawn with reference to the same set of axes. It can be shown that they are as follows:



Note:
We observe that the graph of $\cosh x$ exists only for y greater than or equal to 1; and that graph of $\tanh x$ exists only for y lying between -1 and $+1$. The graph of $\sinh x$, however, covers the whole range of x and y values from $-\infty$ to $+\infty$.

4.1.4 HYPERBOLIC IDENTITIES

It is possible to show that, to every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions though, in some cases, the comparison is more direct than in other cases.

ILLUSTRATIONS

1.

$$e^x \equiv \cosh x + \sinh x.$$

Proof

This follows directly from the definitions of $\cosh x$ and $\sinh x$.

2.

$$e^{-x} \equiv \cosh x - \sinh x.$$

Proof

Again, this follows from the definitions of $\cosh x$ and $\sinh x$.

3.

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

Proof

This follows if we multiply together the results of the previous two illustrations since $e^x.e^{-x} = 1$ and $(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \cosh^2 x - \sinh^2 x$.

Notes:

(i) Dividing throughout by $\cosh^2 x$ gives the identity

$$1 - \tanh^2 x \equiv \text{sech}^2 x.$$

(ii) Dividing throughout by $\sinh^2 x$ gives the identity

$$\coth^2 x - 1 \equiv \text{cosech}^2 x.$$

4.

$$\sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y.$$

Proof:

The right hand side may be expressed in the form

$$\frac{e^x - e^{-x}}{2}.\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}.\frac{e^y - e^{-y}}{2},$$

which expands out to

$$\frac{e^{(x+y)} + e^{(x-y)} - e^{(-x+y)} - e^{(-x-y)}}{4} + \frac{e^{(x+y)} - e^{(x-y)} + e^{(-x+y)} - e^{(-x-y)}}{4};$$

and this simplifies to

$$\frac{2e^{(x+y)} - 2e^{(-x-y)}}{4} \equiv \frac{e^{(x+y)} - e^{-(x+y)}}{2} \equiv \sinh(x + y).$$

5.

$$\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y.$$

Proof

The proof is similar to the previous illustration.

6.

$$\tanh(x+y) \equiv \frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}.$$

Proof

The proof again is similar to that in Illustration No. 4.

4.1.5 OSBORN'S RULE

Many other results, similar to those previously encountered in the standard list of trigonometric identities can be proved in the same way as for Illustration No. 4 above; that is, we substitute the definitions of the appropriate hyperbolic functions.

However, if we merely wish to **write down** a hyperbolic identity without proving it, we may use the following observation due to Osborn:

Starting with any trigonometric identity, change cos to cosh and sin to sinh. Then, if the trigonometric identity contains (or implies) two sine functions multiplied together, change the sign in front of the relevant term from + to − or vice versa.

ILLUSTRATIONS

1.

$$\cos^2 x + \sin^2 x \equiv 1,$$

which leads to the hyperbolic identity

$$\cosh^2 x - \sinh^2 x \equiv 1$$

since the trigonometric identity contains two sine functions multiplied together.

2.

$$\sin(x-y) \equiv \sin x \cos y - \cos x \sin y,$$

which leads to the hyperbolic identity

$$\sinh(x-y) \equiv \sinh x \cosh y - \cosh x \sinh y$$

in which no changes of sign are required.

3.

$$\sec^2 x \equiv 1 + \tan^2 x,$$

which leads to the hyperbolic identity

$$\text{sech}^2 x \equiv 1 - \tanh^2 x$$

since $\tan^2 x$ in the trigonometric identity implies that two sine functions are multiplied together; that is,

$$\tan^2 x \equiv \frac{\sin^2 x}{\cos^2 x}.$$

4.1.6 EXERCISES

1. If

$$L = 2C \sinh \frac{H}{2C},$$

determine the value of L when $H = 63$ and $C = 50$

2. If

$$v^2 = 1.8L \tanh \frac{6.3d}{L},$$

determine the value of v when $d = 40$ and $L = 315$.

3. Use Osborn's Rule to write down hyperbolic identities for

(a)

$$\sinh 2A;$$

(b)

$$\cosh 2A.$$

4. Use the results of the previous question to simplify the expression

$$\frac{1 + \sinh 2A + \cosh 2A}{1 - \sinh 2A - \cosh 2A}.$$

5. Use Osborn's rule to write down the hyperbolic identity which corresponds to the trigonometric identity

$$2 \sin x \sin y \equiv \cos(x - y) - \cos(x + y)$$

and prove your result.

6. If

$$a = c \cosh x \quad \text{and} \quad b = c \sinh x,$$

show that

$$(a + b)^2 e^{-2x} \equiv a^2 - b^2 \equiv c^2.$$

4.1.7 ANSWERS TO EXERCISES

1. 67.25
2. 19.40
3. (a)

$$\sinh 2A \equiv 2 \sinh A \cosh A;$$

(b)

$$\cosh 2A \equiv \cosh^2 A + \sinh^2 A \equiv 2\cosh^2 A - 1 \equiv 1 + 2\sinh^2 A.$$

4.

$$-\coth A.$$

5.

$$-2 \sinh x \sinh y \equiv \cosh(x - y) - \cosh(x + y).$$

"JUST THE MATHS"

UNIT NUMBER

4.2

HYPERBOLIC FUNCTIONS 2
(Inverse hyperbolic functions)

by

A.J.Hobson

UNIT 4.2 - HYPERBOLIC FUNCTIONS 2
INVERSE HYPERBOLIC FUNCTIONS

4.2.1 - INTRODUCTION

The three basic inverse hyperbolic functions are $\text{Cosh}^{-1}x$, $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$.

It may be shown that they are given by the following formulae:

(a)

$$\text{Cosh}^{-1}x = \pm\ln(x + \sqrt{x^2 - 1});$$

(b)

$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1});$$

(c)

$$\text{Tanh}^{-1}x = \frac{1}{2}\ln\frac{1 + x}{1 - x}.$$

Notes:

(i) The positive value of $\text{Cosh}^{-1}x$ is called the **'principal value"** and is denoted by $\cosh^{-1}x$ (using a lower-case c).

(ii) $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$ have only **one** value but, for uniformity, we customarily denote them by $\sinh^{-1}x$ and $\tanh^{-1}x$.

4.2.2 THE PROOFS OF THE STANDARD FORMULAE

(a) Inverse Hyperbolic Cosine

If we let $y = \text{Cosh}^{-1}x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0,$$

which is a quadratic equation in e^y having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Taking natural logarithms of both sides gives

$$y = \ln(x \pm \sqrt{x^2 - 1}).$$

However, the two values $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ are reciprocals of each other, since their product is the value, 1; and so

$$y = \pm \ln(x + \sqrt{x^2 - 1}).$$

(b) Inverse Hyperbolic Sine

If we let $y = \text{Sinh}^{-1}x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Hence,

$$2x = e^y - e^{-y},$$

or

$$(e^y)^2 - 2xe^y - 1 = 0,$$

which is a quadratic equation in e^y having solutions, from the quadratic formula, given by

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

However, $x - \sqrt{x^2+1}$ has a negative value and cannot, therefore, be equated to a power of e, which is positive. Hence, this part of the expression for e^y must be ignored.

Taking natural logarithms of both sides gives

$$y = \ln(x + \sqrt{x^2+1}).$$

(c) Inverse Hyperbolic Tangent

If we let $y = \mathrm{Tanh}^{-1}x$, then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

Hence,

$$x\left(e^{2y}+1\right) = e^{2y} - 1,$$

giving

$$e^{2y} = \frac{1+x}{1-x}.$$

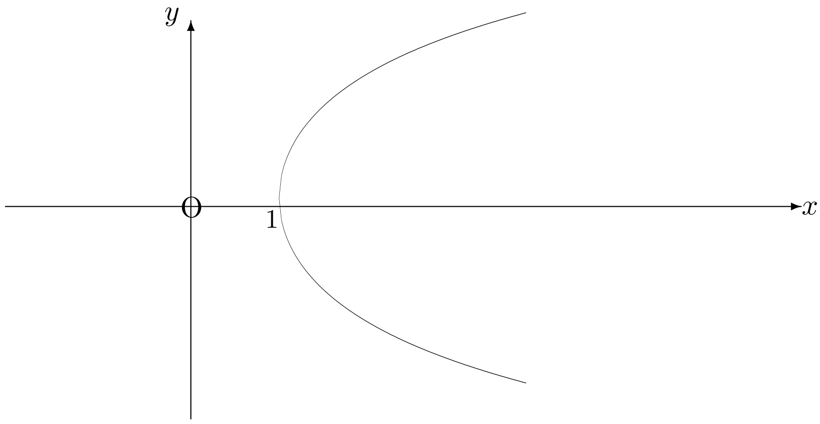
Taking natural logarithms of both sides,

$$y = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

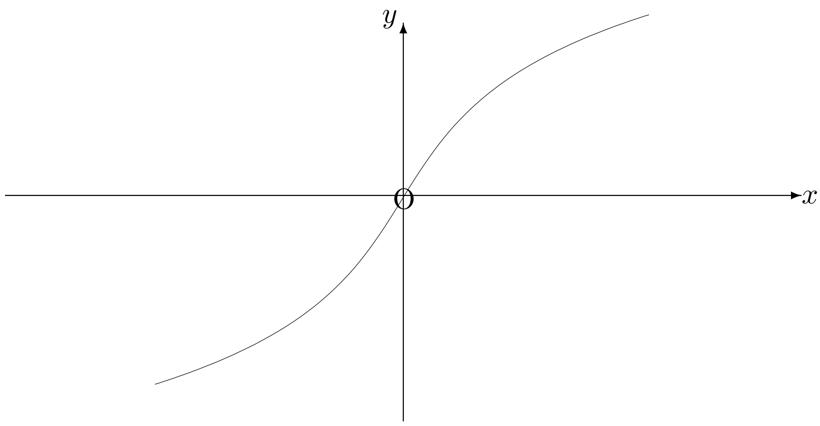
Note:
The graphs of inverse hyperbolic functions are discussed fully in Unit 10.7, but we include them here for the sake of completeness:

The graphs are as follows:

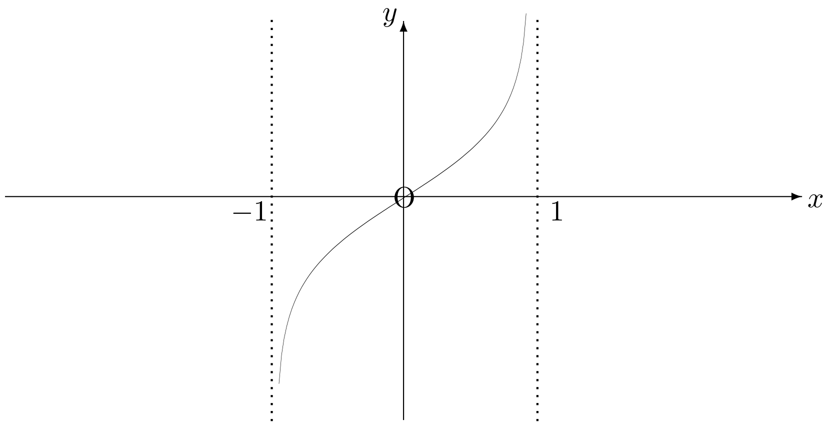
(a) $y = \mathrm{Cosh}^{-1}x$



(b) $y = \mathrm{Sinh}^{-1}x$



(c) $y = \text{Tanh}^{-1}x$



4.2.3 EXERCISES

1. Use the standard formulae to evaluate (a) $\sinh^{-1}7$ and (b) $\cosh^{-1}9$.

2. Express $\cosh 2x$ and $\sinh 2x$ in terms of exponentials and hence solve, for x, the equation

$$2\cosh 2x - \sinh 2x = 2.$$

3. Obtain a formula which equates $\text{cosech}^{-1}x$ to the natural logarithm of an expression in x, distinguishing between the two cases $x > 0$ and $x < 0$.

4. If $t = \tanh(x/2)$, prove that

   (a)

$$\sinh x = \frac{2t}{1-t^2}$$

   and

   (b)

$$\cosh x = \frac{1+t^2}{1-t^2}.$$

   Hence solve, for x, the equation

$$7\sinh x + 20\cosh x = 24.$$

4.2.4 ANSWERS TO EXERCISES

1. (a)

$$\ln(7 + \sqrt{49 + 1}) \simeq 2.644;$$

   (b)

$$\ln(9 + \sqrt{81 - 1}) \simeq 2.887$$

2.

$$\left(e^{2x}\right)^2 - 4e^{2x} + 3 = 0,$$

   which gives $e^{2x} = 1 \;\text{ or }\; 3$ and hence $x = 0 \;\text{ or }\; \frac{1}{2}\ln 3 \simeq 0.549$.

3. If $x > 0$, then

$$\text{cosech}^{-1}x = \ln\frac{1 + \sqrt{1 + x^2}}{x}.$$

   If $x < 0$, then

$$\text{cosech}^{-1}x = \ln\frac{1 - \sqrt{1 + x^2}}{x}.$$

4. Use

$$\sinh x \equiv \frac{2\tanh(x/2)}{\text{sech}^2(x/2)}$$

   and

$$\cosh x \equiv \frac{\left(1 + \tanh^2(x/2)\right)}{\text{sech}^2(x/2)}.$$

   This gives $t = -\frac{1}{2}$ or $t = \frac{2}{11}$ and hence $x \simeq -1.099 \;\text{ or }\; 0.368$ which agrees with the solution obtained using exponentials.

“JUST THE MATHS”

UNIT NUMBER

5.1

GEOMETRY 1
(Co-ordinates, distance & gradient)

by

A.J.Hobson

UNIT 5.1 - GEOMETRY 1

CO-ORDINATES, DISTANCE AND GRADIENT

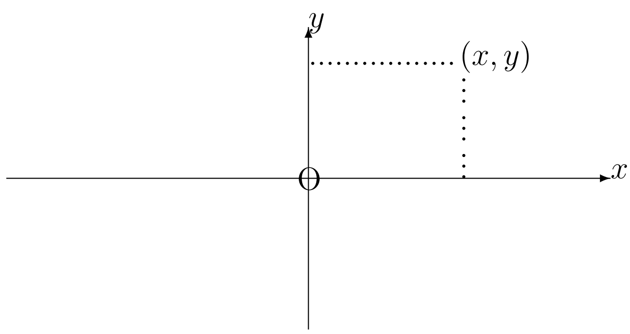
5.1.1 CO-ORDINATES

(a) Cartesian Co-ordinates

The position of a point, P, in a plane may be specified completely if we know its perpendicular distances from two chosen fixed straight lines, where we distinguish between positive distances on one side of each line and negative distances on the other side of each line.

It is not essential that the two chosen fixed lines should be at right-angles to each other, but we usually take them to be so for the sake of convenience.

Consider the following diagram:



The horizontal directed line, Ox, is called the **"x-axis"** and distances to the right of the origin (point O) are taken as positive.

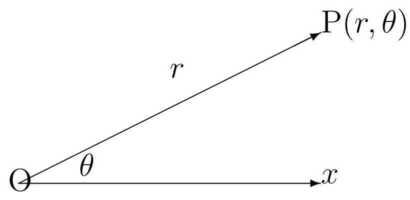
The vertical directed line, Oy, is called the **"y-axis"** and distances above the origin (point O) are taken as positive.

The notation (x, y) denotes a point whose perpendicular distances from Oy and Ox are x and y respectively, these being called the **"cartesian co-ordinates"** of the point.

(b) Polar Co-ordinates

An alternative method of fixing the position of a point P in a plane is to choose first a point, O, called the **"pole"** and directed line , Ox, emanating from the pole in one direction only and called the **"initial line"**.

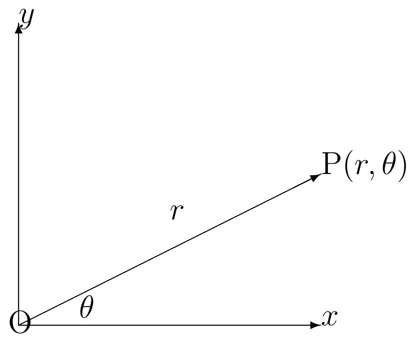
Consider the following diagram:



The position of P is determined by its distance r from the pole and the angle, θ which the line OP makes with the initial line, measuring this angle positively in a counter-clockwise sense or negatively in a clockwise sense from the initial line. The notation (r, θ) denotes the **"polar co-ordinates"** of the point.

5.1.2 THE RELATIONSHIP BETWEEN POLAR AND CARTESIAN CO-ORDINATES

It is convenient to superimpose the diagram for Polar Co-ordinates onto the diagram for Cartesian Co-ordinates as follows:



The trigonometry of the combined diagram shows that

(a) $x = r.\cos\theta$ and $y = r.\sin\theta$;

(b) $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

EXAMPLES

1. Express the equation
$$2x + 3y = 1$$
in polar co-ordinates.

   Solution

   Substituting for x and y separately, we obtain
$$2r\cos\theta + 3r\sin\theta = 1$$
   That is
$$r = \frac{1}{2\cos\theta + 3\sin\theta}$$

2. Express the equation

$$r = \sin\theta$$

in cartesian co-ordinates.

Solution

We could try substituting for r and θ separately, but it is easier, in this case, to rewrite the equation as

$$r^2 = r\sin\theta$$

which gives

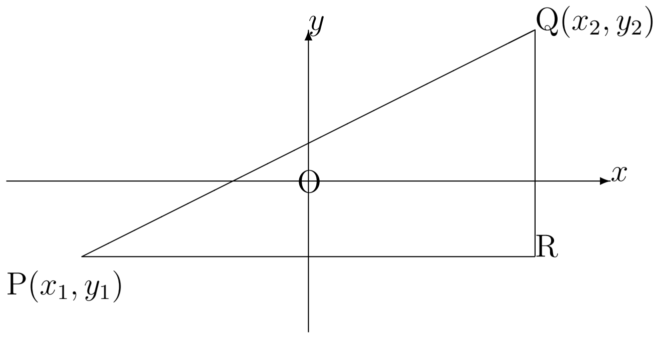
$$x^2 + y^2 = y$$

5.1.3 THE DISTANCE BETWEEN TWO POINTS

Given two points (x_1, y_1) and (x_2, y_2), the quantity $\mid x_2 - x_1 \mid$ is called the **"horizontal separation"** of the two points and the quantity $\mid y_2 - y_1 \mid$ is called the **"vertical separation"** of the two points, assuming, of course, that the x-axis is horizontal.

The expressions for the horizontal and vertical separations remain valid even when one or more of the co-ordinates is negative. For example, the horizontal separation of the points $(5, 7)$ and $(-3, 2)$ is given by $\mid -3 - 5 \mid= 8$ which agrees with the fact that the two points are on opposite sides of the y-axis.

The actual distance between (x_1, y_1) and (x_2, y_2) is easily calculated from Pythagoras' Theorem, using the horizontal and vertical separations of the points.



In the diagram,

$$\text{PQ}^2 = \text{PR}^2 + \text{RQ}^2.$$

That is,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

giving

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note:

We do not need to include the modulus signs of the horizontal and vertical separations because we are squaring them and therefore, any negative signs will disappear. For the same reason, it does not matter which way round the points are labelled.

EXAMPLE
Calculate the distance, d, between the two points $(5, -3)$ and $(-11, -7)$.

Solution

Using the formula, we obtain

$$d = \sqrt{(5+11)^2 + (-3+7)^2}.$$

That is,

$$d = \sqrt{256+16} = \sqrt{272} \cong 16.5$$

5.1.4 GRADIENT

The gradient of the straight-line segment, PQ, joining two points P and Q in a plane is defined to be the tangent of the angle which PQ makes with the positive x-direction.

In practice, when the co-ordinates of the two points are P(x_1, y_1) and Q(x_2, y_2), the gradient, m, is given by either

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

or

$$m = \frac{y_1 - y_2}{x_1 - x_2},$$

both giving the same result.

This is not quite the same as the ratio of the horizontal and vertical separations since we distinguish between positive gradient and negative gradient.

EXAMPLE
Determine the gradient of the straight-line segment joining the two points $(8, -13)$ and $(-2, 5)$ and hence calculate the angle which the segment makes with the positive x-direction.

Solution

$$m = \frac{5+13}{-2-8} = \frac{-13-5}{8+2} = -1.8$$

Hence, the angle, θ, which the segment makes with the positive x-direction is given by

$$\tan\theta = -1.8$$

Thus,

$$\theta = \tan^{-1}(-1.8) \simeq 119^\circ.$$

5.1.5 EXERCISES

1. A square, side d, has vertices O,A,B,C (labelled counter-clockwise) where O is the pole of a system of polar co-ordinates. Determine the polar co-ordinates of A,B and C when

(a) OA is the initial line;

(b) OB is the initial line.

2. Express the following cartesian equations in polar co-ordinates:

(a)

$$x^2 + y^2 - 2y = 0;$$

(b)

$$y^2 = 4a(a - x).$$

3. Express the following polar equations in cartesian co-ordinates:

(a)

$$r^2 \sin 2\theta = 3;$$

(b)

$$r = 1 + \cos\theta.$$

4. Determine the length of the line segment joining the following pairs of points given in cartesian co-ordinates:

(a) $(0, 0)$ and $(3, 4)$;

(b) $(-2, -3)$ and $(1, 1)$;

(c) $(-4, -6)$ and $(-1, -2)$;

(d) $(2, 4)$ and $(-3, 16)$;

(e) $(-1, 3)$ and $(11, -2)$.

5. Determine the gradient of the straight-line segment joining the two points $(-5, -0.5)$ and $(4.5, -1)$.

5.1.6 ANSWERS TO EXERCISES

1. (a) A$(d, 0)$, B$\left(d\sqrt{2}, \frac{\pi}{4}\right)$, C$\left(d, \frac{\pi}{2}\right)$;

   (b) A$\left(d, -\frac{\pi}{4}\right)$, B$(d\sqrt{2}, 0)$, C$\left(d, \frac{\pi}{4}\right)$.

2. (a) $r = 2\sin\theta$;

   (b) $r^2\sin^2\theta = 4a(a - r\cos\theta)$.

3. (a) $xy = \frac{3}{2}$;

   (b) $x^4 + y^4 + 2x^2y^2 - 2x^3 - 2xy^2 - y^2 = 0$.

4. (a) 5; (b) 5; (c) 5; (d) 13; (e) 13.

5. $m = -\frac{1}{19}$.

“JUST THE MATHS”

UNIT NUMBER

5.2

GEOMETRY 2
(The straight line)

by

A.J.Hobson

UNIT 5.2 - GEOMETRY 2

THE STRAIGHT LINE

5.2.1 PREAMBLE

It is not possible to give a satisfactory diagramatic definition of a straight line since the attempt is likely to assume a knowledge of linear measurement which, itself, depends on the concept of a straight line. For example, it is no use defining a straight line as "the shortest path between two points" since the word "shortest" assumes a knowledge of linear measurement.

In fact, the straight line is defined algebraically as follows:

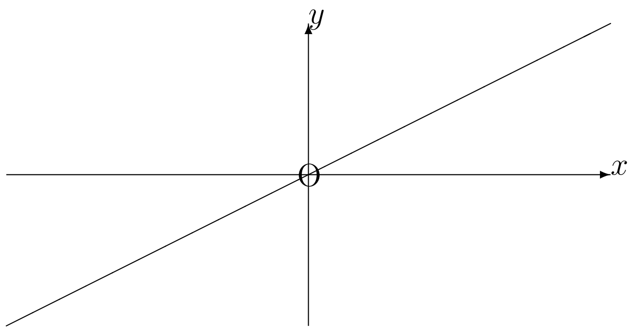
DEFINITION
A straight line is a set of points, (x, y), satisfying an equation of the form

$$ax + by + c = 0$$

where a, b and c are constants. This equation is called a **"linear equation"** and the symbol (x, y) itself, rather than a dot on the page, represents an arbitrary point of the line.

5.2.2 STANDARD EQUATIONS OF A STRAIGHT LINE

(a) Having a given gradient and passing through the origin



Let the gradient be m; then, from the diagram, all points (x, y) on the straight line **(but no others)** satisfy the relationship,

$$\frac{y}{x} = m.$$

That is,

$$\boxed{y = mx}$$

which is the equation of this straight line.

EXAMPLE
Determine, in degrees, the angle, θ, which the straight line,

$$\sqrt{3}y = x,$$

makes with the positive x-direction.

Solution

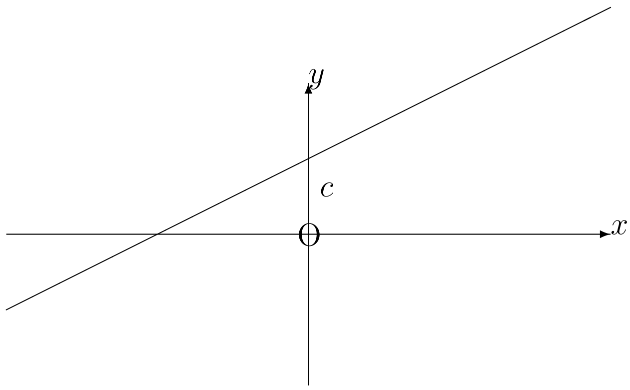
The gradient of the straight line is given by

$$\tan\theta = \frac{1}{\sqrt{3}}.$$

Hence,

$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ.$$

(b) Having a given gradient, and a given intercept on the vertical axis



Let the gradient be m and let the intercept be c; then, in this case we can imagine that the relationship between x and y in the previous section is altered only by adding the number c to all of the y co-ordinates. Hence the equation of the straight line is

$$\boxed{y = mx + c.}$$

EXAMPLE
Determine the gradient, m, and intercept c on the y-axis of the straight line whose equation is

$$7x - 5y - 3 = 0.$$

Solution

On rearranging the equation, we have

$$y = \frac{7}{5}x - \frac{3}{5}.$$

Hence,

$$m = \frac{7}{5}$$

and

$$c = -\frac{3}{5}.$$

This straight line will intersect the y-axis **below** the origin because the intercept is negative.

(c) Having a given gradient and passing through a given point

Let the gradient be m and let the given point be (x_1, y_1). Then,

$$y = mx + c,$$

where

$$y_1 = mx_1 + c.$$

Hence, on subtracting the second of these from the first, we obtain

$$\boxed{y - y_1 = m(x - x_1).}$$

EXAMPLE
Determine the equation of the straight line having gradient $\frac{3}{8}$ and passing through the point $(-7, 2)$.

Solution

From the formula,

$$y - 2 = \frac{3}{8}(x + 7).$$

That is

$$8y - 16 = 3x + 21,$$

giving

$$8y = 3x + 37.$$

(d) Passing through two given points

Let the two given points be (x_1, y_1) and (x_2, y_2). Then, the gradient is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence, from the previous section, the equation of the straight line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1);$$

but this is more usually written

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Note:
The same result is obtained no matter which way round the given points are taken as (x_1, y_1) and (x_2, y_2).

EXAMPLE
Determine the equation of the straight line joining the two points $(-5, 3)$ and $(2, -7)$, stating the values of its gradient and its intercept on the y-axis.

Solution (Method 1).

$$\frac{y-3}{-7-3} = \frac{x+5}{2+5},$$

giving

$$7(y-3) = -10(x+5).$$

That is,

$$10x + 7y + 29 = 0.$$

Solution (Method 2).

$$\frac{y+7}{3+7} = \frac{x-2}{-5-2},$$

giving

$$-7(y+7) = 10(x-2).$$

That is,

$$10x + 7y + 29 = 0,$$

as before.

By rewriting the equation of the line as

$$y = -\frac{10}{7}x - \frac{29}{7}$$

we see that the gradient is $-\frac{10}{7}$ and the intercept on the y-axis is $-\frac{29}{7}$.

(e) The parametric equations of a straight line

In the previous section, the common value of the two fractions

$$\frac{y-y_1}{y_2-y_1} \quad \text{and} \quad \frac{x-x_1}{x_2-x_1}$$

is called the **"parameter"** of the point (x, y) and is usually denoted by t.

By equating each fraction separately to t, we obtain

$$x = x_1 + (x_2 - x_1)t \quad \text{and} \quad y = y_1 + (y_2 - y_1)t.$$

These are called the **"parametric equations"** of the straight line while (x_1, y_1) and (x_2, y_2) are known as the **"base points"** of the parametric representation of the line.

Notes:
(i) In the above parametric representation, (x_1, y_1) has parameter $t = 0$ and (x_2, y_2) has parameter $t = 1$.

(ii) Other parametric representations of the same line can be found by using the given base points in the opposite order, or by using a different pair of points on the line as base points.

EXAMPLES

1. Use parametric equations to find two other points on the line joining $(3,-6)$ and $(-1,4)$.

   Solution

   One possible parametric representation of the line is
   $$x = 3 - 4t \quad y = -6 + 10t.$$
   To find another two points, we simply substitute any two values of t other than 0 or 1. For example, with $t = 2$ and $t = 3$,
   $$x = -5, y = 14 \quad \text{and} \quad x = -9, y = 24.$$
   A pair of suitable points is therefore $(-5, 14)$ and $(-9, 24)$.

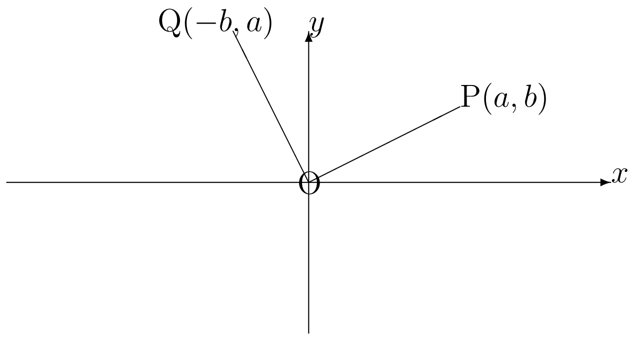
2. The co-ordinates, x and y, of a moving particle are given, at time t, by the equations
   $$x = 3 - 4t \quad \text{and} \quad y = 5 + 2t$$
   Determine the gradient of the straight line along which the particle moves.

   Solution

   Eliminating t, we have
   $$\frac{x-3}{-4} = \frac{y-5}{2}.$$
   That is,
   $$2(x-3) = -4(y-5),$$
   giving
   $$y = -\frac{2}{4}x + \frac{26}{4}.$$
   Hence, the gradient of the line is
   $$-\frac{2}{4} = -\frac{1}{2}.$$

5.2.3 PERPENDICULAR STRAIGHT LINES

The perpendicularity of two straight lines is not dependent on either their length or their precise position in the plane. Hence, without loss of generality, we may consider two straight line segments of equal length passing through the origin. The following diagram indicates appropriate co-ordinates and angles to demonstrate perpendicularity:



In the diagram, the gradient of OP $= \frac{b}{a}$ and the gradient of OQ $= \frac{a}{-b}$.

Hence the **product of the gradients is equal to** -1 or, in other words, **each gradient is minus the reciprocal of the other gradient**.

EXAMPLE
Determine the equation of the straight line which passes through the point $(-2, 6)$ and is perpendicular to the straight line,

$$3x + 5y + 11 = 0.$$

Solution

The gradient of the given line is $-\frac{3}{5}$ which implies that the gradient of a perpendicular line is $\frac{5}{3}$. Hence, the required line has equation

$$y - 6 = \frac{5}{3}(x + 2),$$

giving

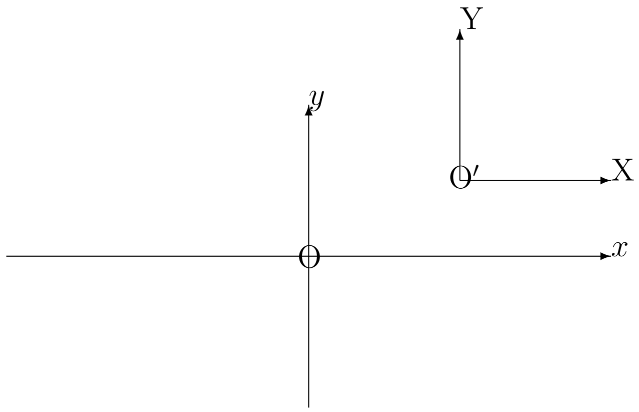
$$3y - 18 = 5x + 10.$$

That is,

$$3y = 5x + 28.$$

5.2.4 CHANGE OF ORIGIN

Given a cartesian system of reference with axes Ox and Oy, it may sometimes be convenient to consider a new set of axes O$'X$ parallel to Ox and O$'Y$ parallel to Oy with new origin at O$'$ whose co-ordinates are (h, k) referred to the original set of axes.



In the above diagram, everything is drawn in the first quadrant, but the relationships obtained between the old and new co-ordinates will apply in all situations. They are

$$X = x - h \quad \text{and} \quad Y = y - k$$

or

$$x = X + h \quad \text{and} \quad y = Y + k.$$

EXAMPLE
Given the straight line,

$$y = 3x + 11,$$

determine its equation referred to new axes with new origin at the point $(-2, 5)$.

Solution

Using

$$x = X - 2 \quad \text{and} \quad y = Y + 5,$$

we obtain

$$Y + 5 = 3(X - 2) + 11.$$

That is,

$$Y = 3X,$$

which is a straight line through the new origin with gradient 3.

Note:
If we had spotted that the point $(-2, 5)$ was **on** the original line, the new line would be bound to pass through the new origin; and its gradient would not alter in the change of origin.

5.2.5 EXERCISES

1. Determine the equations of the following straight lines:
   (a) having gradient 4 and intercept -7 on the y-axis;
   (b) having gradient $\frac{1}{3}$ and passing through the point $(-2, 5)$;
   (c) passing through the two points $(1, 6)$ and $(5, 9)$.
2. Determine the equation of the straight line passing through the point $(1, -5)$ which is perpendicular to the straight line whose cartesian equation is
$$x + 2y = 3.$$
3. Given the straight line
$$y = 4x + 2,$$
referred to axes Ox and Oy, determine its equation referred to new axes O$'X$ and O$'Y$ with new origin at the point where $x = 7$ and $y = -3$ (assuming that Ox is parallel to O$'X$ and Oy is parallel to O$'Y$).
4. Use the parametric equations of the straight line joining the two points $(-3, 4)$ and $(7, -1)$ in order to find its point of intersection with the straight line whose cartesian equation is
$$x - y + 4 = 0.$$

5.2.6 ANSWERS TO EXERCISES

1. (a)
$$y = 4x - 7;$$
   (b)
$$3y = x + 17;$$
   (c)
$$4y = 3x + 21.$$
2.
$$y = 2x - 7.$$
3.
$$Y = 4X + 33.$$
4.
$$x = -3 + 10t \quad y = 4 - 5t,$$
giving the point of intersection (at $t = \frac{1}{5}$) as $(-1, 3)$.

"JUST THE MATHS"

UNIT NUMBER

5.3

GEOMETRY 3
(Straight line laws)

by

A.J.Hobson

UNIT 5.3 - GEOMETRY 3

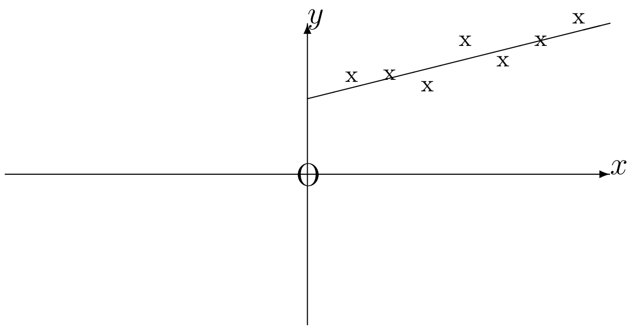
STRAIGHT LINE LAWS

5.3.1 INTRODUCTION

In practical work, the theory of an experiment may show that two variables, x and y, are connected by a straight line equation (or **"straight line law"**) of the form

$$y = mx + c.$$

In order to estimate the values of m and c, we could use the experimental data to plot a graph of y against x and obtain the **"best straight line"** passing through (or near) the plotted points to average out any experimental errors. Points which are obviously out of character with the rest are usually ignored.



It would seem logical, having obtained the best straight line, to measure the gradient, m, and the intercept, c, on the y-axis. However, this not always the wisest way of proceeding and should be avoided in general. The reasons for this are as follows:

(i) Economical use of graph paper may make it impossible to read the intercept, since this part of the graph may be "off the page".

(ii) The use of symbols other than x or y in scientific work may leave doubts as to which is the equivalent of the y-axis and which is the equivalent of the x-axis. Consequently, the gradient may be incorrectly calculated from the graph.

The safest way of finding m and c is to take two sets of readings, (x_1, y_1) and (x_2, y_2), from the best straight line drawn then solve the simultaneous linear equations

$$
\begin{aligned}
y_1 &= mx_1 + c, \\
y_2 &= mx_2 + c.
\end{aligned}
$$

It is a good idea if the two points chosen are as far apart as possible, since this will reduce errors in calculation due to the use of small quantities.

5.3.2 LAWS REDUCIBLE TO LINEAR FORM

Other experimental laws which are not linear can sometimes be reduced to linear form by using the experimental data to plot variables other than x or y, but related to them.

EXAMPLES

1. $y = ax^2 + b$.

   Method

   We let $X = x^2$, so that $y = aX + b$ and hence we may obtain a straight line by plotting y against X.

2. $y = ax^2 + bx$.

   Method

   Here, we need to consider the equation in the equivalent form $\frac{y}{x} = ax + b$ so that, by letting $Y = \frac{y}{x}$, giving $Y = ax + b$, a straight line will be obtained if we plot Y against x.

   Note:
   If one of the sets of readings taken in the experiment happens to be
   $(x, y) = (0, 0)$, we must ignore it in this example.

3. $xy = ax + b$.

   Method

   Two alternatives are available here as follows:
   (a) Letting $xy = Y$, giving $Y = ax + b$, we could plot a graph of Y against x.

   (b) Writing the equation as $y = a + \frac{b}{x}$, we could let $\frac{1}{x} = X$, giving $y = a + bX$, and plot a graph of y against X.

4. $y = ax^b$.

   Method

   This kind of law brings in the properties of logarithms since, if we take logarithms of both sides (base 10 will do here), we obtain

$$\log_{10} y = \log_{10} a + b \log_{10} x.$$

   Letting $\log_{10} y = Y$ and $\log_{10} x = X$, we have

$$Y = \log_{10} a + bX,$$

   so that a straight line will be obtained by plotting Y against X.

5. $y = ab^x$.

   Method

   Here again, logarithms may be used to give

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

   Letting $\log_{10} y = Y$, we have

$$Y = \log_{10} a + x \log_{10} b,$$

   which will give a straight line if we plot Y against x.

6. $y = ae^{bx}$.

   Method

   In this case, it makes sense to take **natural** logarithms of both sides to give

$$\log_e y = \log_e a + bx,$$

   which may also be written

$$\ln y = \ln a + bx$$

   Hence, letting $\ln y = Y$, we can obtain a straight line by plotting a graph of Y against x.

Note:

In all six of the above examples, it is even more important **not** to try to read off the gradient and the intercept from the graph drawn. As before, we should take two sets of readings for x (or X) and y (or Y), substitute them in the straight-line form of the equation and solve two simultaneous linear equations for the constants required.

5.3.3 THE USE OF LOGARITHMIC GRAPH PAPER

In Examples 4,5 and 6 in the previous section, it can be very tedious looking up on a calculator the logarithms of large sets of numbers. We may use, instead, a special kind of graph paper on which there is printed a logarithmic scale (see Unit 1.4) along one or both of the axis directions.

0.1 0.2 0.3 0.4 1 2 3 4 10

Effectively, the logarithmic scale has already looked up the logarithms of the numbers assigned to it provided these numbers are allocated to each **"cycle"** of the scale in successive powers of 10.

Data which includes numbers spread over several different successive powers of ten will need graph paper which has at least that number of cycles in the appropriate axis direction.

For example, the numbers 0.03, 0.09, 0.17, 0.33, 1.82, 4.65, 12, 16, 20, 50 will need **four** cycles on the logarithmic scale.

Accepting these restrictions, which make logarithmic graph paper less economical to use than ordinary graph paper, all we need to do is to plot the **actual** values of the variables whose logarithms we would otherwise have needed to look up. This will give the straight line graph from which we take the usual two sets of readings; these are then substituted into the form of the experimental equation which occurs immediately after taking logarithms of both sides.

It will not matter which base of logarithms is being used since logarithms to two different bases are proportional to each other anyway. The logarithmic graph paper does not, therefore, specify a base.

EXAMPLES

1. $y = ax^b$.

   Method

   (i) Taking logarithms (base 10), $\log_{10} y = \log_{10} a + b \log_{10} x$.

   (ii) Plot a graph of y against x, both on logarithmic scales.

   (iii) Estimate the position of the "best straight line".

   (iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned} \log_{10} y_1 &= \log_{10} a + b \log_{10} x_1, \\ \log_{10} y_2 &= \log_{10} a + b \log_{10} x_2. \end{aligned}$$

If it is possible to choose readings which are powers of 10, so much the better, but this is not essential.

2. $y = ab^x$.

Method

(i) Taking logarithms (base 10), $\log_{10} y = \log_{10} a + x \log_{10} b$.

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the position of the best straight line.

(iv) Read off from the graph two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned} \log_{10} y_1 &= \log_{10} a + x_1 \log_{10} b, \\ \log_{10} y_2 &= \log_{10} a + x_2 \log_{10} b. \end{aligned}$$

If it is possible to choose zero for the x_1 value, so much the better, but this is not essential.

3. $y = ae^{bx}$.

Method

(i) Taking natural logarithms, $\ln y = \ln a + bx$.

(ii) Plot a graph of y against x with y on a logarithmic scale and x on a linear scale.

(iii) Estimate the position of the best straight line.

(iv) Read off two sets of co-ordinates, (x_1, y_1) and (x_2, y_2), as far apart as possible.

(v) Solve for a and b the simultaneous equations

$$\begin{aligned} \ln y_1 &= \ln a + bx_1, \\ \ln y_2 &= \ln a + bx_2. \end{aligned}$$

If it possible to choose zero for the x_1 value, so much the better, but this is not essential.

5.3.5 EXERCISES

In these exercises, use logarithmic graph paper where possible.

1. The following values of x and y can be represented approximately by the law $y = a+bx^2$:

x	0	2	4	6	8	10
y	7.76	11.8	24.4	43.6	71.2	107.0

   Use a straight line graph to find approximately the values of a and b.

2. The following values of x and y are assumed to follow the law $y = ab^x$:

x	0.2	0.4	0.6	0.8	1.4	1.8
y	0.508	0.645	0.819	1.040	2.130	3.420

   Use a straight line graph to find approximately the values of a and b.

3. The following values of x and y are assumed to follow the law $y = ae^{kx}$:

x	0.2	0.5	0.7	1.1	1.3
y	1.223	1.430	1.571	1.921	2.127

   Use a straight line graph to find approximately the values of a and k.

4. The table below gives the pressure, P, and the volume, V, of a certain quantity of steam at maximum density:

P	12.27	17.62	24.92	34.77	47.87	65.06
V	3,390	2,406	1,732	1,264	934.6	699.0

   Assuming that $PV^n = C$, use a straight line graph to find approximately the values of n and C.

5. The coefficient of self induction, L, of a coil, and the number of turns, N, of wire are related by the formula $L = aN^b$, where a and b are constants.
   For the following pairs of observed values, use a straight line graph to calculate approximate values of a and b:

N	25	35	50	75	150	200	250
L	1.09	2.21	5.72	9.60	44.3	76.0	156.0

6. Measurements taken, when a certain gas undergoes compression, give the following values of pressure, p, and temperature, T:

p	10	15	20	25	35	50
T	270	289	303	315	333	353

   Assuming a law of the form $T = ap^n$, use a straight line graph to calculate approximately the values of a and n. Hence estimate the value of T when $p = 32$.

5.3.6 ANSWERS TO EXERCISES

The following answers are approximate; check only that the order of your results are correct. Any slight variations in the position of your straight line could affect the result considerably.

1. $a \simeq 8.0$, $b \simeq 0.99$
2. $a \simeq 0.4$, $b \simeq 3.3$
3. $a \simeq 1.1$, $k \simeq 0.5$
4. $n \simeq 1.06$, $C \simeq 65887$
5. $a \simeq 1.38 \times 10^{-3}$, $b \simeq 2.08$
6. $a \simeq 183.95$, $n \simeq 0.17$

"JUST THE MATHS"

UNIT NUMBER

5.4

GEOMETRY 4
(Elementary linear programming)

by

A.J.Hobson

UNIT 5.4 - GEOMETRY 4

ELEMENTARY LINEAR PROGRAMMING

5.4.1 FEASIBLE REGIONS

(i) The equation, $y = mx + c$, of a straight line is satisfied only by points which lie on the line. But it is useful to investigate the conditions under which a point with co-ordinates (x, y) may lie on one side of the line or the other.

(ii) For example, the inequality $y < mx + c$ is satisfied by points which lie **below** the line and the inequality $y > mx + c$ is satisfied by points which lie **above** the line.

(iii) Linear inequalities of the form $Ax+By+C < 0$ or $Ax+By+C > 0$ may be interpretted in the same way by converting, if necessary, to one of the forms in (ii).

(iv) Weak inequalities of the form $Ax + By + C \leq 0$ or $Ax + By + C \geq 0$ include the points which lie on the line itself as well as those lying on one side of it.

(v) Several simultaneous linear inequalties may be used to determine a region of the xy-plane throughout which all of the inequalities are satisfied. The region is called the **"feasible region"**.

EXAMPLES

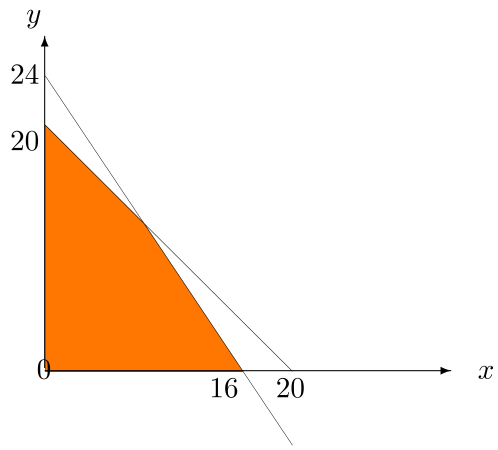
1. Determine the feasible region for the simultaneous inequalities

$$x \geq 0, \; y \geq 0, \; x + y \leq 20, \text{ and } 3x + 2y \leq 48$$

Solution

We require the points of the first quadrant which lie on or below the straight line $y = 20 - x$ and on or below the straight line $y = -\frac{3}{2}x + 16$.

The feasible region is shown as the shaded area in the following diagram:



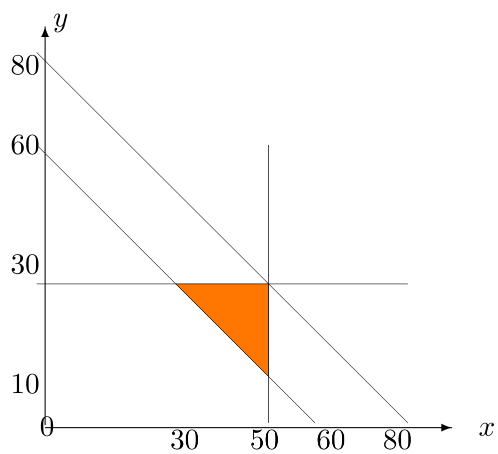
2. Determine the feasible region for the following simultaneous inequalities:

$$0 \leq x \leq 50,\ 0 \leq y \leq 30,\ x + y \leq 80,\ x + y \geq 60$$

Solution

We require the points which lie on or to the left of the straight line $x = 50$, on or below the straight line $y = 30$, on or below the straight line $y = 80 - x$ and on or above the straight line $y = 60 - x$.

The feasible region is shown as the shaded area in the following diagram:



5.4.2 OBJECTIVE FUNCTIONS

An important application of the feasible region discussed in the previous section is that of maximising (or minimising) a linear function of the form $px + qy$ subject to a set of simultaneous linear inequalities. Such a function is known as an **"objective function"**

Essentially, it is required that a straight line with gradient $-\frac{p}{q}$ is moved across the appropriate feasible region until it reaches the highest possible point of that region for a maximum value or the lowest possible point for a minimum value. This will imply that the straight line $px + qy = r$ is such that r is the optimum value required.

However, for convenience, it may be shown that the optimum value of the objective function always occurs at one of the corners of the feasible region so that we simply evaluate it at each corner and choose the maximum (or minimum) value.

EXAMPLES

1. A farmer wishes to buy a number of cows and sheep. Cows cost £18 each and sheep cost £12 each.

   The farmer has accommodation for not more than 20 animals, and cannot afford to pay more than £288.

   If he can reasonably expect to make a profit of £11 per cow and £9 per sheep, how many of each should he buy in order to make his total profit as large as possible ?

   Solution

   Suppose he needs to buy x cows and y sheep; then, his profit is the objective function $P \equiv 11x + 9y$.

   Also,

   $$x \geq 0,\ y \geq 0,\ x + y \leq 20,\ \text{and } 18x + 12y \leq 288 \text{ or } 3x + 2y \leq 48.$$

   Thus, we require to maximize $P \equiv 11x + 9y$ in the feasible region for the first example of the previous section.

   The corners of the region are the points $(0, 0)$, $(16, 0)$, $(0, 20)$ and $(8, 12)$, the last of these being the point of intersection of the two straight lines $x+y = 20$ and $3x+2y = 48$.

   The maximum value occurs at the point $(8, 12)$ and is equal to $88 + 108 = 196$. Hence, the farmer should buy 8 cows and 12 sheep.

2. A cement manufacturer has two depots, D_1 and D_2, which contain current stocks of 80 tons and 20 tons of cement respectively.

   Two customers C_1 and C_2 place orders for 50 and 30 tons respectively.

   The transport cost is £1 per ton, per mile and the distances, in miles, between D_1, D_2, C_1 and C_2 are given by the following table:

	C_1	C_2
D_1	40	30
D_2	10	20

   From which depots should the orders be dispatched in order to minimise the transport costs ?

   Solution

   Suppose that D_1 distributes x tons to C_1 and y tons to C_2; then D_2 must distribute $50 - x$ tons to C_1 and $30 - y$ tons to C_2.

   All quantities are positive and the following inequalties must be satisfied:

$$x \leq 50,\ \ y \leq 30,\ \ x + y \leq 80,\ \ 80 - (x + y) \leq 20 \text{ or } x + y \geq 60$$

   The total transport costs, T, are made up of $40x,\ 30y,\ 10(50 - x)$ and $20(30 - y)$.

   That is,

$$T \equiv 30x + 10y + 1100,$$

   and this is the objective function to be minimised.

   From the diagram in the second example of the previous section, we need to evaluate the objective function at the points $(30, 30),\ (50, 30)$ and $(50, 10)$.

   The minimum occurs, in fact, at the point $(30, 30)$ so that D_1 should send 30 tons to C_1 and 30 tons to C_2 while D_2 should send 20 tons to C_1 but 0 tons to C_2.

5.4.3 EXERCISES

1. Sketch, on separate diagrams, the regions of the xy-plane which correspond to the following inequalities (assuming that $x \geq 0$ and $y \geq 0$):

   (a)
$$x + y \leq 6;$$
   (b)
$$x + y \geq 4;$$
   (c)
$$3x + y \geq 6;$$
   (d)
$$x + 3y \geq 6.$$

2. Sketch the feasible region for which all the inequalities in question 1 are satisfied.

3. Maximise the objective function $5x + 7y$ subject to the simultaneous linear inequalities
$$x \geq 0,\ \ y \geq 0,\ \ 3x + 2y \geq 6 \text{ and } x + y \leq 4.$$

4. A mine manager has contracts to supply, weekly,

   100 tons of grade 1 coal,
   700 tons of grade 2 coal,
   2000 tons of grade 3 coal,
   4500 tons of grade 4 coal.

   Two seams, A and B, are being worked at a cost of £4000 and £10,000, respectively, per shift, and the yield, in tons per shift, from each seam is given by the following table:

	Grade 1	Grade 2	Grade 3	Grade 4
A	200	100	200	400
B	100	100	500	1500

   How many shifts per week should each seam be worked, in order to fulfill the contracts most economically ?

5. A manufacturer employs 5 skilled and 10 semi-skilled workers to make an article in two qualities, standard and deluxe.

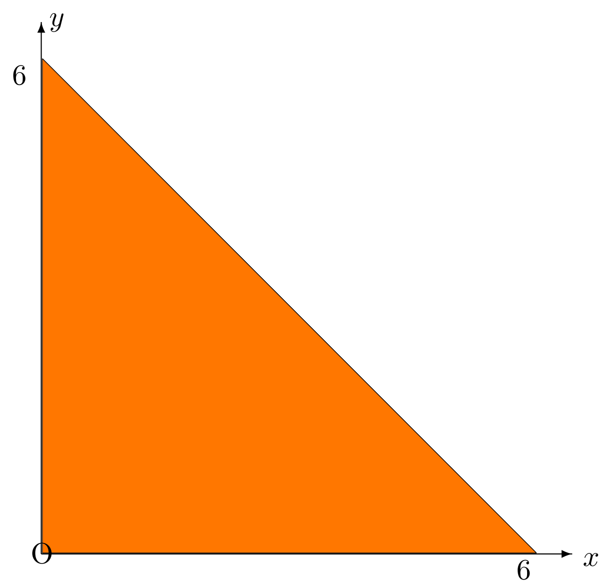
   The deluxe model requires 2 hour's work by skilled workers; the standard model requires 1 hour's work by skilled workers and 3 hour's work by semi-skilled workers.

   No worker works more than 8 hours per day and profit is £10 on the deluxe model and £8 on the standard model.

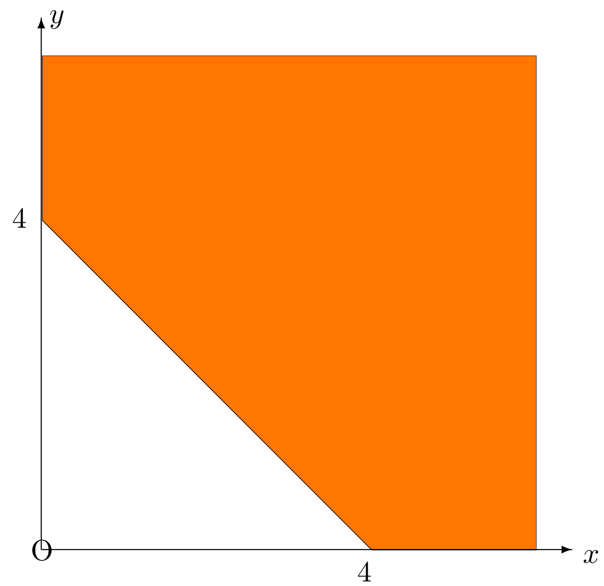
   How many of each type, per day, should be made in order to maximise profits ?

5.4.4 ANSWERS TO EXERCISES

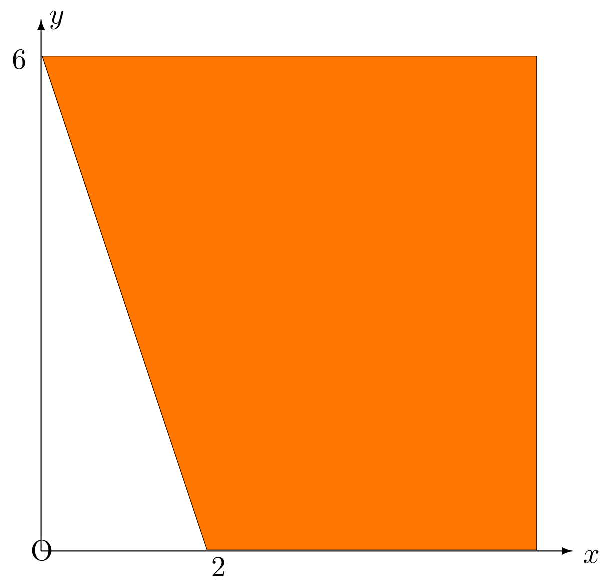
1. (a) The region is as follows:



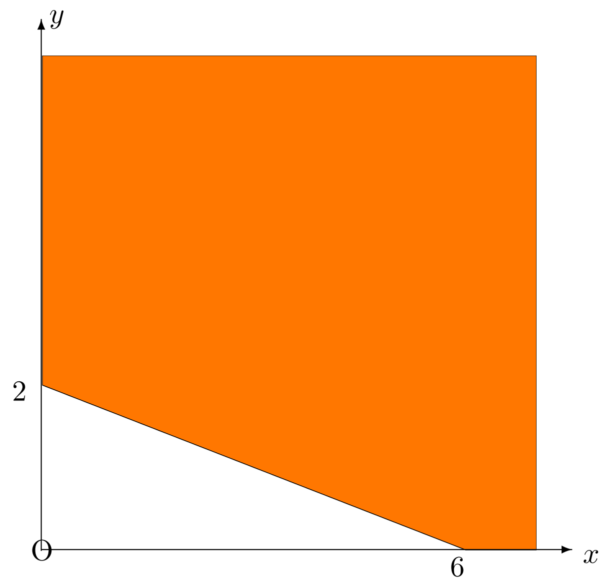
(b) The region is as follows:



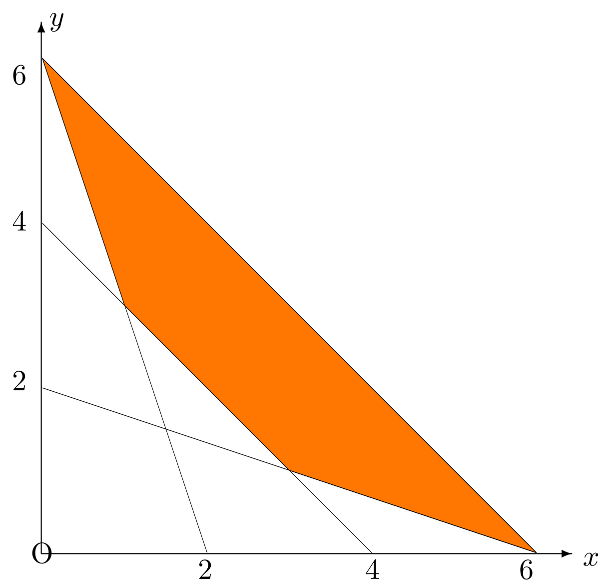
(c) The region is as follows:



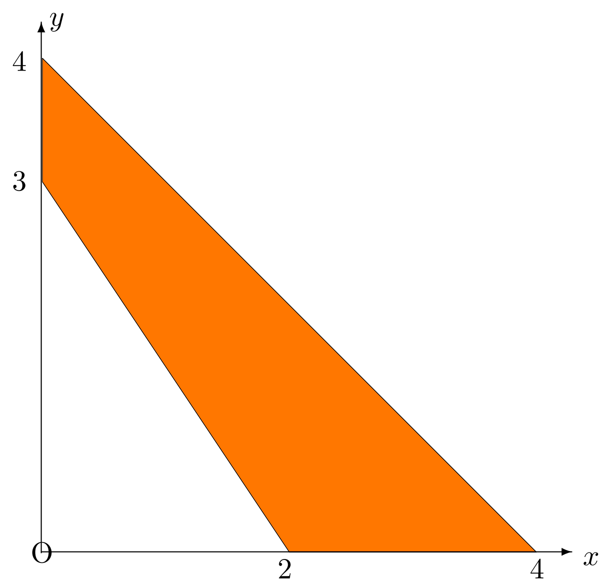
(d) The region is as follows:



2. The feasible region is as follows:



3. The feasible region is as follows:



The maximum value of $5x + 7y$ occurs at the point $(0, 4)$ and is equal to 28.

4. Subject to the simultaneous inequalities

$$x \geq 0,\ y \geq 0,\ 2x + y \geq 10,\ x + y \geq 7,\ 2x + 5y \geq 20 \text{ and } 4x + 5y \geq 45,$$

the function $2x + 5y$ has minimum value 20 at any point on the line $2x + 5y = 20$.

5. Subject to the simultaneous inequalities

$$x \geq 0,\ y \geq 0,\ x + 2y \leq 0 \text{ and } 3x + 2y \leq 80,$$

the objective function $P \equiv 8x + 10y$ has maximum value 260 at the point $(20, 10)$.

"JUST THE MATHS"

UNIT NUMBER

5.5

GEOMETRY 5
(Conic sections - the circle)

by

A.J.Hobson

UNIT 5.5 - GEOMETRY 5

CONIC SECTIONS - THE CIRCLE

5.5.1 INTRODUCTION

In this and the following three units, we shall investigate some of the geometry of four standard curves likely to be encountered in the scientific applications of Mathematics. They are the Circle, the Parabola, the Ellipse and the Hyperbola.

These curves could be generated, if desired, by considering plane sections through a cone; and, because of this, they are often called **"conic sections"** or even just **"conics"**. We shall not discuss this interpretation further, but rather use a more analytical approach.

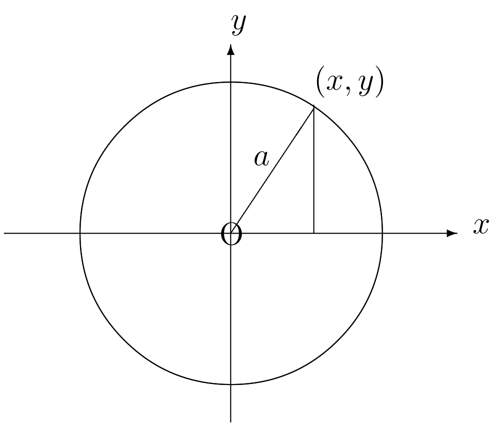
The properties of the four standard conics to be included here will be restricted to those required for simple applications work and, therefore, these notes will not provide an extensive course on elementary co-ordinate geometry.

Useful results from previous work which will be used in these units include the Change of Origin technique (Unit 5.2) and the method of Completing the Square (Unit 1.5). These results should be reviewed, if necessary, by the student.

DEFINITION
A circle is the path traced out by (or **"locus"** of) a point which moves at a fixed distance, called the **"radius"**, from a fixed point, called the **"centre"**.

5.5.2 STANDARD EQUATIONS FOR A CIRCLE

(a) Circle with centre at the origin and having radius a.



Using Pythagoras's Theorem in the diagram, the equation which is satisfied by every point (x, y) on the circle, but no other points in the plane of the axes, is

$$x^2 + y^2 = a^2.$$

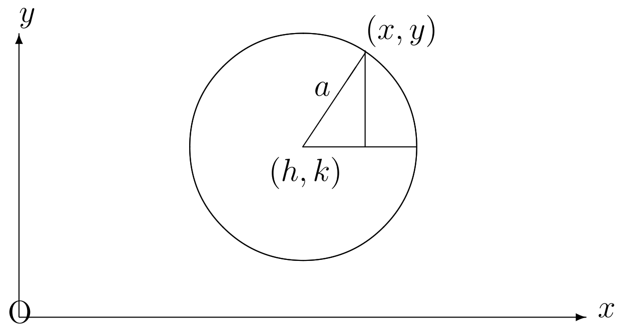
This is therefore the cartesian equation of a circle with centre $(0, 0)$ and radius a.

Note:
The angle θ in the diagram could be used as a parameter for the point (x, y) to give the parametric equations

$$x = a\cos\theta, \quad y = a\sin\theta.$$

Each point on the curve has infinitely many possible parameter values, all differing by a multiple of 2π; but it is usually most convenient to choose the value which lies in the interval $-\pi < \theta \leq \pi$.

(b) Circle with centre (h, k) having radius a.



If we were to consider a temporary change of origin to the point (h, k) with X-axis and Y-axis, the circle would have equation

$$X^2 + Y^2 = a^2,$$

with reference to the new axes. But, from previous work,

$$X = x - h \text{ and } Y = y - k.$$

Hence, with reference to the original axes, the circle has equation

$$(x - h)^2 + (y - k)^2 = a^2;$$

or, in its expanded form,

$$x^2 + y^2 - 2hx - 2ky + c = 0,$$

where

$$c = h^2 + k^2 - a^2.$$

Notes:
(i) The parametric equations of this circle with reference to the temporary new axes through the point (h, k) would be

$$X = a\cos\theta, \quad Y = a\sin\theta.$$

Hence, the parametric equations of the circle with reference to the original axes are

$$x = h + a\cos\theta, \quad y = k + a\sin\theta.$$

(ii) If the equation of a circle is encountered in the form

$$(x-h)^2 + (y-k)^2 = a^2,$$

it is very easy to identify the centre, (h, k) and the radius, a. If the equation is encountered in its expanded form, the best way to identify the centre and radius is to **complete the square in the x and y terms** in order to return to the first form.

EXAMPLES

1. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$x^2 + y^2 + 4x + 6y + 4 = 0.$$

Solution

Completing the square in the x terms,

$$x^2 + 4x \equiv (x+2)^2 - 4.$$

Completing the square in the y terms,

$$y^2 + 6y \equiv (y+3)^2 - 9.$$

The equation of the circle therefore becomes

$$(x+2)^2 + (y+3)^2 = 9.$$

Hence the centre is the point $(-2, -3)$ and the radius is 3.

2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is

$$5x^2 + 5y^2 - 10x + 15y + 1 = 0.$$

Solution

Here it is best to divide throughout by the coefficient of the x^2 and y^2 terms, even if some of the new coefficients become fractions. We obtain

$$x^2 + y^2 - 2x + 3y + \frac{1}{5} = 0.$$

Completing the square in the x terms,

$$x^2 - 2x \equiv (x-1)^2 - 1.$$

Completing the square in the y terms,

$$y^2 + 3y \equiv \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}.$$

The equation of the circle therefore becomes

$$(x-1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{61}{20}.$$

Hence the centre is the point $\left(1, -\frac{3}{2}\right)$ and the radius is $\sqrt{\frac{61}{20}} \cong 1.75$

Note:
Not every equation of the form

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

represents a circle because, for some combinations of h, k and c, the radius would not be a real number. In fact,

$$a = \sqrt{h^2 + k^2 - c},$$

which could easily turn out to be unreal.

5.5.3 EXERCISES

1. Write down the equation of the circle with centre $(4, -3)$ and radius 2.

2. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is
$$x^2 + y^2 - 2x + 4y - 11 = 0.$$

3. Determine the co-ordinates of the centre and the value of the radius of the circle whose equation is
$$36x^2 + 36y^2 - 36x - 24y - 131 = 0.$$

4. Determine the equation of the circle passing through the point $(4, -3)$ and having centre $(2, 1)$. What is the radius of the circle and what are its parametric equations ?

5. Use the parametric equations of the straight line joining the two points $(2, 4)$ and $-4, 2)$ to find its points of intersection with the circle whose equation is
$$x^2 + y^2 + 4x - 2y = 0.$$

   Hint:
   Substitute the parametric equations of the straight line into the equation of the circle and find two solutions for the parameter.

5.5.4 ANSWERS TO EXERCISES

1. The equation of the circle is either
$$(x-4)^2+(y+3)^2=4,$$
or
$$x^2+y^2-8x+6y+21=0.$$

2. The centre is $(1,-2)$ and the radius is 4.

3. The centre is $\left(\frac{1}{2},\frac{1}{3}\right)$ and radius is 2.

4. The equation is
$$(x-2)^2+(y-1)^2=20$$
and the radius is $\sqrt{20}$.
The parametric equations are
$$x=2+\sqrt{20}\cos\theta,\quad y=1+\sqrt{20}\sin\theta.$$

5. From
$$x=2-6t,\quad \text{and}\quad y=4-2t,$$
we obtain
$$40t^2-60t+20=0,$$
giving $t=\frac{1}{2}$ and $t=1$.
The points of intersection are $(-1,3)$ and $(-4,2)$.

"JUST THE MATHS"

UNIT NUMBER

5.6

GEOMETRY 6
(Conic sections - the parabola)

by

A.J.Hobson

UNIT 5.6 - GEOMETRY 6

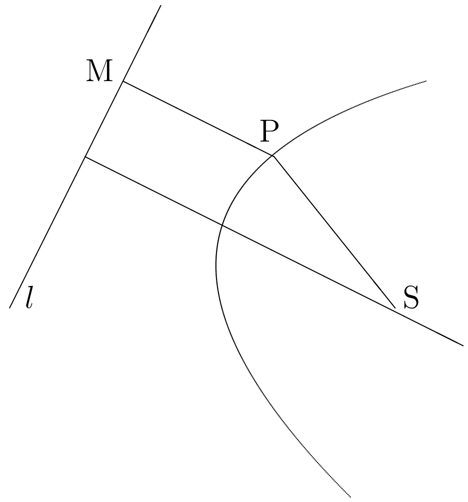
CONIC SECTIONS - THE PARABOLA

5.5.1 INTRODUCTION

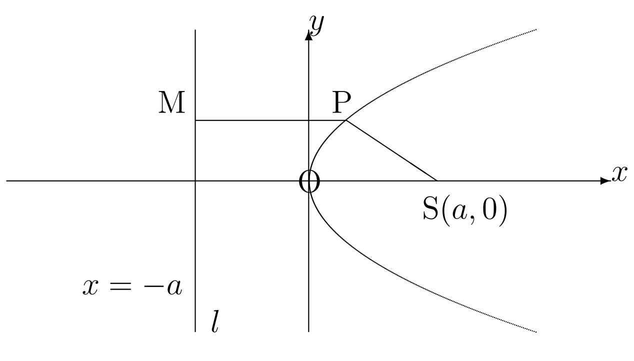
The Standard Form for the equation of a Parabola

DEFINITION

A parabola is the path traced out by (or **"locus"** of) a point, P, whose distance, SP, from a fixed point, S, called the **"focus"**, is equal to its perpendicular distance, PM, from a fixed line, l, called the **"directrix"**.



For convenience, we may take the directrix, l, to be a vertical line - with the perpendicular onto it from the focus, S, being the x-axis. We could take the y-axis to be the directrix itself, but the equation of the parabola turns out to be simpler if we use a different line; namely the line parallel to the directrix passing through the mid-point of the perpendicular from the focus onto the directrix. This point is one of the points on the parabola so that we make the curve pass through the origin.



Letting the focus be the point $(a, 0)$ (since it lies on the x-axis) the definition of the parabola implies that SP = PM. That is,

$$\sqrt{(x-a)^2 + y^2} = x + a.$$

Squaring both sides gives

$$(x-a)^2 + y^2 = x^2 + 2ax + a^2,$$

or

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2.$$

This reduces to

$$y^2 = 4ax$$

and is the standard equation of a parabola with **"vertex"** at the origin, focus at $(a, 0)$ and axis of symmetry along the x-axis. All other versions of the equation of a parabola which we shall consider will be based on this version.

Notes:
(i) If a is negative, the bowl of the parabola faces in the opposite direction towards negative x values.

(ii) Any equation of the form $y^2 = kx$, where k is a constant, represents a parabola with vertex at the origin and axis of symmetry along the x-axis. Its focus will lie at the point $\left(\frac{k}{4}, 0\right)$; it is worth noting this observation for future reference.

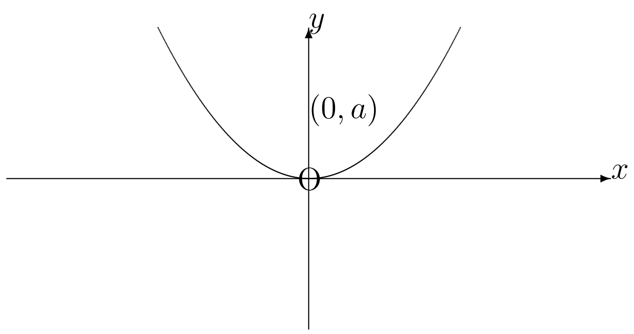
(iii) The parabola $y^2 = 4ax$ may be represented parametrically by the pair of equations

$$x = at^2, \quad y = 2at;$$

but the parameter, t, has no significance in the diagram such as was the case for the circle.

5.6.2 OTHER FORMS OF THE EQUATION OF A PARABOLA

(a) Vertex at $(0,0)$ with focus at $(0,a)$



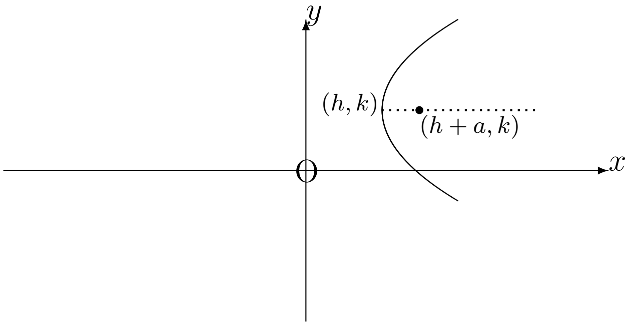
This parabola is effectively the same as the standard one except that the roles of x and y have been reversed. We may assume, therefore that the curve has equation

$$x^2 = 4ay$$

with associated parametric equations

$$x = 2at, \quad y = at^2$$

(b) Vertex at (h, k) with focus at $(h + a, k)$



If we were to consider a temporary change of origin to the point (h, k), with X-axis and Y-axis, the parabola would have equation

$$Y^2 = 4aX.$$

With reference to the original axes, therefore, the parabola has equation

$$(y - k)^2 = 4a(x - h).$$

Notes:
(i) Such a parabola will often be encountered in the expanded form of this equation, containing quadratic terms in y and linear terms in x. Conversion to the stated form by completing the square in the y terms will make it possible to identify the vertex and focus.

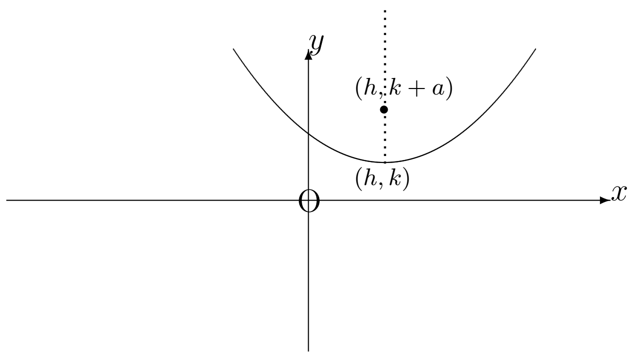
(ii) With reference to the new axes with origin at the point (h, k), the parametric equations of the parabola would be

$$X = at^2, \quad Y = 2at.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + at^2, \quad y = k + 2at.$$

(c) Vertex at (h, k) with focus at $(h, k + a)$



If we were to consider a temporary change of origin to the point (h, k) with X-axis and Y-axis, the parabola would have equation

$$X^2 = 4aY.$$

With reference to the original axes, therefore, the parabola has equation

$$(x - h)^2 = 4a(y - k).$$

Notes:
(i) Such a parabola will often be encountered in the expanded form of this equation, containing quadratic terms in x and linear terms in y. Conversion to the stated form by completing the square in the x terms will make it possible to identify the vertex and focus.

(ii) With reference to the new axes with origin at the point (h, k), the parametric equations of the parabola would be

$$X = 2at, \quad Y = at^2.$$

Hence, with reference to the original axes, the parametric equations are

$$x = h + 2at, \quad y = k + at^2.$$

EXAMPLES

1. Give a sketch of the parabola whose cartesian equation is

$$y^2 - 6y + 3x = 10,$$

showing the co-ordinates of the vertex, focus and intesections with the x-axis and y-axis.

Solution

First, we must complete the square in the y terms obtaining

$$y^2 - 6y \equiv (y - 3)^2 - 9.$$

Hence, the equation becomes

$$(y-3)^2 - 9 + 3x = 10.$$

That is,

$$(y-3)^2 = 19 - 3x,$$

or

$$(y-3)^2 = 4.\left(-\frac{3}{4}\right)\left(x - \frac{19}{3}\right).$$

Thus, the vertex lies at the point $\left(\frac{19}{3}, 3\right)$ and the focus lies at the point $\left(\frac{19}{3} - \frac{3}{4}, 3\right)$; that is, $\left(\frac{67}{12}, 3\right)$.

The parabola intersects the x-axis where $y = 0$, i.e.

$$3x = 10,$$

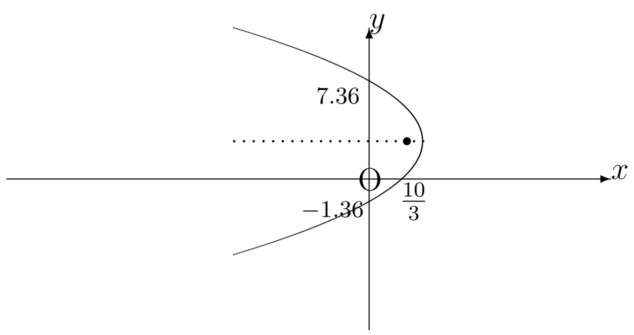
giving $x = \frac{10}{3}$.

The parabola intersects the y-axis where $x = 0$; that is,

$$y^2 - 6y - 10 = 0,$$

which is a quadratic equation with solutions

$$y = \frac{6 \pm \sqrt{36+40}}{2} \cong 7.36 \text{ or } -1.36$$



2. Use the parametric equations of the parabola

$$x^2 = 8y$$

to determine its points of intesection with the straight line

$$y = x + 6.$$

Solution

The parametric equations are $x = 4t$, $y = 2t^2$.

Substituting these into the equation of the straight line, we have

$$2t^2 = 4t + 6.$$

That is,

$$t^2 - 2t - 3 = 0,$$

or

$$(t-3)(t+1) = 0,$$

which is a quadratic equation in t having solutions $t = 3$ and $t = -1$.

The points of intersection are therefore $(12, 18)$ and $(-4, 2)$.

5.6.3 EXERCISES

1. For the following parabolae, determine the co-ordinates of the vertex, the focus and the points of intersection with the x- axis and y-axis where appropriate:

   (a)
   $$(y-1)^2 = 4(x-2);$$

   (b)
   $$(x+1)^2 = 8(y-3);$$

   (c)
   $$2x = y^2 + 4y + 6;$$

   (d)
   $$x^2 + 4x - 4y + 6 = 0.$$

2. Use the parametric equations of the parabola
   $$y^2 = 12x$$
   to determine its points of intersection with the straight line
   $$6x + 5y - 12 = 0.$$

5.6.4 ANSWERS TO EXERCISES

1. (a) Vertex $(2, 1)$, Focus $(3, 1)$, Intersection $(\frac{9}{4}, 0)$ with the x-axis;
   (b) Vertex $(-1, 3)$, Focus $(-1, 5)$, Intersection $(0, \frac{25}{8})$ with the y-axis;
   (c) Vertex $(1, -2)$, Focus $(\frac{3}{2}, -2)$, Intersection $(3, 0)$ with the x-axis;
   (d) Vertex $(-2, \frac{1}{2})$, Focus $(-2, \frac{3}{2})$, Intersection $(0, \frac{3}{2})$ with the y-axis.

2. $x = 3t^2$ and $y = 6t$ give $18t^2 + 30t - 12 = 0$ with solutions $t = \frac{1}{3}$ and $t = -2$. Hence the points of intersection are $(\frac{1}{3}, 2)$ and $(12, -12)$.

"JUST THE MATHS"

UNIT NUMBER

5.7

GEOMETRY 7
(Conic sections - the ellipse)

by

A.J.Hobson

UNIT 5.7 - GEOMETRY 7

CONIC SECTIONS - THE ELLIPSE

5.7.1 INTRODUCTION

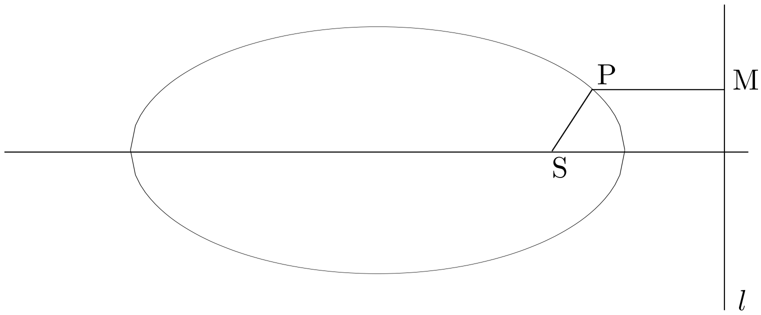
The Standard Form for the equation of an Ellipse

DEFINITION

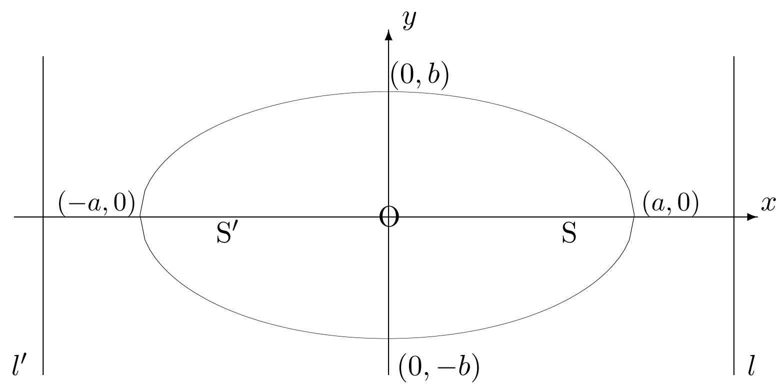
The Ellipse is the path traced out by (or **"locus** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

where $\epsilon < 1$ is a constant called the **"eccentricity"** of the ellipse. The fixed line, l, is called a **"directrix"** of the ellipse and the fixed point, S, is called a **"focus"** of the ellipse.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to l and the perpendicular line from S onto l. The diagram below illustrates two foci, S and S′, together with two directrices, l and l'. The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\cos\theta, \quad y = b\sin\theta.$$

The curve clearly intersects the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$. Whichever is the larger of a and b defines the length of the **"semi-major axis"** and whichever is the smaller defines the length of the **"semi-minor axis"**.

For the sake of completeness, it may further be shown that the eccentricity, ϵ, is obtainable from the formula

$$b^2 = a^2\left(1 - \epsilon^2\right)$$

and, having done so, the foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$. However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of an ellipse.

5.7.2 A MORE GENERAL FORM FOR THE EQUATION OF AN ELLIPSE

The equation of an ellipse, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is easily obtainable from the standard form of equation by a temporary change of origin to the point (h, k). We obtain

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with associated parametric equations

$$x = h + a\cos\theta, \quad y = k + b\sin\theta.$$

Ellipses will usually be encountered in the expanded form of the above cartesian equation and it will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the ellipse. The expanded form will be similar in appearance to that of a circle but the coefficients of x^2 and y^2, though both of the same sign, will not be equal to each other.

EXAMPLE

Determine the co-ordinates of the centre and the lengths of the semi-axes of the ellipse whose equation is

$$3x^2 + y^2 + 12x - 2y + 1 = 0.$$

Solution

Completing the square in the x terms gives

$$3x^2 + 12x \equiv 3\left[x^2 + 4x\right] \equiv 3\left[(x+2)^2 - 4\right] \equiv 3(x+2)^2 - 12.$$

Completing the square in the y terms gives

$$y^2 - 2y \equiv (y-1)^2 - 1.$$

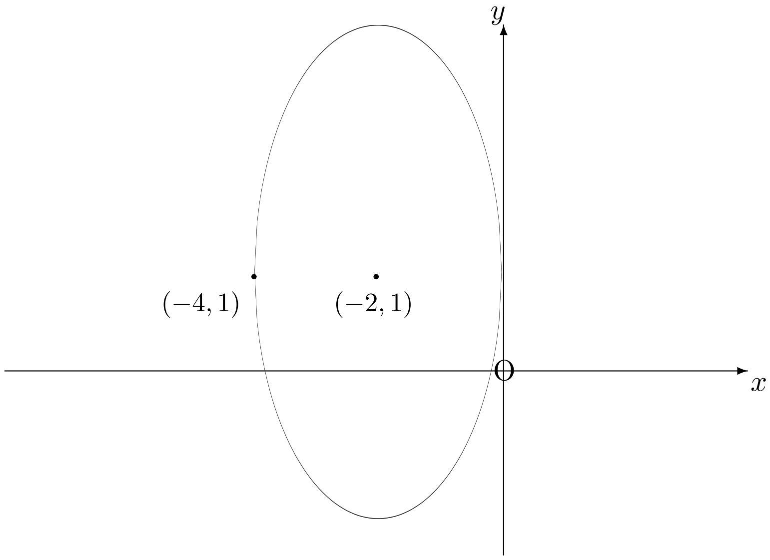
Hence, the equation of the ellipse becomes

$$3(x+2)^2 + (y-1)^2 = 12.$$

That is,

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{12} = 1.$$

The centre is thus located at the point $(-2, 1)$ and the semi-axes have lengths $a = 2$ and $b = \sqrt{12}$.



5.7.3 EXERCISES

1. For each of the following ellipses, determine the co-ordinates of the centre and give a sketch of the curve:

   (a)

   $$x^2 + 4y^2 - 4x - 8y + 4 = 0;$$

   (b)

   $$x^2 + 4y^2 + 16y + 12 = 0;$$

   (c)

   $$x^2 + 4y^2 + 6x - 8y + 9 = 0.$$

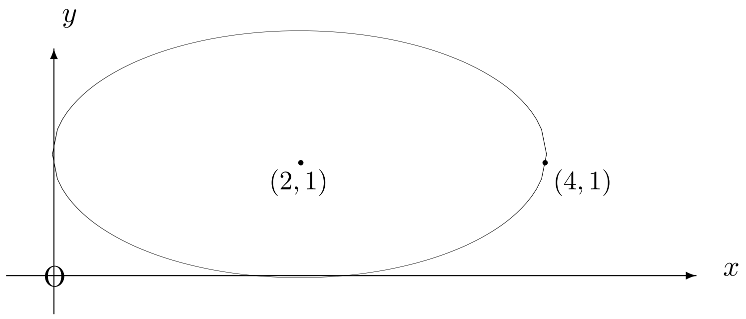
2. Determine the lengths of the semi-axes of the ellipse whose equation is

   $$9x^2 + 25y^2 = 225$$

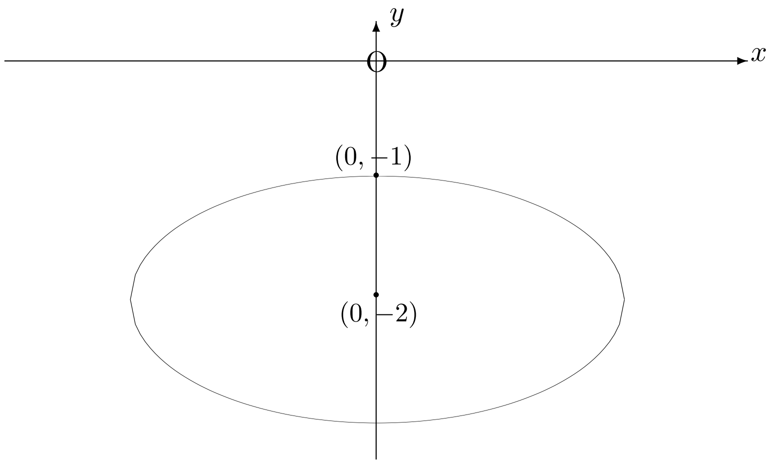
   and write down also a pair of parametric equations for this ellipse.

5.7.4 ANSWERS TO EXERCISES

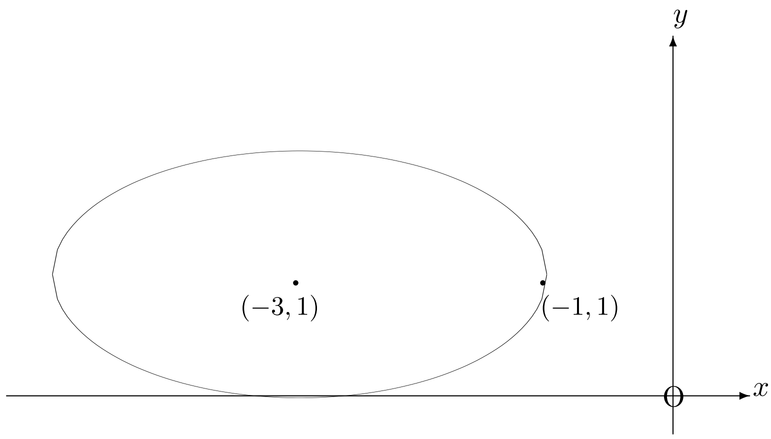
1. (a) Centre $(2,1)$ with $a=2$ and $b=1$.



(b) Centre $(0,-2)$ with $a=2$ and $b=1$.



(c) Centre $(-3,1)$ with $a=2$ and $b=1$.



2. $a=5$ and $b=3$, giving the parametric equations $x=5\cos\theta,\quad y=3\sin\theta$.

"JUST THE MATHS"

UNIT NUMBER

5.8

GEOMETRY 8
(Conic sections - the hyperbola)

by

A.J.Hobson

UNIT 5.8 - GEOMETRY 8

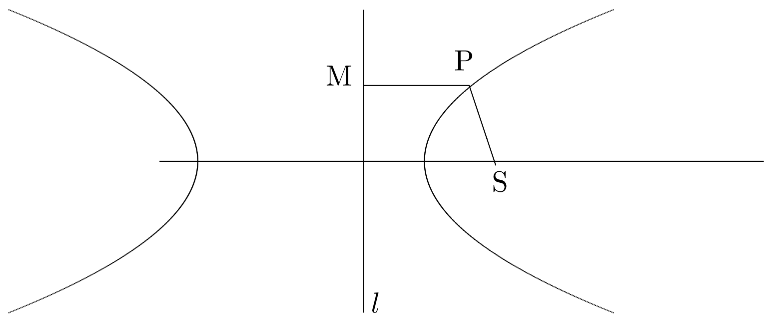
CONIC SECTIONS - THE HYPERBOLA

5.8.1 INTRODUCTION

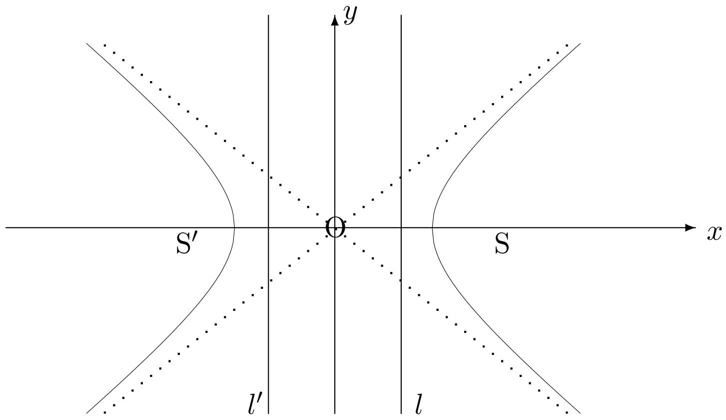
DEFINITION
The hyperbola is the path traced out by (or **"locus"** of) a point, P, for which the distance, SP, from a fixed point, S, and the perpendicular distance, PM, from a fixed line, l, satisfy a relationship of the form

$$\text{SP} = \epsilon.\text{PM},$$

where $\epsilon > 1$ is a constant called the **"eccentricity"** of the hyperbola. The fixed line, l, is called a **"directrix"** of the hyperbola and the fixed point, S, is called a **"focus"** of the hyperbola.



In fact, there are two foci and two directrices because the curve turns out to be symmetrical about a line parallel to l and the perpendicular line from S onto l. The diagram below illustrates two foci S and S′ together with two directrices l and l'. The axes of symmetry are taken as the co-ordinate axes.



It can be shown that, with this system of reference, the hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with associated parametric equations

$$x = a\sec\theta, \quad y = b\tan\theta$$

although, for students who meet "hyperbolic functions", a better set of parametric equations would be

$$x = a\cosh t, \quad y = b\sinh t.$$

The curve clearly intersects the x-axis at $(\pm a, 0)$ but does not intersect the y-axis at all.

For the sake of completeness, it may further be shown that the eccentricity, ϵ, is obtainable from the formula

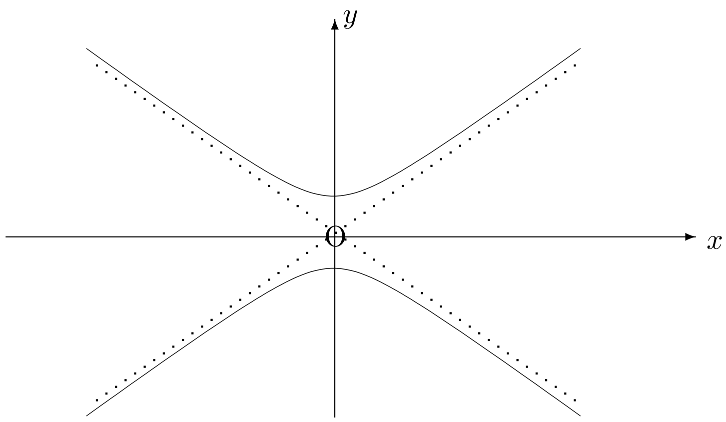
$$b^2 = a^2\left(\epsilon^2 - 1\right)$$

and, having done so, the foci lie at $(\pm a\epsilon, 0)$ with directrices at $x = \pm\frac{a}{\epsilon}$. However, in these units, students will not normally be expected to determine the eccentricity, foci or directrices of a hyperbola

Note:
A similar hyperbola to the one above, but intersecting the y-axis rather than the x-axis, has equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

The roles of x and y are simply reversed.



5.8.2 ASYMPTOTES

A special property of the hyperbola is that, at infinity, it approaches two straight lines through the centre of the hyperbola called **"asymptotes"**.

It can be shown that both of the hyperbolae

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

have asymptotes whose equations are:

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

These are easily obtained by factorising the **left hand side** of the equation of the hyperbola, then equating each factor to zero.

5.8.3 MORE GENERAL FORMS FOR THE EQUATION OF A HYPERBOLA

The equation of a hyperbola, with centre (h, k) and axes of symmetry parallel to Ox and Oy respectively, is easily obtainable from one of the standard forms of equation by a temporary change of origin to the point (h, k). We obtain either

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

with associated parametric equations

$$x = h + a\text{sec}\theta, \quad y = k + b\tan\theta$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1,$$

with associated parametric equations

$$x = h + a\tan\theta, \quad y = k + b\text{sec}\theta.$$

Hyperbolae will usually be encountered in the expanded form of the above cartesian equations and it will be necessary to complete the square in both the x terms and the y terms in order to locate the centre of the hyperbola. The expanded form will be similar in appearance to that of a circle but the coefficients of x^2 and y^2 will be different numerically and opposite in sign.

EXAMPLE

Determine the co-ordinates of the centre and the equations of the asymptotes of the hyperbola whose equation is

$$4x^2 - y^2 + 16x + 6y + 6 = 0.$$

Solution

Completing the square in the x terms gives

$$4x^2 + 16x \equiv 4\left[x^2 + 4x\right] \equiv 4\left[(x+2)^2 - 4\right] \equiv 4(x+2)^2 - 16.$$

Completing the square in the y terms gives

$$-y^2 + 6y \equiv -\left[y^2 - 6y\right] \equiv -\left[(y-3)^2 - 9\right] \equiv -(y-3)^2 + 9.$$

Hence the equation of the hyperbola becomes

$$4(x+2)^2 - (y-3)^2 = 1$$

or

$$\frac{(x+2)^2}{\left(\frac{1}{2}\right)^2} - \frac{(y-3)^2}{1^2} = 1.$$

The centre is thus located at the point $(-2, 3)$.

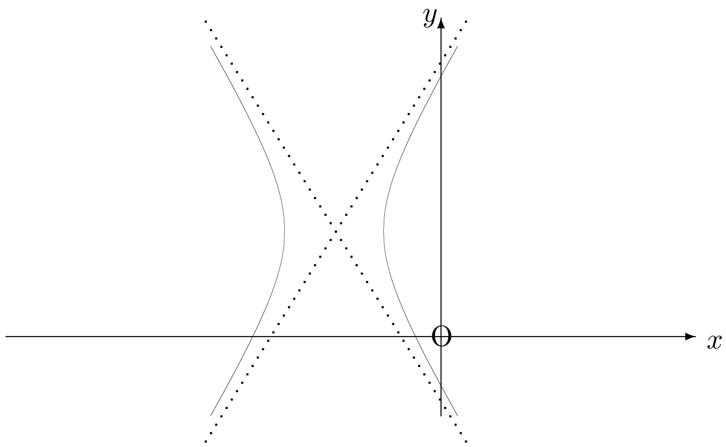
The asymptotes are best obtained by factorising the left hand side of the penultimate version of the equation of the hyperbola, then equating each factor to zero. We obtain

$$2(x+2) - (y-3) = 0 \quad \text{and} \quad 2(x+2) + (y-3) = 0.$$

In other words,

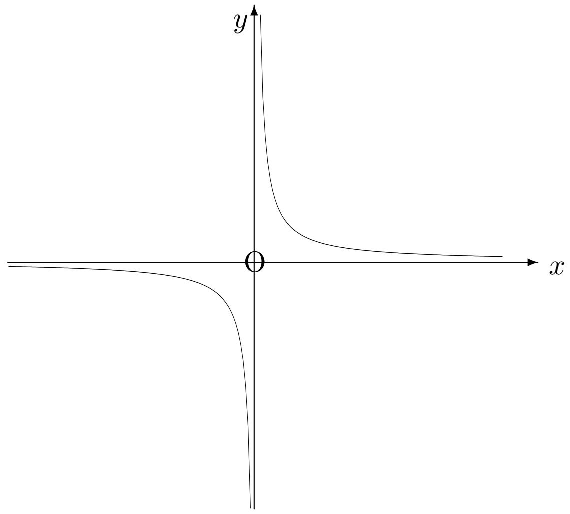
$$2x - y + 7 = 0 \quad \text{and} \quad 2x + y + 1 = 0.$$

To sketch the graph of a hyperbola, it is not always enough to have the position of the centre and the equations of the asymptotes. It may also be necessary to investigate some of the intersections of the curve with the co-ordinate axes. In the current example, by substituting first $y = 0$ and then $x = 0$ into the equation of the hyperbola, it is possible to determine intersections at $(-0.84, 0)$, $(-7.16, 0)$, $(0, -0.87)$ and $(0, 6.87)$.

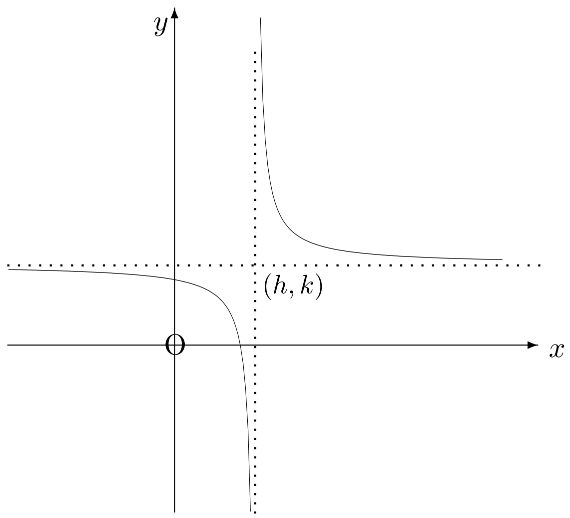


5.8.4 THE RECTANGULAR HYPERBOLA

For some hyperbolae, it will turn out that the asymptotes are at right-angles to each other; in which case, the **asymptotes themselves** could be used as the x-axis and y-axis. When this choice of reference system is made for a hyperbola with centre at the origin, it can be shown that hyperbola has the simpler equation $xy = C$, where C is a constant.



Similarly, for a rectangular hyperbola with centre at the point (h, k) and asymptotes used as the axes of reference, the equation will be $(x - h)(y - k) = C$.



Note:
A suitable pair of parametric equations for the rectangular hyperbola $(x-h)(y-k) = C$ are

$$x = t + h, \quad y = k + \frac{C}{t}.$$

EXAMPLES

1. Determine the centre of the rectangular hyperbola whose equation is

$$7x - 3y + xy - 31 = 0.$$

Solution

The equation factorises into the form

$$(x-3)(y+7)=10.$$

Hence, the centre is located at the point $(3,-7)$.

2. A certain rectangular hyperbola has parametric equations

$$x=1+t, \quad y=3-\frac{1}{t}.$$

Determine its points of intersection with the straight line $x+y=4$.

Solution

Substituting for x and y into the equation of the straight line, we obtain

$$1+t+3-\frac{1}{t}=4 \ \text{ or } \ t^2-1=0.$$

Hence, $t=\pm1$ giving points of intersection at $(2,2)$ and $(0,4)$.

5.8.5 EXERCISES

1. For each of the following hyperbolae, determine the co-ordinates of the centre and the equations of the asympotes. Give a sketch of the curve, indicating where appropriate, the co-ordinates of its points of intersection with the x-axis and y-axis:

(a)

$$x^2-y^2-2y=0;$$

(b)

$$y^2-x^2-6x=10;$$

(c)

$$x^2-y^2-2x-2y=4;$$

(d)

$$y^2-x^2-6x+4y=14;$$

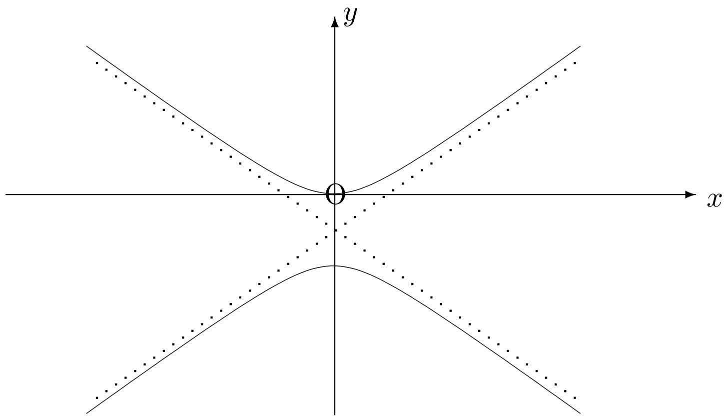
(e)

$$9x^2-4y^2+18x-16y=43.$$

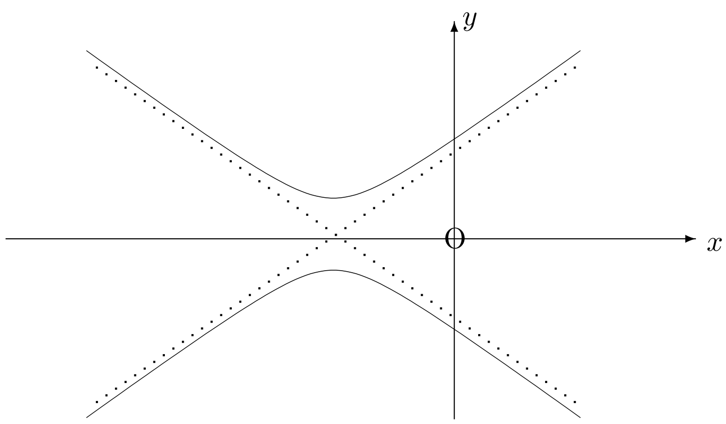
2. Determine a pair of parametric equations for the rectangular hyperbola whose equation is $xy-x+2y-6=0$ and hence obtain its points of intersection with the straight line $y=x+3$. Sketch the hyperbola and the straight line on the same diagram.

5.8.6 ANSWERS TO EXERCISES

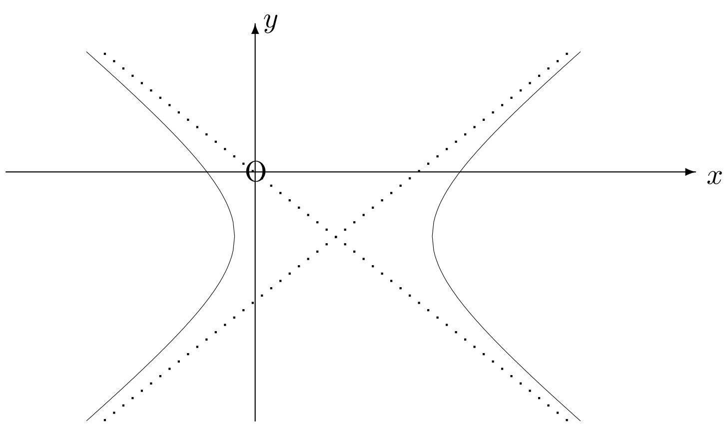
1. (a) Centre $(0, -1)$ with asymptotes $y = x - 1$ and $y = -x - 1$;



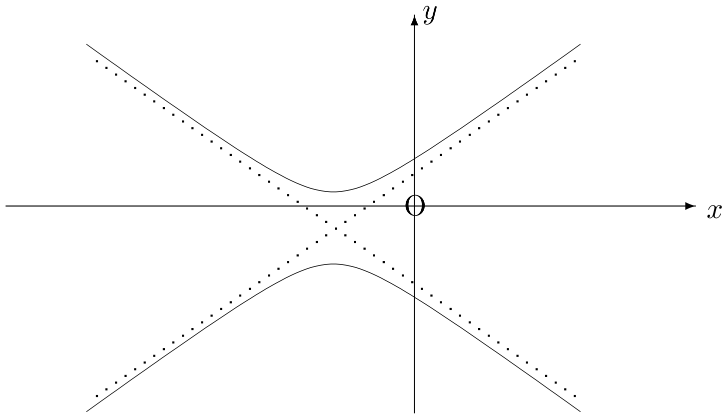
(b) Centre $(-3, 0)$ with asymptotes $y = x + 3$ and $y = -x - 3$;



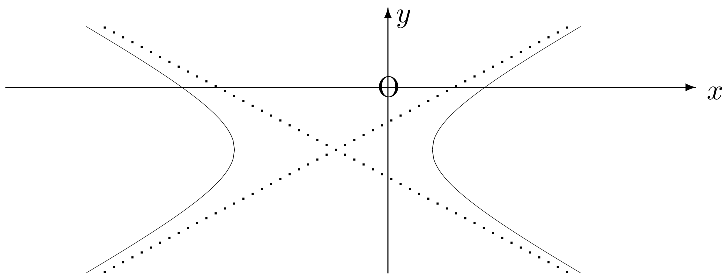
(c) Centre $(1, -1)$ with asymptotes $y = -x$ and $y = x - 2$;



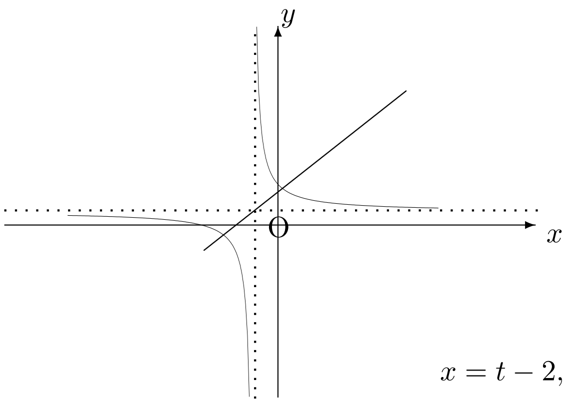
(d) Centre $(-3,-2)$ with asymptotes $y = x + 1$ and $y = -x - 5$;



(e) Centre $(-1,-2)$ with asymptotes $3x - 2y = 1$ and $3x + 2y = -7$.



2. Centre $(-2, 1)$ with asymptotes $x = -2$ and $y = 1$. Intersections $(0, 3)$ and $(-4, -1)$.



$x = t - 2, \quad y = \frac{4}{t} + 1$

"JUST THE MATHS"

UNIT NUMBER

5.9

GEOMETRY 9
(Curve sketching in general)

by

A.J.Hobson

UNIT 5.9 - GEOMETRY 9

CURVE SKETCHING IN GENERAL

Introduction

The content of the present section is concerned with those situations where it is desirable to find out the approximate shape of a curve whose equation is known, but not necessarily to determine an accurate "plot" of the curve.

In becoming accustomed to the points discussed below, the student should not feel that **every one** has to be used for a particular curve; merely enough of them to give a satisfactory impression of what the curve looks like.

5.9.1 SYMMETRY

A curve is symmetrical about the x-axis if its equation contains only even powers of y. It is symmetrical about the y-axis if its equation contains only even powers of x.

We say also that a curve is symmetrical with respect to the origin if its equation is unaltered when both x and y are changed in sign. In other words, if a point (x, y) lies on the curve, so does the point $(-x, -y)$.

ILLUSTRATIONS

1. The curve whose equation is
$$x^2\left(y^2 - 2\right) = x^4 + 4$$
is symmetrical about to both the x-axis and the y axis. This means that, once the shape of the curve is known in the first quadrant, the rest of the curve is obtained from this part by reflecting it in both axes.

   The curve is also symmetrical with respect to the origin.

2. The curve whose equation is
$$xy = 5$$
is symmetrical with respect to the origin but not about either of the co-ordinate axes.

5.9.2 INTERSECTIONS WITH THE CO-ORDINATE AXES

Any curve intersects the x-axis where $y = 0$ and the y-axis where $x = 0$; but sometimes the curve has no intersection with one or more of the co-ordinate axes. This will be borne out by an inability to solve for x when $y = 0$ or for y when $x = 0$ (or both).

ILLUSTRATION

The circle
$$x^2 + y^2 - 4x - 2y + 4 = 0$$
meets the x-axis where
$$x^2 - 4x + 4 = 0.$$
That is,
$$(x - 2)^2 = 0,$$
giving a double intersection at the point $(2, 0)$. This means that the circle **touches** the x-axis at $(2, 0)$.

The circle meets the y-axis where
$$y^2 - 2y + 4 = 0.$$
That is,
$$(y - 1)^2 = -3,$$
which is impossible, since the left hand side is bound to be positive when y is a real number.

Thus, there are no intersections with the y-axis.

5.9.3 RESTRICTIONS ON THE RANGE OF EITHER VARIABLE

It is sometimes possible to detect a range of x values or a range of y values outside of which the equation of a curve would be meaningless in terms of real geometrical points of the cartesian diagram. Usually, this involves ensuring that neither x nor y would have to assume complex number values; but other kinds of restriction can also occur.

ILLUSTRATIONS

1. The curve whose equation is
$$y^2 = 4x$$
requires that x shall not be negative; that is, $x \geq 0$.

2. The curve whose equation is
$$y^2 = x\left(x^2 - 1\right)$$
requires that the right hand side shall not be negative; and from the methods of Unit 1.10, this will be so when either $x \geq 1$ or $-1 \leq x \leq 0$.

5.9.4 THE FORM OF THE CURVE NEAR THE ORIGIN

For small values of x (or y), the higher powers of the variable can often be usefully neglected to give a rough idea of the shape of the curve near to the origin.

This method is normally applied to curves which pass **through** the origin, although the behaviour near to other points can be considered by using a temporary change of origin.

ILLUSTRATION

The curve whose equation is

$$y = 3x^3 - 2x$$

approximates to the straight line

$$y = -2x$$

for very small values of x.

5.9.5 ASYMPTOTES

DEFINITION

An **"asymptote"** is a straight line which is approached by a curve at a very great distance from the origin.

(i) Asymptotes Parallel to the Co-ordinate Axes

Consider, by way of illustration, the curve whose equation is

$$y^2 = \frac{x^3(3 - 2y)}{x - 1}.$$

(a) By inspection, we see that the straight line $x = 1$ "meets" this curve at an infinite value of y, making it an asymptote parallel to the y-axis.

(b) Now suppose we re-write the equation as

$$x^3 = \frac{y^2(x - 1)}{3 - 2y}.$$

Inspection, this time, suggests that the straight line $y = \frac{3}{2}$ "meets" the curve at an infinite value of x, making it an asymptote parallel to the x axis.

(c) Another way of arriving at the conclusions in (a) and (b) is to write the equation of the curve in a form without fractions, namely

$$y^2(x - 1) - x^3(3 - 2y) = 0,$$

then equate to zero the coefficients of the highest powers of x and y. That is,

The coefficient of y^2 gives $x - 1 = 0$.

The coefficient of x^3 gives $3 - 2y = 0$.

It can be shown that this method may be used with any curve to find asymptotes parallel to the co-ordinate axes. Of course, there may not be any, in which case the method will not work.

(ii) Asymptotes in General for a Polynomial Curve

This paragraph requires a fairly advanced piece of algebraical argument, but an outline proof will be included, for the sake of completeness.

Suppose a given curve has an equation of the form

$$P(x, y) = 0$$

where $P(x, y)$ is a polynomial in x and y.

Then, to find the intersections with this curve of a straight line

$$y = mx + c,$$

we substitute $mx + c$ in place of y into the equation of the curve, obtaining a polynomial equation in x, say

$$a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = 0.$$

In order for the line $y = mx + c$ to be an asymptote, this polynomial equation must have **coincident solutions at infinity**.

But now let us replace x by $\frac{1}{u}$ giving, after multiplying throughout by u^n, the new polynomial equation

$$a_0 u^n + a_1 u^{n-1} + a_2 u^{n-2} + ... + a_{n-1} u + a_n = 0$$

This equation must have coincident solutions at $u = 0$ which will be the case provided

$$a_n = 0 \quad \text{and} \quad a_{n-1} = 0.$$

Conclusion
To find the asymptotes (if any) to a polynomial curve, we first substitute $y = mx + c$ into

the equation of the curve. Then, in the polynomial equation obtained, we **equate to zero the two leading coefficents**; (that is, the coeffiecients of the highest two powers of x) and solve for m and c.

EXAMPLE

Determine the equations of the asymptotes to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solution

Substituting $y = mx + c$ gives

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1.$$

That is,

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0.$$

Equating to zero the two leading coefficients; that is, the coefficients of x^2 and x, we obtain

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \text{ and } \frac{2mc}{b^2} = 0.$$

No solution is obtainable if we let $m = 0$ in the second of these statements since it would imply $\frac{1}{a^2} = 0$ in the first statement, which is impossible. Therefore we must let $c = 0$ in the second statement, and $m = \pm\frac{b}{a}$ in the first statement.

The asymptotes are therefore

$$y = \pm\frac{b}{a}x.$$

In other words,

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ or } \frac{x}{a} + \frac{y}{b} = 0,$$

as used earlier in the section on the hyperbola.

5.9.6 EXERCISES

1. Sketch the graphs of the following equations:

   (a)
$$y = x + \frac{1}{x};$$

   (b)
$$y = \frac{1}{x^2+1};$$

   (c)
$$y^2 = \frac{x}{x-2};$$

   (d)
$$y = \frac{(x-1)(x+4)}{(x-2)(x-3)};$$

   (e)
$$y(x+2) = (x+3)(x-4);$$

   (f)
$$x^2\left(y^2-25\right) = y;$$

   (g)
$$y = 6 - e^{-2x}.$$

2. For each of the following curves, determine the equations of the asymptotes which are parallel to either the x-axis or the y-axis:

   (a)
$$xy^2 + x^2 - 1 = 0;$$

   (b)
$$x^2y^2 = 4\left(x^2+y^2\right);$$

(c)

$$y = \frac{x^2 - 3x + 5}{x - 3}.$$

3. Determine all the asymptotes of the following curves:

(a)

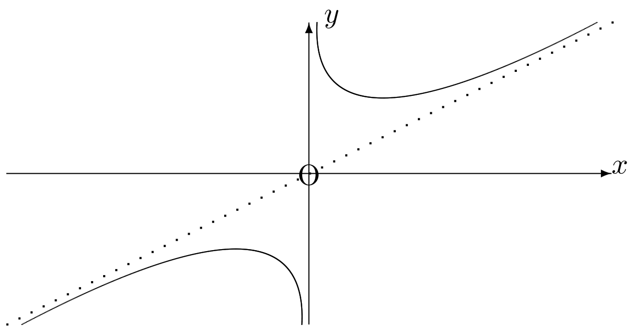
$$x^3 - xy^2 + 4x - 16 = 0;$$

(b)

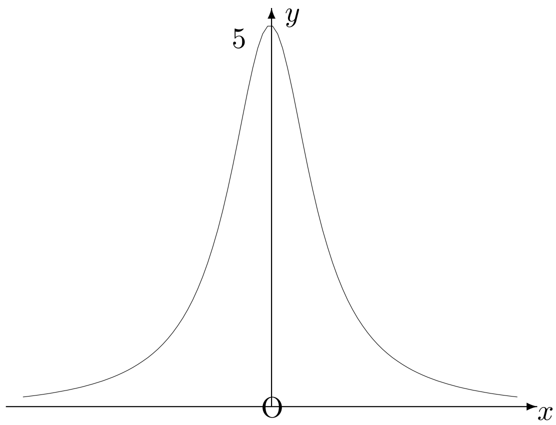
$$y^3 + 2y^2 - x^2y + y - x + 4 = 0.$$

5.9.7 ANSWERS TO EXERCISES

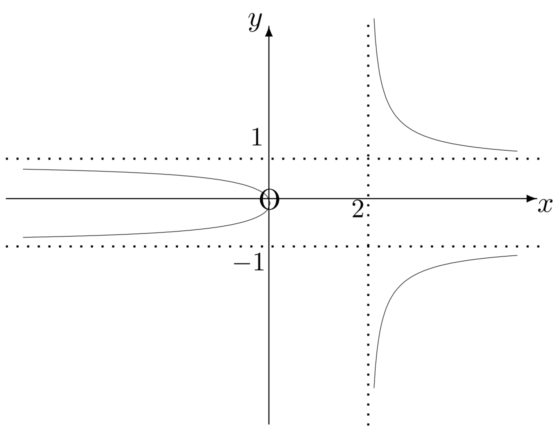
1.(a)



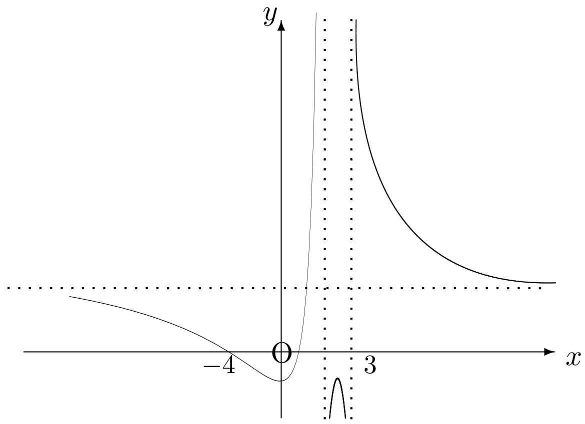
1.(b)



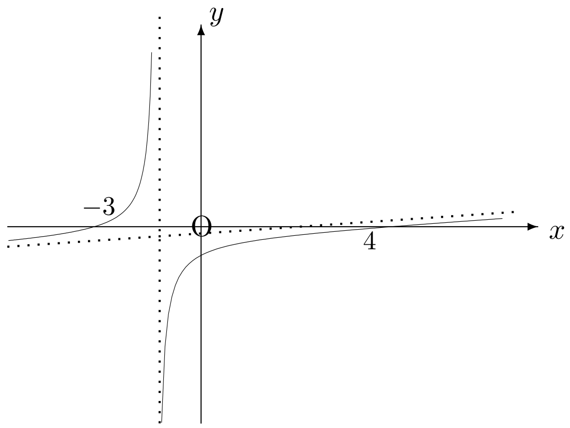
1.(c)



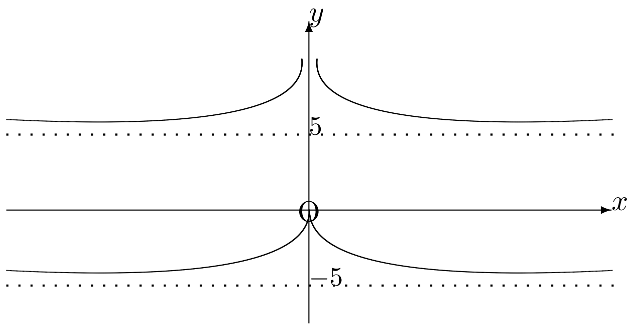
1.(d)



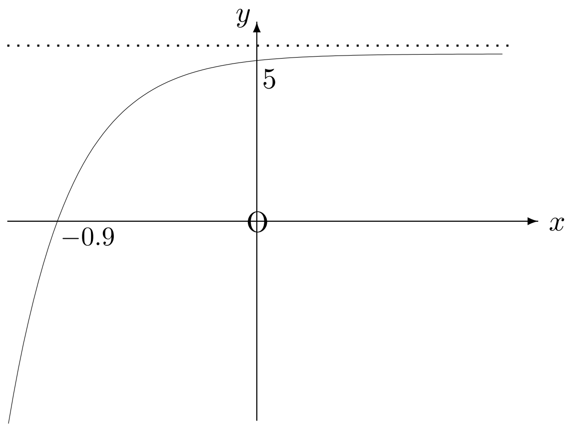
1.(e)



1.(f)



1.(g)



2.(a) $x = 0$, (b) $x = \pm 2$ and $y = \pm 2$, (c) $x = 3$

3.(a) $y = x$, $y = -x$ and $x = 0$; (b) $y = 0$, $y = x - 1$ and $y = -x - 1$.

"JUST THE MATHS"

UNIT NUMBER

5.10

GEOMETRY 10
(Graphical solutions)

by

A.J.Hobson

UNIT 5.10 - GEOMETRY 10

GRAPHICAL SOLUTIONS

5.10.1 INTRODUCTION

An algebraic equation in a variable quantity, x, may be written in the general form

$$f(x) = 0,$$

where $f(x)$ is an algebraic expression involving x; we call it a **"function of x"** (see Unit 10.1).

In the work which follows, $f(x)$ will usually be either a **linear** function of the form $ax + b$, where a and b are constants, or a **quadratic** function of the form $ax^2 + bx + c$ where a, b and c are constants.

The solutions of the equation $f(x) = 0$ consist of those values of x which, when substituted into the function $f(x)$, cause it to take the value zero.

The solutions may also be interpreted as the values of x for which the graph of the equation

$$y = f(x)$$

meets the x-axis since, at any point of this axis, y is equal to zero.

5.10.2 THE GRAPHICAL SOLUTION OF LINEAR EQUATIONS

To solve the equation

$$ax + b = 0,$$

we may plot the graph of the equation $y = ax + b$ to find the point at which it meets the x-axis.

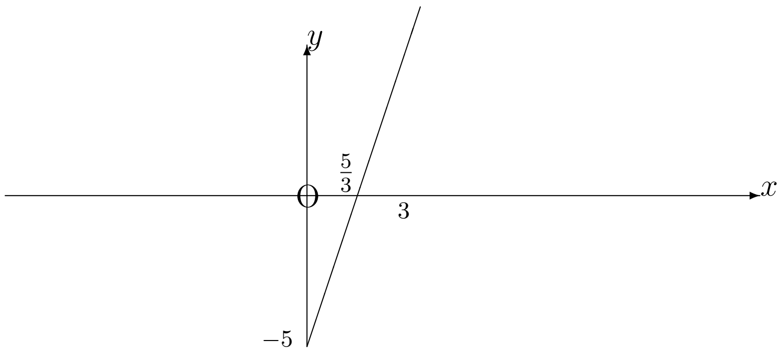
EXAMPLES

1. By plotting the graph of $y = 3x - 5$ from $x = 0$ to $x = 3$, solve the linear equation

$$3x - 5 = 0.$$

Solution

x	0	1	2	3
y	-5	-2	1	4



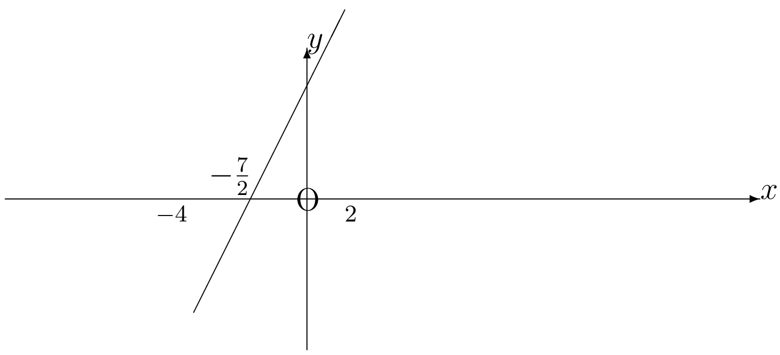
Hence $x \simeq 1.7$

2. By plotting the graph of $y = 2x + 7$ from $x = -4$ to $x = 2$, solve the linear equation

$$2x + 7 = 0$$

Solution

x	-4	-3	-2	-1	0	1	2
y	-1	1	3	5	7	10	11



Hence $x = -3.5$

5.10.3 THE GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

To solve the quadratic equation

$$ax^2 + bx + c = 0$$

by means of a graph, we may plot the graph of the equation $y = ax^2 + bx + c$ and determine the points at which it crosses the x-axis.

An alternative method is to plot the graphs of the <u>two</u> equations $y = ax^2 + bx$ and $y = -c$ in order to determine their points of intersection. This method is convenient since the first graph has the advantage of passing through the origin.

EXAMPLE

By plotting the graph of $y = x^2 - 4x$ from $x = -2$ to $x = 6$, solve the quadratic equations

(a)

$$x^2 - 4x = 0;$$

(b)

$$x^2 - 4x + 2 = 0;$$

(c)

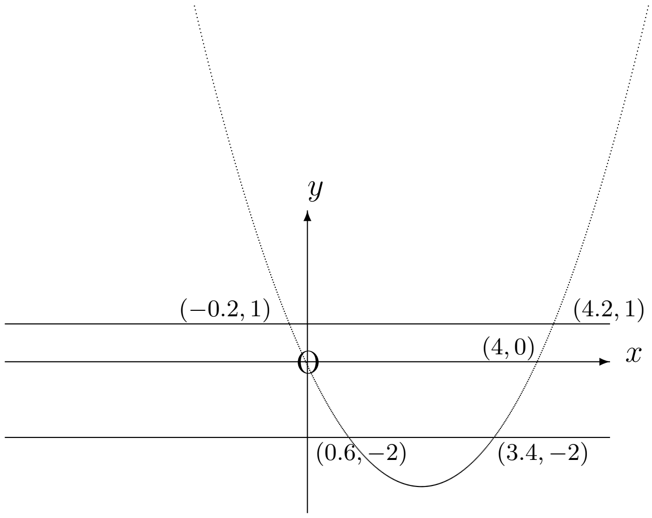
$$x^2 - 4x - 1 = 0.$$

Solution

A table of values for the graph of $y = x^2 - 4x$ is

x	-2	-1	0	1	2	3	4	5	6
y	12	5	0	-3	-2	-3	0	5	12

For parts (b) and (c), we shall also need the graphs of $y = -2$ and $y = 1$.



Hence, the three sets of solutions are:

(a)
$$x = 0 \text{ and } x = 4;$$

(b)
$$x \simeq 3.4 \text{ and } x \simeq 0.6;$$

(c)
$$x \simeq 4.2 \text{ and } x \simeq -0.2$$

5.10.4 THE GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

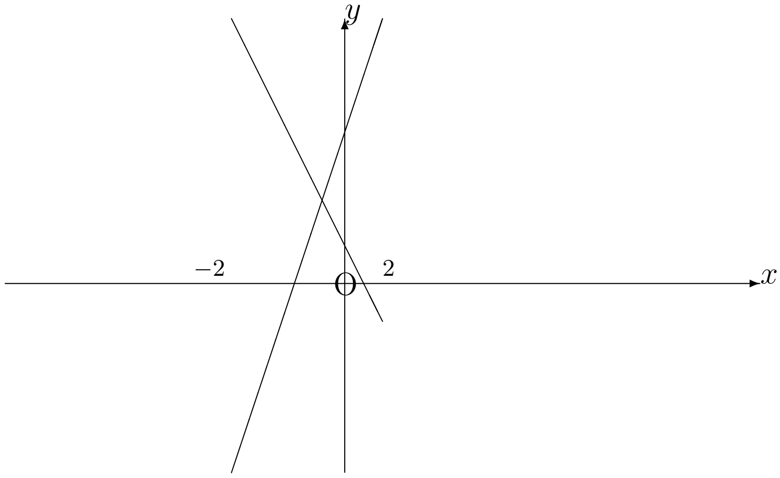
A simple extension of the ideas covered in the previous paragraphs is to solve either a pair of simultaneous linear equations or a pair of simultaneous equations consisting of one linear and one quadratic equation. More complicated cases can also be dealt with by a graphical method but we shall limit the discussion to the simpler ones.

EXAMPLES

1. By plotting the graphs of $5x + y = 2$ and $-3x + y = 6$ from $x = -2$ to $x = 2$, determine the common solution of the two equations.

   Solution

x	-2	-1	0	1	2
$y_1 = 2 - 5x$	12	7	2	-3	-8
$y_2 = 6 + 3x$	0	3	6	9	12



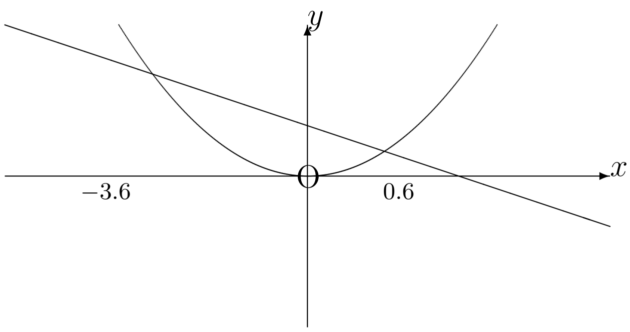
Hence, $x = -0.5$ and $y = 4.5$.

2. By plotting the graphs of the equations $y = x^2$ and $y = 2 - 3x$ from $x = -4$ to $x = 2$ determine their common solutions and hence solve the quadratic equation

$$x^2 + 3x - 2 = 0.$$

Solution

x	-4	-3	-2	-1	0	1	2
$y_1 = x^2$	16	9	4	1	0	1	4
$y_2 = 2 - 3x$	14	11	8	5	2	-1	-4



Hence $x \simeq 0.6$ and $x \simeq -3.6$.

5.10.5 EXERCISES

In these exercises, state your answers correct to one place of decimals.

1. Use a graphical method to solve the following linear equations:

   (a)
   $$8x - 3 = 0;$$

   (b)
   $$8x = 7.$$

2. Use a graphical method to solve the following quadratic equations:

   (a)
   $$2x^2 - x = 0;$$

   (b)
   $$2x^2 - x + 3 = 10;$$

   (c)
   $$2x^2 - x = 11.$$

3. Use a graphical method to solve the follwing pairs of simultaneous equations:

   (a)
   $$3x - y = 6 \quad \text{and} \quad x + y = 0;$$

   (b)
   $$x + 2y = 13 \quad \text{and} \quad 2x - 3y = 14;$$

   (c)
   $$y = 3x^2 \quad \text{and} \quad y = -5x + 1.$$

5.10.6 ANSWERS TO EXERCISES

1. (a)

$$x \simeq 0.4;$$

   (b)

$$x \simeq 0.9$$

2. (a)

$$x = 0 \quad \text{and} \quad x = 2;$$

   (b)

$$x \simeq 2.1 \quad \text{and} \quad x \simeq -1.6;$$

   (c)

$$x \simeq 2.6 \quad \text{and} \quad x \simeq -2.1$$

3. (a)

$$x = 1.2 \quad \text{and} \quad y = -1.2;$$

   (b)

$$x \simeq 9.6 \quad \text{and} \quad y \simeq 1.7;$$

   (c)

$$x \simeq 0.18 \quad \text{and} \quad y \simeq 0.1 \quad \text{or} \quad x \simeq -1.8 \quad \text{and} \quad y \simeq 10.2$$

“JUST THE MATHS”

UNIT NUMBER

5.11

GEOMETRY 11
(Polar curves)

by

A.J.Hobson

UNIT 5.11 - GEOMETRY 11 - POLAR CURVES

5.11.1 INTRODUCTION

The concept of polar co-ordinates was introduced in Unit 5.1 as an alternative method, to cartesian co-ordinates, of specifying the position of a point in a plane. It was also seen that a relationship between cartesian co-ordinates, x and y, may be converted into an equivalent relationship between polar co-ordinates, r and θ by means of the formulae,

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta,$$

while the reverse process may be carried out using the formulae

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Sometimes the reverse process may be simplified by using a mixture of both sets of formulae.

In this Unit, we shall consider the graphs of certain relationships between r and θ without necessarily refering to the equivalent of those relationships in cartesian co-ordinates. The graphs obtained will be called **"polar curves"**.

Note:

In Unit 5.1, no consideration was given to the possibility of **negative** values of r; in fact, when polar co-ordinates are used in the subject of complex numbers (see Units 6.1 - 6.6) r is **not** allowed to take negative values.

However, for the present context it will be necessary to assign a meaning to a point (r, θ), in polar co-ordinates, when r is negative.

We simply plot the point at a distance of $|r|$ along the $\theta - 180^\circ$ line; and, of course, this implies that, when r is negative, the point (r, θ) is the same as the point $(|r|, \theta - 180^\circ)$

5.11 2 THE USE OF POLAR GRAPH PAPER

For equations in which r is expressed in terms of θ, it is convenient to plot values of r against values of θ using a special kind of graph paper divided into small cells by concentric circles and radial lines.

The radial lines are usually spaced at intervals of 15° and the concentric circles allow a scale to be chosen by which to measure the distances, r, from the pole.

We illustrate with examples:

EXAMPLES

1. Sketch the graph of the equation

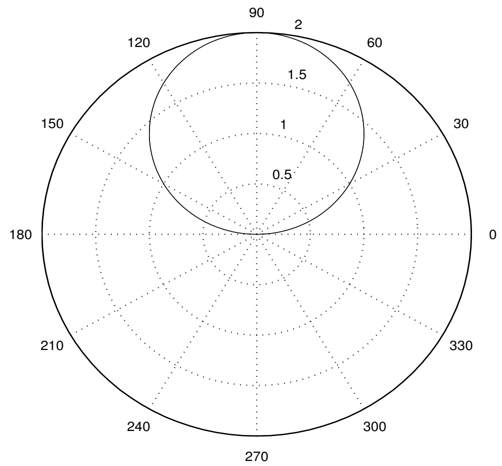
$$r = 2\sin\theta.$$

Solution

First we construct a table of values of r and θ, in steps of $15°$, from $0°$ to $360°$.

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$	$195°$
r	0	0.52	1	1.41	1.73	1.93	2	1.93	1.73	1.41	1	0.52	0	-0.52

θ	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	-1	-1.41	-1.73	-1.93	-2	-1.93	-1.73	-1.41	-1	-0.52	0



Notes:

(i) The curve, in this case, is a circle whose cartesian equation turns out to be

$$x^2 + y^2 - 2y = 0.$$

(ii) The fact that half of the values of r are negative means, here, that the circle is described twice over. For example, the point $(-0.52, 195°)$ is the same as the point $(0.52, 15°)$.

2. Sketch the graph of the following equations:

(a)

$$r = 2(1 + \cos\theta);$$

(b)

$$r = 1 + 2\cos\theta;$$

(c)

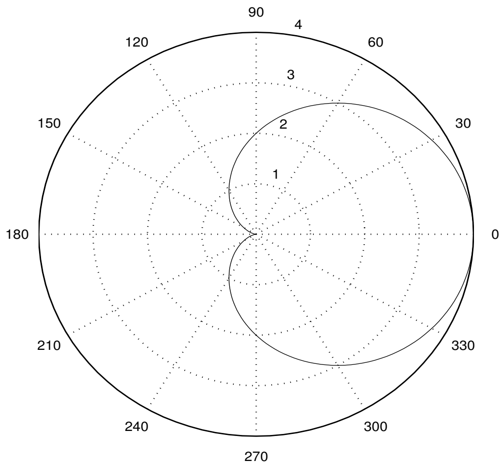
$$r = 5 + 3\cos\theta.$$

Solution

(a) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	4	3.93	3.73	3.42	3	2.52	2	1.48	1	0.59	0.27	0.07	0

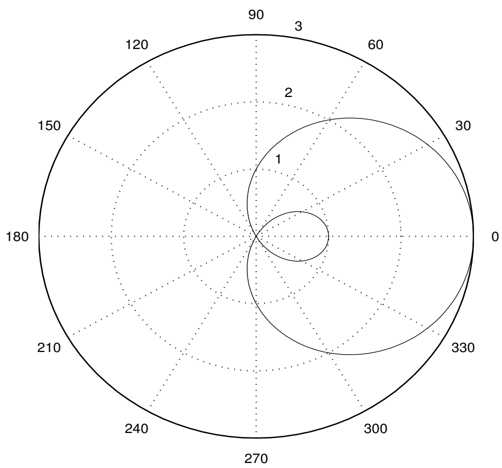
θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	0.07	0.27	0.59	1	1.48	2	2.52	3	3.42	3.73	3.93	4



(b) The table of values is as follows:

θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$	$105°$	$120°$	$135°$	$150°$	$165°$	$180°$
r	3	2.93	2.73	2.41	2	1.52	1	0.48	0	-0.41	-0.73	-0.93	-1

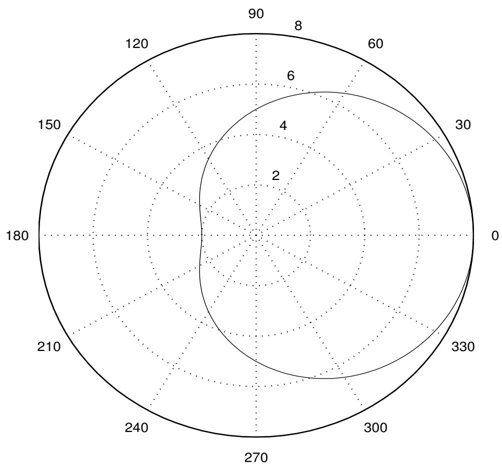
θ	$195°$	$210°$	$225°$	$240°$	$255°$	$270°$	$285°$	$300°$	$315°$	$330°$	$345°$	$360°$
r	-0.93	-0.73	-0.41	0	0.48	1	1.52	2	2.41	2.73	2.93	3



(c) The table of values is as follows:

θ	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
r	8	7.90	7.60	7.12	6.5	5.78	5	4.22	3.5	2.88	2.40	2.10	2

θ	195°	210°	225°	240°	255°	270°	285°	300°	315°	330°	345°	360°
r	2.10	2.40	2.88	3.5	4.22	5	5.78	6.5	7.12	7.60	7.90	8



Note:

Each of the three curves in the above example is known as a **"limacon"** and they illustrate special cases of the more general curve, $r = a + b\cos\theta$, as follows:

(i) If $a = b$, the limacon may also be called a **"cardioid"**; that is, a heart-shape. At the pole, the curve possesses a **"cusp"**.

(ii) If $a < b$, the limacon contains a **"re-entrant loop"**.

(iii) If $a > b$, the limacon contains neither a cusp nor a re-entrant loop.

Other well-known polar curves, together with any special titles associated with them, may be found in the answers to the exercises at the end of this unit.

5.11.3 EXERCISES

Plot the graphs of the following polar equations:

1. $$r = 3\cos\theta.$$

2. $$r = \sin 3\theta.$$

3. $$r = \sin 2\theta.$$

4. $$r = 4\cos 3\theta.$$

5. $$r = 5\cos 2\theta.$$

6. $$r = 2\sin^2\theta.$$

7. $$r = 2\cos^2\theta.$$

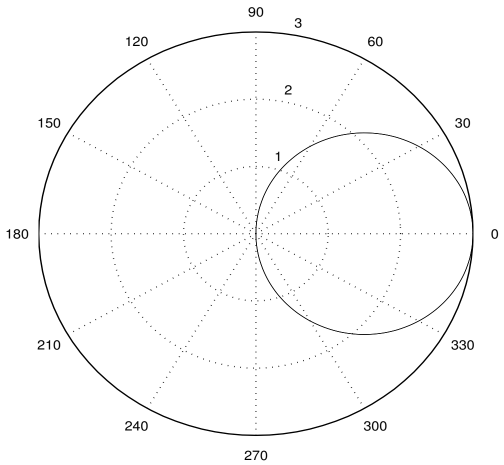
8. $$r^2 = 25\cos 2\theta.$$

9. $$r^2 = 16\sin 2\theta.$$

10. $$r = 2\theta.$$

5.11.4 ANSWERS TO EXERCISES

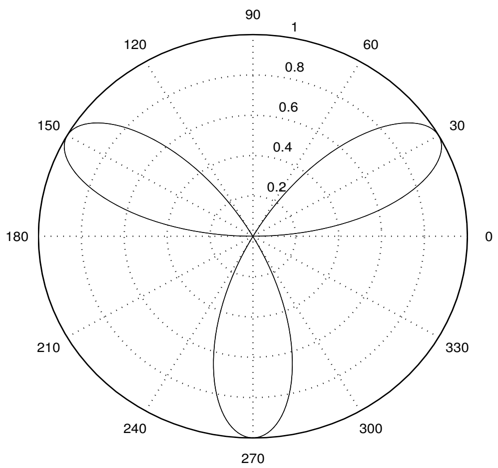
1. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\cos\theta$, which is a circle.

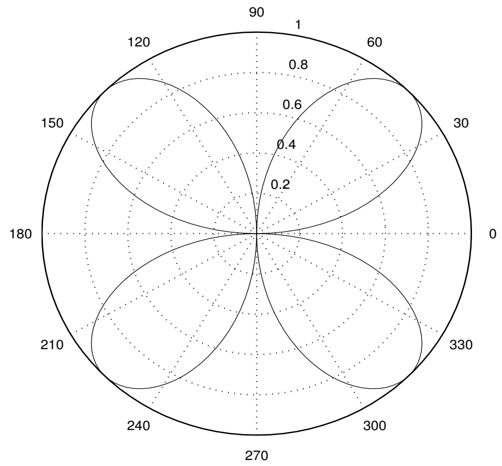
2. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\sin n\theta$, where n is **odd**. It is an **"n-leaved rose"**.

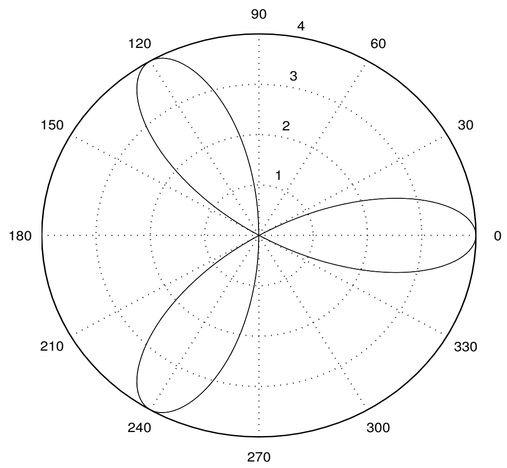
3. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\sin n\theta$, where n is **even**. It is a **"$2n$-leaved rose"**.

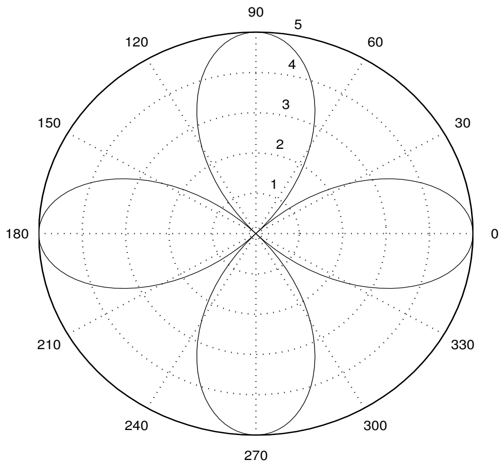
4. The graph is as follows:



Note:

This is an example od the more general curve, $r = a\cos n\theta$, where n is **odd**. It is an **"n-leaved rose"**.

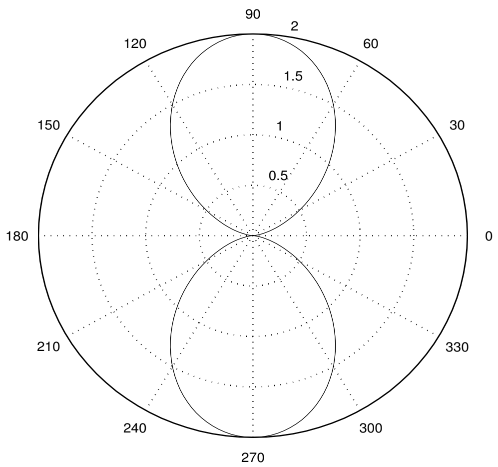
5. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\cos n\theta$, where n is **even**. It is a **"$2n$-leaved rose"**.

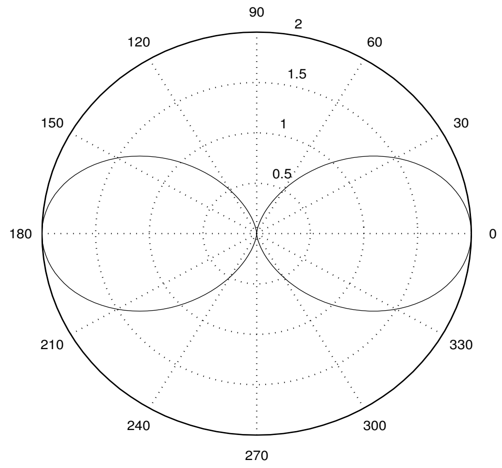
6. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\sin^2\theta$, which is called a **"lemniscate"**.

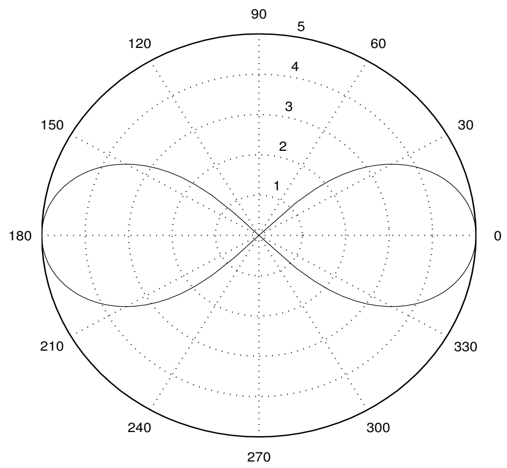
7. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\cos^2\theta$, which is also called a **"lemniscate"**.

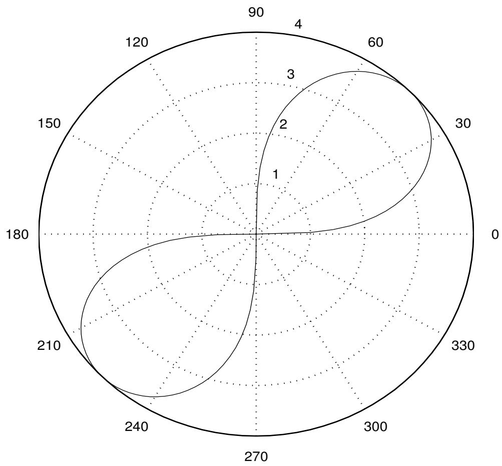
8. The graph is as follows:



Note:

This is an example of the more general curve, $r^2 = a^2\cos 2\theta$. It is another example of a **"lemniscate"**; but, since r^2 cannot be negative, there are no points on the curve in the intervals $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ and $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$.

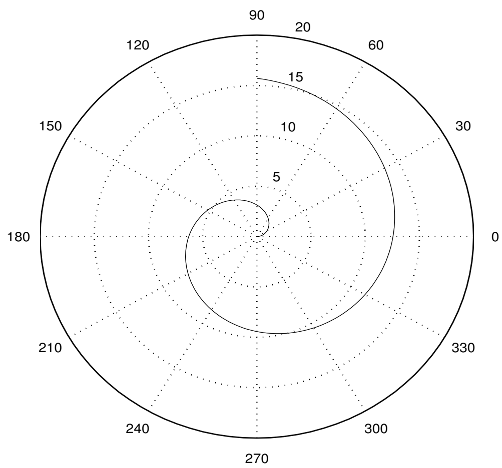
9. The graph is as follows:



Note:

This is an example of the more general curve, $r^2 = a^2 \sin 2\theta$. It is another example of a **"lemniscate"**; but, since r^2 cannot be negative, there are no points on the curve in the intervals $\frac{\pi}{2} < \theta < \pi$ and $\frac{3\pi}{2} < \theta < 2\pi$.

10. The graph is as follows:



Note:

This is an example of the more general curve, $r = a\theta$, $a > 0$, which is called an **"Archimedean spiral"**.

"JUST THE MATHS"

UNIT NUMBER

6.1

COMPLEX NUMBERS 1
(Definitions and algebra)

by

A.J.Hobson

UNIT 6.1 - COMPLEX NUMBERS 1 - DEFINITIONS AND ALGEBRA

6.1.1 THE DEFINITION OF A COMPLEX NUMBER

Students who are already familiar with the Differential Calculus may appreciate that equations of the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

which are called "Differential Equations", have wide-reaching applications in science and engineering. They are particularly applicable to problems involving either electrical circuits or mechanical vibrations.

It is possible to show that, in order to determine a formula (without derivatives) giving the variable y in terms of the variable x, one method is to solve, first, the quadratic equation whose coefficents are a, b and c and whose solutions are therefore

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:
Students who are **not** already familiar with the Differential Calculus should consider only the quadratic equation whose coefficients are a, b and c, ignoring references to differential equations.

ILLUSTRATION

One method of solving the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13 = 2\sin x$$

would be to solve, first, the quadratic equation whose coefficients are 1, -6 and 13.

Its solutions are

$$\frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

which clearly do not exist since we cannot find the square root of a negative number in elementary arithmetic.

However, if we assume that the differential equation represents a genuine scientific problem with a genuine scientific solution, we cannot simply dismiss the result obtained from the quadratic formula.

The difficulty seems to be, not so much with the -16 but with the minus sign in front of the 16. We shall therefore write the solutions in the form

$$\frac{6 \pm 4\sqrt{-1}}{2} = 3 \pm 2\sqrt{-1}.$$

Notes:
(i) The symbol $\sqrt{-1}$ will be regarded as an **"imaginary"** number.

(ii) In mathematical work, $\sqrt{-1}$ is normally denoted by i but, in scientific work it is denoted by j in order to avoid confusion with other quantities (eg. electric current) which could be denoted by the same symbol.

(iii) Whenever the imaginary quantity $j = \sqrt{-1}$ occurs in the solutions of a quadratic equation, those solutions will always be of the form $a + bj$ (or $a + jb$), where a and b are ordinary numbers of elementary arithmetic.

DEFINITIONS

1. The term **"complex number"** is used to denote any expression of the form $a + bj$ or $a + jb$ where a and b are ordinary numbers of elementary arithmetic (including zero) and j denotes the imaginary number $\sqrt{-1}$; i.e. $j^2 = -1$.

2. If the value a happens to be zero, then the complex number $a + bj$ or $a + jb$ is called **"purely imaginary"** and is written bj or jb.

3. If the value b happens to be zero, then the complex number $a + bj$ or $a + jb$ is defined to be the same as the number a and is called **"real"**. That is $a + j0 = a + 0j = a$.

4. For the complex number $a + bj$ or $a + jb$, the value a is called the **"real part"** and the value b is called the **"imaginary part"**. Notice that the imaginary part is b and **not** jb.

5. The complex numbers $a \pm bj$ are said to form a pair of **"complex conjugates"** and similarly $a \pm jb$ form a pair of complex conjugates. Alternatively, we may say, for instance, that $a - jb$ is the complex conjugate of $a + jb$ and $a + jb$ is the complex conjugate of $a - jb$.

Note:
In some work on complex numbers, especially where many complex numbers may be under

discussion at the same time, it is convenient to denote real and imaginary parts by the symbols x and y respectively, rather than a and b. It is also convenient, on some occasions, to denote the whole complex number $x + jy$ by the symbol z in which case the conjugate, $x - jy$, will be denoted by $\overline{z}$.

6.1.2 THE ALGEBRA OF COMPLEX NUMBERS

INTRODUCTION

An **"Algebra"** (coming from the Arabic word AL-JABR) refers to any mathematical system which uses the concepts of equality, addition, subtraction, multiplication and division. For example, the algebra of real numbers is what we normally call **"arithmetic"**; but algebraical concepts can be applied to other mathematical systems of which the system of complex numbers is one.

In meeting a new mathematical system for the first time, the concepts of equality, addition, subtraction, multiplication and division need to be properly defined, and that is the purpose of the present section. In some cases, the definitions are fairly obvious, but need to be made without contradicting ideas already established in the system of real numbers which complex numbers include.

(a) EQUALITY

Unlike a real number, a complex number does not have a "value"; and so the word "equality" must take on a meaning, here, which is different from that used in elementary arithmetic. In fact two complex numbers are defined to be equal if they have the same real part <u>and</u> the same imaginary part.

That is

$$a + jb = c + jd \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d$$

EXAMPLE

Determine x and y such that

$$(2x - 3y) + j(x + 5y) = 11 - j14.$$

Solution

From the definition of equality, we may

EQUATE REAL AND IMAGINARY PARTS.

Thus,

$$\begin{aligned} 2x - 3y &= 11, \\ x + 5y &= -14 \end{aligned}$$

These simultaneous linear equations are satisfied by $x = 1$ and $y = -3$.

(b) ADDITION AND SUBTRACTION

These two concepts are very easily defined. We simply add (or subtract) the real parts and the imaginary parts of the two complex numbers whose sum (or difference) is required.

That is,

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

and

$$(a + jb) - (c + jd) = (a - c) + j(b - d).$$

EXAMPLE

$$(-7 + j2) + (10 - j5) = 3 - j3 = 3(1 - j)$$

and

$$(-7 + j2) - (10 - j5) = -17 + j7.$$

(c) MULTIPLICATION

The definition of multiplication essentially treats j in the same way as any other algebraic symbol, but uses the fact that $j^2 = -1$.

Thus,

$$(a + jb)(c + jd) = (ac - bd) + j(bc + ad);$$

but this is not so much a formula to be learned off-by-heart as a technique to be applied in future examples.

EXAMPLES

1. $$(5 + j9)(2 + j6) = (10 - 54) + j(18 + 30) = -44 + j48.$$

2. $$(3 - j8)(1 + j4) = (3 + 32) + j(-8 + 12) = 35 + j4.$$

3. $$(a + jb)(a - jb) = a^2 + b^2.$$

Note:
The third example above will be useful in the next section. It shows that **the product of a complex number and its complex conjugate is always a real number consisting of the sum of the squares of the real and imaginary parts**.

(d) DIVISION

The objective here is to make a definition which provides the real and imaginary parts of the complex expression

$$\frac{a + jb}{c + jd}.$$

Once again, we make this definition in accordance with what would be obtained algebraically by treating j in the same way as any other algebraic symbol, but using the fact that $j^2 = -1$.

The method is to multiply both the numerator and the denominator of the complex ratio by the conjugate of the denominator giving

$$\frac{a+jb}{c+jd}.\frac{c-jd}{c-jd} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2}.$$

The required definition is thus

$$\frac{a+jb}{c+jd} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2},$$

which, again, is not so much a formula to be learned off-by-heart as a technique to be applied in future examples.

EXAMPLES

1.

$$\frac{5+j3}{2+j7} = \frac{5+j3}{2+j7}.\frac{2-j7}{2-j7}$$

$$= \frac{(10+21)+j(6-35)}{2^2+7^2} = \frac{31-j29}{53}.$$

Hence, the real part is $\frac{31}{53}$ and the imaginary part is $-\frac{29}{53}$.

2.

$$\frac{6+j}{j2-4} = \frac{6+j}{j2-4}.\frac{-j2-4}{-j2-4}$$

$$= \frac{(-24+2)+j(-4-12)}{(-2)^2+(-4)^2} = \frac{-22-j16}{20}.$$

Hence, the real part is $-\frac{22}{20} = -\frac{11}{10}$ and the imaginary part is $-\frac{16}{20} = -\frac{4}{5}$.

6.1.3 EXERCISES

1. Simplify the following:
   (a) j^3; (b) j^4; (c) j^5; (d) j^{15}; (e) j^{22}.
2. If $z_1 = 2 - j5$, $z_2 = 1 + j7$ and $z_3 = -3 - j4$, determine the following in the form $a + jb$:
   (a)
   $$z_1 - z_2 + z_3;$$
   (b)
   $$2z_1 + z_2 - z_3;$$
   (c)
   $$z_1 - (4z_2 - z_3);$$
   (d)
   $$\frac{z_1}{z_2};$$
   (e)
   $$\frac{z_2}{z_3};$$
   (f)
   $$\frac{z_3}{z_1}.$$
3. Determine the values of x and y such that
   $$(3x - 5y) + j(x + 3y) = 20 + j2.$$
4. Determine the real and imaginary parts of the expression
   $$(1 - j3)^2 + j(2 + j5) - \frac{3(4 - j)}{1 - j}.$$
5. If $z \equiv x + jy$ and $\overline{z} \equiv x - jy$ are conjugate complex numbers, determine the values of x and y such that
   $$4z\overline{z} - 3(z - \overline{z}) = 2 + j.$$

6.1.4 ANSWERS TO EXERCISES

1. (a) $-j$; (b) 1; (c) j; (d) $-j$; (e) -1.
2. (a)
$$4 - j16;$$
(b)
$$8 + j;$$
(c)
$$-5 - j37;$$
(d)
$$-0.66 - j0.38;$$
(e)
$$-1.24 - j0.68;$$
(f)
$$0.48 - j0.79$$
3.
$$x = 5 \quad \text{and} \quad y = -1.$$
4. The real part $= -20.5$; the imaginary part $= -8.5$
5.
$$x = \pm\frac{1}{2} \quad \text{and} \quad y = -\frac{1}{2}.$$

"JUST THE MATHS"

UNIT NUMBER

6.2

COMPLEX NUMBERS 2
(The Argand Diagram)

by

A.J.Hobson

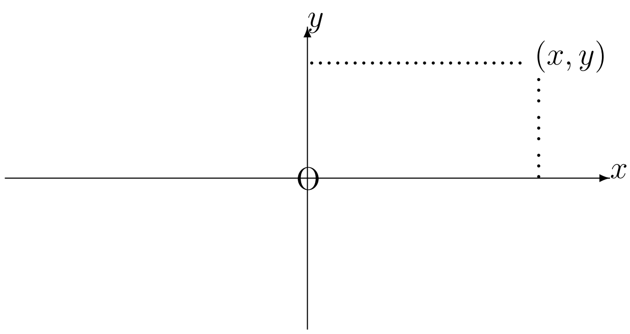
UNIT 6.2 - COMPLEX NUMBERS 2

THE ARGAND DIAGRAM

6.2.1 INTRODUCTION

It may be observed that a complex number $x + jy$ is completely specified if we know the values of x and y in the correct order. But the same is true for the cartesian co-ordinates, (x, y), of a point in two dimesions. There is therefore a **"one-to-one correspondence"** between the complex number $x + jy$ and the point with co-ordinates (x, y).

Hence it is possible to represent the complex number $x+jy$ by the point (x, y) in a geometrical diagram called the Argand Diagram:



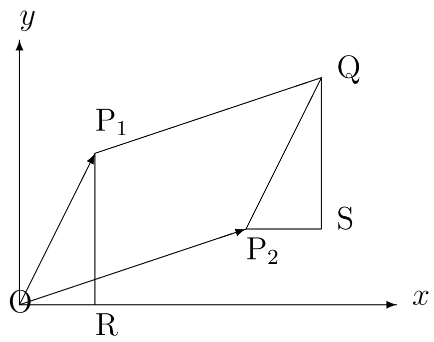
DEFINITIONS:

1. The x-axis is called the **"real axis"** since the points on it represent real numbers.
2. The y-axis is called the **"imaginary axis"** since the points on it represent purely imaginary numbers.

6.2.2 GRAPHICAL ADDITON AND SUBTRACTION

If two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, are represented in the Argand Diagram by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively, then the sum, $z_1 + z_2$, of the complex numbers will be represented by the point $Q(x_1 + x_2, y_1 + y_2)$.

If O is the origin, it is possible to show that Q is the fourth vertex of the parallelogram having OP_1 and OP_2 as adjacent sides.



In the diagram, the triangle ORP_1 has exactly the same shape as the triangle P_2SQ. Hence, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

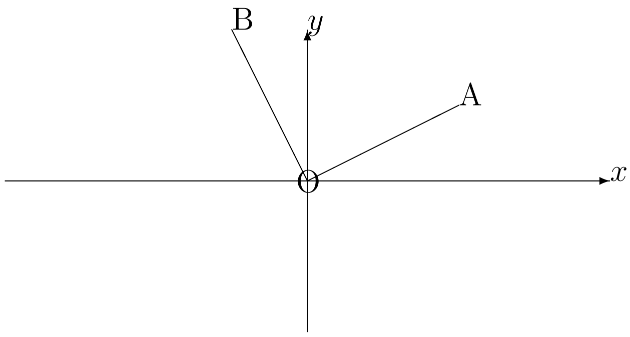
Note:
The difference $z_1 - z_2$ of the two complex numbers may similarly be found by completing the parallelogram of which two adjacent sides are the straight line segments joining the origin to the points with co-ordinates (x_1, y_1) and $(-x_2, -y_2)$.

6.2.3 MULTIPLICATION BY j OF A COMPLEX NUMBER

Given any complex number $z = x + jy$, we observe that

$$jz = j(x + jy) = -y + jx.$$

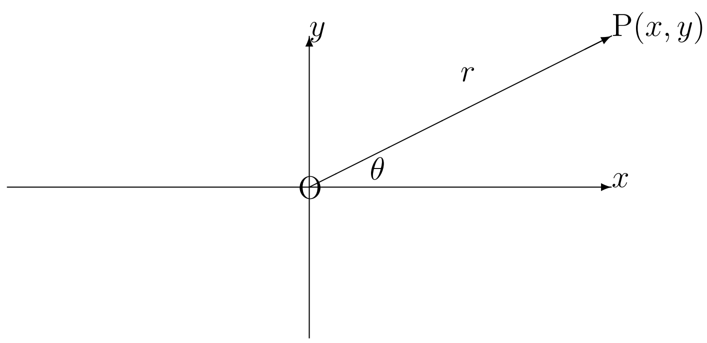
Thus, if z is represented in the Argand Diagram by the point with co-ordinates $A(x, y)$, then jz is represented by the point with co-ordinates $B(-y, x)$.



But OB is in the position which would be occupied by OA if it were rotated through 90° in a counter-clockwise direction.

We conclude that, in the Argand Diagram, multiplication by j of a complex number rotates, through 90° in a counter-clockwise direction, the straight line segment joining the origin to the point representing the complex number.

6.2.4 MODULUS AND ARGUMENT



(a) Modulus

If a complex number, $z = x + jy$ is represented in the Argand Diagram by the point, P, with cartesian co-ordinates (x, y) then the distance, r, of P from the origin is called the **"modulus"** of z and is denoted by either $|z|$ or $|x + jy|$.

Using the theorem of Pythagoras in the diagram, we conclude that

$$r = |z| = |x + jy| = \sqrt{x^2 + y^2}.$$

Note:
This definition of modulus is consistent with the definition of modulus for real numbers (which are included in the system of complex numbers). For any real number x, we may say that

$$|x| = |x + j0| = \sqrt{x^2 + 0^2} = \sqrt{x^2},$$

giving the usual numerical value of x.

ILLUSTRATIONS

1.
$$|3 - j4| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

2.
$$|1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

3.
$$|j7| = |0 + j7| = \sqrt{0^2 + 7^2} = \sqrt{49} = 7.$$

Note:
The result of the last example above is obvious from the Argand Diagram since the point on the y-axis representing $j7$ is a distance of exactly 7 units from the origin. In the same way, a real number is represented by a point on the x-axis whose distance from the origin is the numerical value of the real number.

(b) Argument

The **"argument"** (or **"amplitude"**) of a complex number, z, is defined to be the angle, θ, which the straight line segment OP makes with the positive real axis (measuring θ positively from this axis in a counter-clockwise sense).

In the diagram,

$$\tan\theta = \frac{y}{x}; \quad \text{that is,} \quad \theta = \tan^{-1}\frac{y}{x}.$$

Note:
For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of 360°. The complete set of possible values is denoted by Arg$z$, using an upper-case A.

The particular value of the argument which lies in the interval $-180^\circ < \theta \leq 180^\circ$ is called the **"principal value"** of the argument and is denoted by $\arg z$ using a lower-case a. The particular value, 180°, in preference to -180°, represents the principal value of the argument of a negative real number.

ILLUSTRATIONS

1.

$$\text{Arg}(\sqrt{3} + j) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ + k360^\circ,$$

where k may be any integer. But we note that

$$\arg(\sqrt{3} + j) = 30^\circ \quad \text{only.}$$

2.

$$\text{Arg}(-1 + j) = \tan^{-1}(-1) = 135^\circ + k360^\circ$$

but **not** $-45^\circ + k360^\circ$, since the complex number $-1 + j$ is represented by a point in the second quadrant of the Argand Diagram.

We note also that

$$\arg(-1 + j) = 135^\circ \quad \text{only.}$$

3.

$$\text{Arg}(-1 - j) = \tan^{-1}(1) = 225^\circ + k360^\circ \quad \text{or} \quad -135^\circ + k360^\circ$$

but **not** $45^\circ + k360^\circ$ since the complex number $-1 - j$ is represented by a point in the third quadrant of the Argand Diagram.

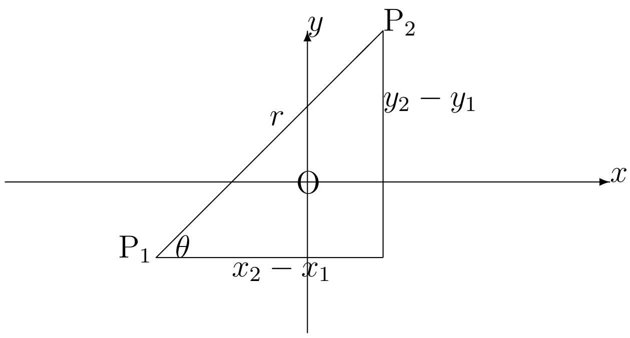
We note also that

$$\arg(-1 - j) = -135^\circ \quad \text{only.}$$

Note:
It is worth mentioning here that, in the Argand Diagram, the directed straight line segment described from the point P_1 (representing the complex number $z_1 = x_1 + jy_1$) to the point P_2 (representing the complex number $z_2 = x_2 + jy_2$) has length, r, equal to $|z_2 - z_1|$, and is inclined to the positive direction of the real axis at an angle, θ, equal to $\arg(z_2 - z_1)$. This follows from the relationship

$$z_2 - z_1 = (x_2 - x_1) + j(y_2 - y_1)$$

in which $x_2 - x_1$ and $y_2 - y_1$ are the distances separating the two points, parallel to the real axis and the imaginary axis respectively.



6.2.5 EXERCISES

1. Determine the modulus (in decimals, where appropriate, correct to three significant figures) and the principal value of the argument (in degrees, correct to the nearest degree) of the following complex numbers:

   (a)
   $$1 - j;$$

   (b)
   $$-3 + j4;$$

   (c)
   $$-\sqrt{2} - j\sqrt{2};$$

(d)

$$\frac{1}{2} - j\frac{\sqrt{3}}{2};$$

(e)

$$-7 - j9.$$

2. If $z = 4 - j5$, verify that jz has the same modulus as z but that the principal value of the argument of jz is greater, by 90° than the principal value of the argument of z.

3. Illustrate the following statements in the Argand Diagram:

   (a)

$$(6 - j11) + (5 + j3) = 11 - j8;$$

   (b)

$$(6 - j11) - (5 + j3) = -1 - j14.$$

6.2.6 ANSWERS TO EXERCISES

1. (a) 1.41 and -45°;
   (b) 5 and 127°;
   (c) 2 and -135°;
   (d) 1 and -60°;
   (e) 11.4 and -128°.
2. $4 - j5$ has modulus $\sqrt{41}$ and argument -51°;
   $j(4 - j5) = 5 + j4$ has modulus $\sqrt{41}$ and argument $39^\circ = -51^\circ + 90^\circ$.
3. Construct the graphical sum and difference of the two complex numbers.

"JUST THE MATHS"

UNIT NUMBER

6.3

COMPLEX NUMBERS 3
(The polar & exponential forms)

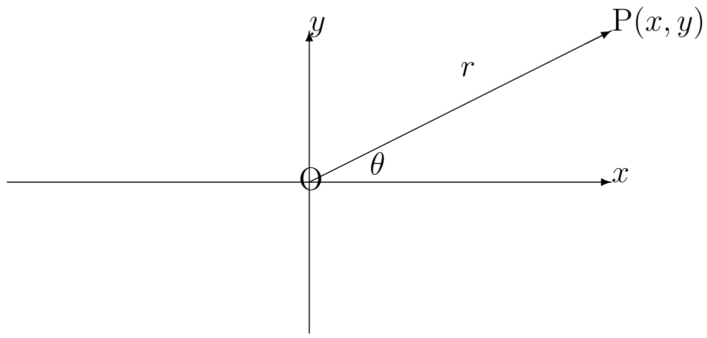
by

A.J.Hobson

UNIT 6.3 - COMPLEX NUMBERS 3

THE POLAR AND EXPONENTIAL FORMS

6.3.1 THE POLAR FORM



From the above diagram, we may observe that

$$\frac{x}{r} = \cos\theta \ \text{ and } \ \frac{y}{r} = \sin\theta.$$

Hence, the relationship between x, y, r and θ may also be stated in the form

$$x = r\cos\theta, \quad y = r\sin\theta,$$

which means that the complex number $x + jy$ may be written as $r\cos\theta + jr\sin\theta$. In other words,

$$x + jy = r(\cos\theta + j\sin\theta).$$

The left-hand-side of this relationship is called the **"rectangular form"** or **"cartesian form"** of the complex number while the right-hand-side is called the **"polar form"**.

Note:
For convenience, the polar form may be abbreviated to $r\angle\theta$, where θ may be positive, negative or zero and may be expressed in either degrees or radians.

EXAMPLES

1. Express the complex number $z = \sqrt{3} + j$ in polar form.
   Solution

$$|z| = r = \sqrt{3+1} = 2$$

   and

$$\text{Arg}z = \theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ + k360^\circ,$$

   where k may be any integer.
   Alternatively, using radians,

$$\text{Arg}z = \frac{\pi}{6} + k2\pi,$$

   where k may be any integer.
   Hence, in polar form,

$$z = 2(\cos[30^\circ + k360^\circ] + j\sin[30^\circ + k360^\circ]) = 2\angle[30^\circ + k360^\circ]$$

   or

$$z = 2\left(\cos\left[\frac{\pi}{6} + k2\pi\right] + j\sin\left[\frac{\pi}{6} + k2\pi\right]\right) = 2\angle\left[\frac{\pi}{6} + k2\pi\right].$$

2. Express the complex number $z = -1 - j$ in polar form.
   Solution

$$|z| = r = \sqrt{1+1} = \sqrt{2}$$

   and

$$\text{Arg}z = \theta = \tan^{-1}(1) = -135^\circ + k360^\circ,$$

   where k may be any integer.

Alternatively,

$$\text{Arg}z = -\frac{3\pi}{4} + k2\pi,$$

where k may be any integer.
Hence, in polar form,

$$z = \sqrt{2}(\cos[-135^\circ + k360^\circ] + j\sin[-135^\circ + k360^\circ]) = \sqrt{2}\angle[-135^\circ + k360^\circ]$$

or

$$z = \sqrt{2}\left(\cos\left[-\frac{3\pi}{4} + k2\pi\right] + j\sin\left[-\frac{3\pi}{4} + k2\pi\right]\right) = \sqrt{2}\angle\left[-\frac{3\pi}{4} + k2\pi\right].$$

Note:
If it is required that the polar form should contain only the **principal** value of the argument, θ, then, provided $-180^\circ < \theta \leq 180^\circ$ or $-\pi < \theta \leq \pi$, the component $k360^\circ$ or $k2\pi$ of the result is simply omitted.

6.3.2 THE EXPONENTIAL FORM

Using some theory from the differential calculus of complex variables (not included here) it is possible to show that, for any complex number, z,

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

These are, in fact, taken as the **definitions** of the functions $e^z, \sin z$ and $\cos z$.

Students who are already familiar with the differential calculus of a real variable, x, may recognise similarities between the above formulae and the "MacLaurin Series" for the functions $e^x, \sin x$ and $\cos x$. In the case of the series for $\sin x$ and $\cos x$, the value, x, must be expressed in **radians and not degrees**.

A useful deduction can be made from the three formulae if we make the substitution $z = j\theta$ into the first one, obtaining:

$$e^{j\theta} = 1 + \frac{j\theta}{1!} + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

and, since $j^2 = -1$, this gives

$$e^{j\theta} = 1 + j\frac{\theta}{1!} - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ...$$

On regrouping this into real and imaginary parts, then using the sine and cosine series, we obtain

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

provided θ is expressed in radians and not degrees.

The complex number $x+jy$, having modulus r and argument $\theta+k2\pi$, may thus be expressed not only in polar form but also in

the exponential form, $re^{j\theta}$.

ILLUSTRATIONS

Using the examples of the previous section

1.
$$\sqrt{3} + j = 2e^{j\left(\frac{\pi}{6}+k2\pi\right)}.$$

2.
$$-1 + j = \sqrt{2}e^{j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

3.
$$-1 - j = \sqrt{2}e^{-j\left(\frac{3\pi}{4}+k2\pi\right)}.$$

Note:
If it is required that the exponential form should contain only the **principal** value of the argument, θ, then, provided $-\pi < \theta \leq \pi$, the component $k2\pi$ of the result is simply omitted.

6.3.3 PRODUCTS AND QUOTIENTS IN POLAR FORM

Let us suppose that two complex numbers z_1 and z_2 have already been expressed in polar form, so that

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) = r_1\angle\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) = r_2\angle\theta_2.$$

It is then possible to establish very simple rules for determining both the product and the quotient of the two complex numbers. The explanation is as follows:

(a) The Product

$$z_1.z_2 = r_1.r_2(\cos\theta_1 + j\sin\theta_1).(\cos\theta_2 + j\sin\theta_2).$$

That is,

$$z_1.z_2 = r_1.r_2([\cos\theta_1.\cos\theta_2 - \sin\theta_1.\sin\theta_2] + j[\sin\theta_1.\cos\theta_2 + \cos\theta_1.\sin\theta_2]).$$

Using trigonometric identities, this reduces to

$$z_1.z_2 = r_1.r_2(\cos[\theta_1 + \theta_2] + j\sin[\theta_1 + \theta_2]) = r_1.r_2\angle[\theta_1 + \theta_2].$$

We have shown that, to determine the product of two complex numbers in polar form, we construct the product of their modulus values and the sum of their argument values.

(b) The Quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + j\sin\theta_1)}{(\cos\theta_2 + j\sin\theta_2)}.$$

On multiplying the numerator and denominator by $\cos\theta_2 - j\sin\theta_2$, we obtain

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}([\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2] + j[\sin\theta_1.\cos\theta_2 - \cos\theta_1.\sin\theta_2]).$$

Using trigonometric identities, this reduces to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos[\theta_1 - \theta_2] + j\sin[\theta_1 - \theta_2]) = \frac{r_1}{r_2}\angle[\theta_1 - \theta_2].$$

We have shown that, to determine the quotient of two complex numbers in polar form, we construct the quotient of their modulus values and the difference of their argument values.

ILLUSTRATIONS

Using results from earlier examples:

1.
$$(\sqrt{3} + j).(-1 - j) = 2\angle 30^\circ.\sqrt{2}\angle(-135^\circ) = 2\sqrt{2}\angle(-105^\circ).$$

   We notice that, for all of the complex numbers in this example, including the result, the argument appears as the principal value.

2.
$$\frac{\sqrt{3} + j}{-1 - j} = \frac{2\angle 30^\circ}{\sqrt{2}\angle(-135^\circ)} = \sqrt{2}\angle 165^\circ.$$

   Again, for all of the complex numbers in this example, including the result, the argument appears as the principal value.

   Note:
   It will not always turn out that the argument of a product or quotient of two complex numbers appears as the principal value. For instance,

3.
$$(-1 - j).(-\sqrt{3} - j) = \sqrt{2}\angle(-135^\circ).2\angle(-150^\circ) = 2\sqrt{2}\angle(-285^\circ),$$

   which must be converted to $2\sqrt{2}\angle(75^\circ)$ if the principal value of the argument is required.

6.3.4 EXERCISES

In the following cases, express the complex numbers z_1 and z_2 in

(a) the polar form, $r\angle\theta$

and

(b) the exponential form, $re^{j\theta}$

using only the principal value of θ.

(c) For each case, determine also the product, $z_1.z_2$, and the quotient, $\frac{z_1}{z_2}$, in polar form using only the principal value of the argument.

1.
$$z_1 = 1 + j, \quad z_2 = \sqrt{3} - j.$$

2.
$$z_1 = -\sqrt{2} + j\sqrt{2}, \quad z_2 = -3 - j4.$$

3.
$$z_1 = -4 - j5, \quad z_2 = 7 - j9.$$

6.3.5 ANSWERS TO EXERCISES

1. (a)
$$z_1 = \sqrt{2}\angle 45^\circ \quad z_2 = 2\angle(-30^\circ);$$

(b)
$$z_1 = \sqrt{2}e^{j\frac{\pi}{4}} \quad z_2 = 2e^{-j\frac{\pi}{6}};$$

(c)
$$z_1.z_2 = 2\sqrt{2}\angle 15^\circ \quad \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\angle 75^\circ.$$

2. (a)
$$z_1 = 2\angle(135^\circ) \quad z_2 = 5\angle(-127^\circ);$$

(b)

$$z_1 = 2e^{j\frac{3\pi}{4}} \quad z_2 = 5e^{-j2.22};$$

(c)

$$z_1.z_2 = 10\angle 8^\circ \quad \frac{z_1}{z_2} = \frac{2}{5}\angle(-98^\circ).$$

3. (a)

$$z_1 = 6.40\angle(-128.66^\circ) \quad z_2 = 11.40\angle(-55.13^\circ);$$

(b)

$$z_1 = 6.40e^{-j2.25} \quad z_2 = 11.40e^{-j0.96};$$

(c)

$$z_1.z_2 = 72.96\angle 176.21^\circ \quad \frac{z_1}{z_2} = 0.56\angle(-73.53^\circ).$$

"JUST THE MATHS"

UNIT NUMBER

6.4

COMPLEX NUMBERS 4
(Powers of complex numbers)

by

A.J.Hobson

UNIT 6.4 - COMPLEX NUMBERS 4

POWERS OF COMPLEX NUMBERS

6.4.1 POSITIVE WHOLE NUMBER POWERS

As an application of the rule for multiplying together complex numbers in polar form, it is a simple matter to multiply a complex number by itself any desired number of times.

Suppose that

$$z = r\angle\theta.$$

Then,

$$z^2 = r.r\angle(\theta + \theta) = r^2\angle 2\theta;$$

$$z^3 = z.z^2 = r.r^2\angle(\theta + 2\theta) = r^3\angle 3\theta;$$

and, by continuing this process,

$$z^n = r^n\angle n\theta.$$

This result is due to De Moivre, but other aspects of it will need to be discussed before we may formalise what is called **"De Moivre's Theorem"**.

EXAMPLE

$$\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)^{19} = (1\angle\left[\frac{\pi}{4}\right])^{19} = 1\angle\left[\frac{19\pi}{4}\right] = 1\angle\left[\frac{3\pi}{4}\right] = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}.$$

6.4.2 NEGATIVE WHOLE NUMBER POWERS

If n is a <u>negative</u> whole number, we shall suppose that

$$n = -m,$$

where m is a <u>positive</u> whole number.

Thus, if $z = r\angle\theta$,

$$z^n = z^{-m} = \frac{1}{z^m} = \frac{1}{r^m \angle m\theta}.$$

In more detail,

$$z^n = \frac{1}{r^m(\cos m\theta + j\sin m\theta)},$$

giving

$$z^n = \frac{1}{r^m} . \frac{(\cos m\theta - j\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}(\cos[-m\theta] + j\sin[-m\theta]).$$

But $-m = n$, and so

$$z^n = r^n(\cos n\theta + j\sin n\theta) = r^n \angle n\theta,$$

showing that the result of the previous section remains true for negative whole number powers.

EXAMPLE

$$(\sqrt{3} + j)^{-3} = (2\angle 30^\circ)^{-3} = \frac{1}{8}\angle(-90^\circ) = -\frac{j}{8}.$$

6.4.3 FRACTIONAL POWERS AND DE MOIVRE'S THEOREM

To begin with, here, we consider the complex number

$$z^{\frac{1}{n}},$$

where n is a positive whole number and $z = r\angle\theta$.

We define $z^{\frac{1}{n}}$ to be any complex number which gives z itself when raised to the power n. Such a complex number is called **"an n-th root of z"**.

Certainly one such possibility is

$$r^{\frac{1}{n}} \angle \frac{\theta}{n},$$

by virtue of the paragraph dealing with positive whole number powers.

But the general expression for z is given by

$$z = r\angle(\theta + k360^\circ),$$

where k may be any integer; and this suggests other possibilities for $z^{\frac{1}{n}}$, namely

$$r^{\frac{1}{n}} \angle \frac{\theta + k360^\circ}{n}.$$

However, this set of n-th roots is not an infinite set because the roots which are given by $k = 0, 1, 2, 3$............$n - 1$ are also given by $k = n, n+1, n+2, n+3,$ ...., $2n-1, 2n,$ $2n+1, 2n+2, 2n+3,$ ..... and so on, respectively.

We conclude that there are precisely n n-th roots given by $k = 0, 1, 2, 3$........., $n - 1$.

EXAMPLE

Determine the cube roots (i.e. 3rd roots) of the complex number $j8$.

Solution

We first write

$$j8 = 8\angle(90^\circ + k360^\circ).$$

Hence,

$$(j8)^{\frac{1}{3}} = 8^{\frac{1}{3}} \angle \frac{(90^\circ + k360^\circ)}{3},$$

where $k = 0, 1, 2$

The three distinct cube roots are therefore

$$2\angle 30^\circ, 2\angle 150^\circ \text{ and } 2\angle 270^\circ = 2\angle(-90^\circ).$$

They all have the same modulus of 2 but their arguments are spaced around the Argand Diagram at regular intervals of $\frac{360^\circ}{3} = 120^\circ$.

Notes:

(i) In general, the n-th roots of a complex number will all have the same modulus, but their arguments will be spaced at regular intervals of $\frac{360^\circ}{n}$.

(ii) Assuming that $-180^\circ < \theta \leq 180^\circ$; that is, assuming that the polar form of z uses the principal value of the argument, then the particular n-th root of z which is given by $k = 0$ is called the **"principal n-th root"**.

(iii) If $\frac{m}{n}$ is a fraction in its lowest terms, we define

$$z^{\frac{m}{n}}$$

to be either $\left(z^{\frac{1}{n}}\right)^m$ or $(z^m)^{\frac{1}{n}}$ both of which turn out to give the same set of n distinct results.

The discussion, so far, on powers of complex numbers leads us to the following statement:

DE MOIVRE'S THEOREM

If $z = r\angle\theta$, then, for any rational number n, **one value** of z^n is $r^n\angle n\theta$.

6.4.4 EXERCISES

1. Determine the following in the form $a + jb$, expressing a and b in decimals correct to four significant figures:

   (a)
   $$(1 + j\sqrt{3})^{10};$$
   (b)
   $$(2 - j5)^{-4}.$$

2. Determine the fourth roots of $j81$ in exponential form $re^{j\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$.

3. Determine the fifth roots of the complex number $-4 + j4$ in the form $a + jb$ expressing a and b in decimals, where appropriate, correct to two places. State also which root is the principal root.

4. Determine all the values of

$$(3 + j4)^{\frac{3}{2}}$$

in polar form.

6.4.5 ANSWERS TO EXERCISES

1. (a)

$$(1 + j\sqrt{3})^{10} = -512.0 - j886.8;$$

(b)

$$(2 - j5)^{-4} = 5.796 - j1.188$$

2. The fourth roots are

$$3e^{-\frac{\pi}{8}},\ \ 3e^{\frac{3\pi}{8}},\ \ 3e^{\frac{7\pi}{8}},\ \ 3e^{-\frac{5\pi}{8}}.$$

3. The fifth roots are

$$1.26 + j0.64, \quad -0.22 + j1.40, \quad -1.40 + j0.22, \quad -0.64 - j1.26, \quad 1 - j.$$

The principal root is $1.26 + j0.64$.

4. There are two values, namely

$$11.18\angle 79.695^\circ \ \text{ and } \ 11.18\angle(-100.305^\circ).$$

"JUST THE MATHS"

UNIT NUMBER

6.5

COMPLEX NUMBERS 5
(Applications to trigonometric identities)

by

A.J.Hobson

UNIT 6.5 - COMPLEX NUMBERS 5

APPLICATIONS TO TRIGONOMETRIC IDENTITIES

6.5.1 INTRODUCTION

It will be useful for the purposes of this section to restate the result known as **"Pascal's Triangle"** previously discussed in Unit 2.2.

If n is a positive whole number, the diagram

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

provides the coefficients in the expansion of $(A + B)^n$ which contains the sequence of terms

$$A^n,\ A^{n-1}B,\ A^{n-2}B^2,\ A^{n-3}B^3, ..., B^n.$$

6.5.2 EXPRESSIONS FOR $\cos n\theta$ AND $\sin n\theta$ IN TERMS OF $\cos\theta$ AND $\sin\theta$.

From De Moivre's Theorem

$$(\cos\theta + j\sin\theta)^n \equiv \cos n\theta + j\sin n\theta,$$

from which we may deduce that, in the expansion of the left-hand-side, using Pascal's Triangle, the real part will coincide with $\cos n\theta$ and the imaginary part will coincide with $\sin n\theta$.

EXAMPLE

$$(\cos\theta + j\sin\theta)^3 \equiv \cos^3\theta + 3\cos^2\theta.(j\sin\theta) + 3\cos\theta.(j\sin\theta)^2 + (j\sin\theta)^3.$$

That is,

$$\cos 3\theta \equiv \cos^3\theta - 3\cos\theta.\sin^2\theta \quad \text{or} \quad 4\cos^3\theta - 3\cos\theta,$$

using $\sin^2\theta \equiv 1 - \cos^2\theta$;

and

$$\sin 3\theta \equiv 3\cos^2\theta . \sin\theta - \sin^3\theta \quad \text{or} \quad 3\sin\theta - 4\sin^3\theta,$$

using $\cos^2\theta \equiv 1 - \sin^2\theta$.

6.5.3 EXPRESSIONS FOR $\cos^n\theta$ AND $\sin^n\theta$ IN TERMS OF SINES AND COSINES OF WHOLE MULTIPLES OF θ.

The technique described here is particularly useful in calculus problems when we are required to integrate an integer power of a sine function or a cosine function. It does stand, however, as a self-contained application to trigonometry of complex numbers.

Suppose

$$z \equiv \cos\theta + j\sin\theta \quad - \quad (1)$$

Then, by De Moivre's Theorem, or by direct manipulation,

$$\frac{1}{z} \equiv \cos\theta - j\sin\theta \quad - \quad (2).$$

Adding (1) and (2) together, then subtracting (2) from (1), we obtain

$z + \frac{1}{z} \equiv 2\cos\theta$	$z - \frac{1}{z} \equiv j2\sin\theta$

Also, by De Moivre's Theorem,

$$z^n \equiv \cos n\theta + j\sin n\theta \quad - \quad (3)$$

and

$$\frac{1}{z^n} \equiv \cos n\theta - j\sin n\theta \quad - \quad (4).$$

Adding (3) and (4) together, then subtracting (4) from (3), we obtain

$z^n + \frac{1}{z^n} \equiv 2\cos n\theta$	$z^n - \frac{1}{z^n} \equiv j2\sin n\theta$

We are now in a position to discuss some examples on finding trigonometric identities for whole number powers of $\sin\theta$ or $\cos\theta$.

EXAMPLES

1. Determine an identity for $\sin^3\theta$.

   Solution

   We use the result

$$j^3 2^3 \sin^3\theta \equiv \left(z - \frac{1}{z}\right)^3,$$

   where $z \equiv \cos\theta + j\sin\theta$.

   That is,

$$-j8\sin^3\theta \equiv z^3 - 3z^2.\frac{1}{z} + 3z.\left(\frac{1}{z}\right)^2 - \frac{1}{z^3}$$

   or, after cancelling common factors,

$$-j8\sin^3\theta \equiv z^3 - 3z + \frac{3}{z} - \frac{1}{z^3} \equiv \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right),$$

   which gives

$$-j8\sin^3\theta \equiv j2\sin 3\theta - j6\sin\theta.$$

   Hence,

$$\sin^3\theta \equiv \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right).$$

2. Determine an identity for $\cos^4\theta$.

Solution

We use the result

$$2^4\cos^4\theta \equiv \left(z + \frac{1}{z}\right)^4,$$

where $z \equiv \cos\theta + j\sin\theta$.

That is,

$$16\cos^4\theta \equiv z^4 + 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 + 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

or, after cancelling common factors,

$$16\cos^4\theta \equiv z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\cos^4\theta \equiv 2\cos 4\theta + 8\cos 2\theta + 6.$$

Hence,

$$\cos^4\theta \equiv \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$$

6.5.4 EXERCISES

1. Use a complex number method to determine identities for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.
2. Use a complex number method to determine an identity for $\sin^5\theta$ in terms of sines of whole multiples of θ.
3. Use a complex number method to deternine an identity for $\cos^6\theta$ in terms of cosines of whole multiples of θ.

6.5.5 ANSWERS TO EXERCISES

1.

$$\cos 4\theta \equiv \cos^4\theta - 6\cos^2\theta.\sin^2\theta$$

and

$$\sin 4\theta \equiv 4\cos^3\theta.\sin\theta - 4\cos\theta.\sin^3\theta.$$

2.

$$\sin^5\theta \equiv \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta].$$

3.

$$\cos^6\theta \equiv \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

"JUST THE MATHS"

UNIT NUMBER

6.6

COMPLEX NUMBERS 6
(Complex loci)

by

A.J.Hobson

UNIT 6.6 - COMPLEX NUMBERS 6

COMPLEX LOCI

6.6.1 INTRODUCTION

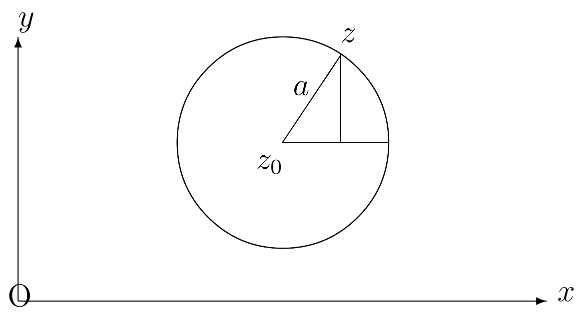
In Unit 6.2, it was mentioned that the directed line segment joining the point representing a complex number z_1 to the point representing a complex number z_2 is of length equal to $|z_2 - z_1|$ and is inclined to the positive direction of the real axis at an angle equal to $\arg(z_2 - z_1)$.

This observation now has significance when discussing variable complex numbers which are constrained to move along a certain path (or **"locus"**) in the Argand Diagram. For many practical applications, such paths (or **"loci"**) will normally be either straight lines or circles and two standard types of example appear in what follows.

In both types, we shall assume that $z = x + jy$ denotes a **variable** complex number (represented by the point (x, y) in the Argand Diagram), while $z_0 = x_0 + jy_0$ denotes a **fixed** complex number (represented by the point (x_0, y_0) in the Argand Diagram).

6.6.2 THE CIRCLE

Suppose that the moving point representing z moves on a circle, with radius a, whose centre is at the fixed point representing z_0.



Then the distance between these two points will always be equal to a. In other words,

$$|z - z_0| = a$$

and this is the standard equation of the circle in terms of complex numbers.

Note:
By substituting $z = x + jy$ and $z_0 = x_0 + jy_0$ in the above equation, we may obtain the equivalent equation in terms of cartesian co-ordinates, namely,

$$|(x - x_0) + j(y - y_0)| = a.$$

That is,

$$(x - x_0)^2 + (y - y_0)^2 = a^2.$$

ILLUSTRATION

The equation

$$|z - 3 + j4| = 7$$

represents a circle, with radius 7, whose centre is the point representing the complex number $3 - j4$.

In cartesian co-ordinates, it is the circle with equation

$$(x - 3)^2 + (y + 4)^2 = 49.$$

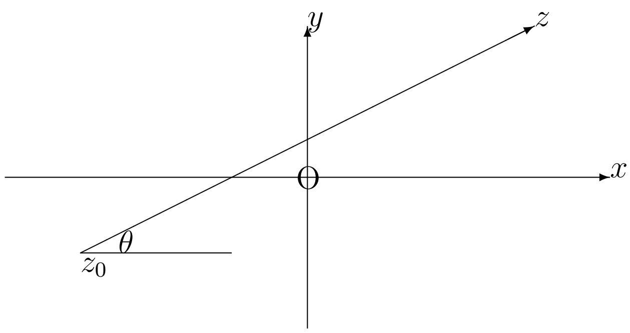
6.6.3 THE HALF-STRAIGHT-LINE

Suppose now that the **"directed"** straight line segment described **from** the fixed point representing z_0 **to** the moving point representing z is inclined at an angle θ to the positive direction of the real axis.

Then,

$$\arg(z - z_0) = \theta$$

and this equation is satisfied by **all** of the values of z for which the inclination of the directed line segment is genuinely θ and **not** $180° - \theta$. The latter angle would correspond to points on the other half of the straight line joining the two points.



Note:
If we substitute $z = x + jy$ and $z_0 = x_0 + jy_0$, we obtain

$$\arg([x - x_0] + j[y - y_0]) = \theta.$$

That is,

$$\tan^{-1}\frac{y - y_0}{x - x_0} = \theta$$

or

$$y - y_0 = \tan\theta(x - x_0),$$

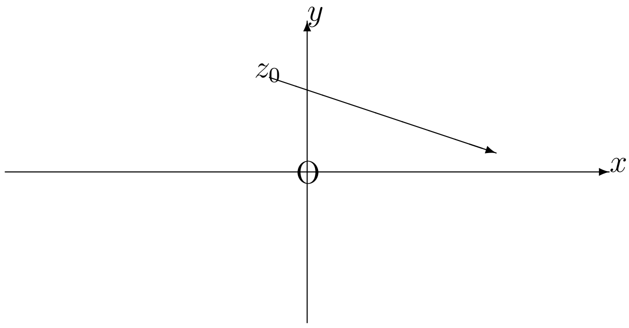
which is certainly the equation of a straight line with gradient $\tan\theta$ passing through the point (x_0, y_0); but it represents only that half of the straight line for which $x - x_0$ and $y - y_0$ correspond, in sign as well as value, to the real and imaginary parts of a complex number whose argument is genuinely θ and not $180° - \theta$.

ILLUSTRATION

The equation

$$\arg(z + 1 - j5) = -\frac{\pi}{6}$$

represents the half-straight-line described from the point representing $z_0 = -1 + j5$ to the point representing $z = x + jy$ and inclined to the positive direction of the real axis at an angle of $-\frac{\pi}{6}$.



In terms of cartesian co-ordinates,

$$\arg([x + 1] + j[y - 5]) = -\frac{\pi}{6},$$

in which it must be true that $x + 1 > 0$ and $y - 5 < 0$ in order that the argument of $[x + 1] + j[y - 5]$ may be a negative acute angle.

We thus have the half-straight-line with equation

$$y - 5 = \tan\left(-\frac{\pi}{6}\right)(x + 1) = -\frac{1}{\sqrt{3}}(x + 1)$$

which lies to the right of, and below the point $(-1, 5)$.

6.6.4 MORE GENERAL LOCI

Certain types of locus problem may be encountered which cannot be identified with either of the two standard types discussed above. The secret, in such problems is to substitute $z = x + jy$ in order to obtain the cartesian equation of the locus. We have already seen that this method is applicable to the two standard types anyway.

ILLUSTRATIONS

1. The equation

$$\left|\frac{z-1}{z+2}\right| = 3$$

may be written

$$|z-1| = 3|z+2|.$$

That is,

$$(x-1)^2 + y^2 = 3[(x+2)^2 + y^2],$$

which simplifies to

$$2x^2 + 2y^2 + 14x + 13 = 0$$

or

$$\left(x + \frac{7}{2}\right)^2 + y^2 = \frac{23}{4},$$

representing a circle with centre $\left(-\frac{7}{2}, 0\right)$ and radius $\sqrt{\frac{23}{4}}$.

2. The equation

$$\arg\left(\frac{z-3}{z}\right) = \frac{\pi}{4}$$

may be written

$$\arg(z-3) - \arg z = \frac{\pi}{4}.$$

That is,

$$\arg([x-3]+jy)-\arg(x+jy)=\frac{\pi}{4}.$$

Taking tangents of both sides and using the trigonometric identity for $\tan(A-B)$, we obtain

$$\frac{\frac{y}{x-3}-\frac{y}{x}}{1+\frac{y}{x-3}\frac{y}{x}}=1.$$

On simplification, the equation becomes

$$x^2+y^2-3x-3y=0$$

or

$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{9}{2},$$

the equation of a circle with centre $\left(\frac{3}{2},\frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$.
However, we observe that the original complex number,

$$\frac{z-3}{z},$$

cannot have an argument of $\frac{\pi}{4}$ unless its real and imaginary parts are **both** positive. In fact,

$$\frac{z-3}{z}=\frac{(x-3)+jy}{x+jy}.\frac{x-jy}{x-jy}=\frac{x(x-3)+y^2+j3}{x^2+y^2}$$

which requires, therefore, that

$$x(x-3)+y^2>0.$$

That is,

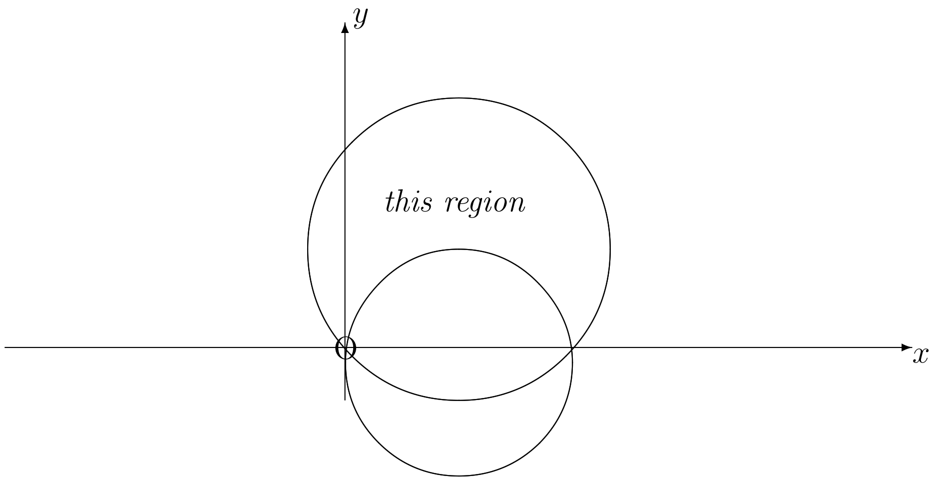
$$x^2+y^2-3x>0$$

or

$$\left(x-\frac{3}{2}\right)^2+y^2>\frac{9}{4}.$$

Conclusion

The locus is that part of the circle with centre $\left(\frac{3}{2},\frac{3}{2}\right)$ and radius $\frac{3}{\sqrt{2}}$ which lies **outside** the circle with centre $\left(\frac{3}{2},0\right)$ and radius $\frac{3}{2}$.



6.6.5 EXERCISES

1. Identify the loci whose equations are

   (a)
   $$|z-3|=4;$$
   (b)
   $$|z-4+j7|=2.$$

2. Identify the loci whose equations are

   (a)
   $$\arg(z+1)=\frac{\pi}{3};$$
   (b)
   $$\arg(z-2-j3)=\frac{3\pi}{2}.$$

3. Identify the loci whose equations are

   (a)
   $$\left|\frac{z+j2}{z-j3}\right|=1;$$
   (b)
   $$\arg\left(\frac{z+j}{z-1}\right)=-\frac{\pi}{4}.$$

6.6.6 ANSWERS TO EXERCISES

1. (a) A circle with centre $(3, 0)$ and radius 4;

   (b) A circle with centre $4, -7)$ and radius 2.

2. (a) A half-straight-line to the right of, and above the point $(-1, 0)$ inclined at an angle of $\frac{\pi}{3}$ to the positive direction of the real axis;

   (b) A half-straight-line below the point $(2, 3)$ and perpendicular to the real axis.

3. (a) The straight line $y = \frac{1}{2}$;

   (b) That part of the circle $x^2 + y^2 = 1$ which lies outside the circle with centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$ **and** above the straight line whose equation is $y = x - 1$.

Note:
Examples like No. 3(b) are often quite difficult and will not normally be included in the more elementary first year courses in mathematics.

"JUST THE MATHS"

UNIT NUMBER

7.1

DETERMINANTS 1
(Second order determinants)

by

A.J.Hobson

UNIT 7.1 - DETERMINANTS 1

SECOND ORDER DETERMINANTS

7.1.1 PAIRS OF SIMULTANEOUS LINEAR EQUATIONS

The subject of Determinants may be introduced by considering, first, a set of two simultaneous linear equations in two unknowns. We shall take them in the form

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0. - - - - - - - - -(2) \end{aligned}$$

If we subtract equation (2) $\times$ b_1 from equation (1) $\times$ b_2, we obtain

$$a_1b_2x - a_2b_1x + c_1b_2 - c_2b_1 = 0.$$

Hence,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$.

Similarly, if we subtract equation (2) $\times$ a_1 from equation (1) $\times$ a_2, we obtain

$$a_2b_1y - a_1b_2y + a_2c_1 - a_1c_2 = 0.$$

Hence,

$$y = -\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$.

Note:
Other arrangements of the solutions for x and y are possible, but the above arrangements

have been stated for a particular purpose which will be made clear shortly under "Observations".

The Symmetrical Form of the solution

The two separate solutions for x and y may be conveniently written in **"symmetrical"** form as follows:

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1},$$

provided that $a_1b_2 - a_2b_1 \neq 0$.

7.1.2 THE DEFINITION OF A SECOND ORDER DETERMINANT

Each of the denominators in the symmetrical form of the previous section has the same general appearance, namely the difference of the products of two pairs of numbers; and we shall rewrite each denominator in a new form using a mathematical symbol called a **"second order determinant"** and defined by the statement:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC.$$

The symbol on the left-hand-side may be called either a second order determinant or a 2×2 determinant; it has two **"rows"** (horizontally), two **"columns"** (vertically) and four **"elements"** (the numbers inside the determinant).

7.1.3 CRAMER'S RULE FOR TWO SIMULTANEOUS LINEAR EQUATIONS

The symmetrical solution to the two simultaneous linear equations may now be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

provided that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$;

or, in an abbreviated form,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

provided that $\Delta_0 \neq 0$.

This determinant rule for solving two simultaneous linear equations is called **"Cramer's Rule"** and has equivalent forms for a larger number of equations.

Note:
The interpretation of Cramer's Rule in the case when $a_1b_2 - a_2b_1 = 0$ will be dealt with as a special case after some elementary examples:

Observations

In Cramer's Rule,

1. The determinant underneath x can be remembered by covering up the x terms in the original simultaneous equations, then using the coefficients of y and the constant terms in the pattern which they occupy on the page.
2. The determinant underneath y can be remembered by covering up the y terms in the original simultaneous equations, then using the coefficients of x and the constant terms in the pattern they occupy on the page.
3. The determinant underneath 1 can be remembered by covering up the constant terms in the original simultaneous equations, then using the coefficients of x and y in the pattern they occupy on the page.
4. The final determinant is labelled Δ_0 as a reminder to evaluate it **first**; because, if it happens to be zero, there is no point in evaluating Δ_1 and Δ_2.

EXAMPLES

1. Evaluate the determinant
$$\Delta = \begin{vmatrix} 7 & -2 \\ 4 & 5 \end{vmatrix}.$$

   Solution

$$\Delta = 7 \times 5 - 4 \times (-2) = 35 + 8 = 43$$

2. Express the value of the determinant
$$\Delta = \begin{vmatrix} -p & -q \\ p & -q \end{vmatrix}$$
   in terms of p and q.

Solution

$$\Delta = (-p) \times (-q) - p \times (-q) = p.q + p.q = 2pq.$$

3. Use Cramer's Rule to solve for x and y the simultaneous linear equations

$$\begin{aligned} 5x - 3y &= -3, \\ 2x - y &= -2. \end{aligned}$$

Solution

We may first rearrange the equations in the form

$$\begin{aligned} 5x - 3y + 3 &= 0, \\ 2x - y + 2 &= 0. \end{aligned}$$

Hence, by Cramer's Rule,

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0},$$

where

$$\Delta_0 = \begin{vmatrix} 5 & -3 \\ 2 & -1 \end{vmatrix} = -5 + 6 = 1;$$

$$\Delta_1 = \begin{vmatrix} -3 & 3 \\ -1 & 2 \end{vmatrix} = -6 + 3 = -3;$$

$$\Delta_2 = \begin{vmatrix} 5 & 3 \\ 2 & 2 \end{vmatrix} = 10 - 6 = 4.$$

Thus,

$$x = \frac{\Delta_1}{\Delta_0} = -3 \ \text{ and } \ y = -\frac{\Delta_2}{\Delta_0} = -4.$$

Special Cases

When using Cramer's Rule, the determinant Δ_0 must not have the value zero. But if it **does** have the value zero, then, the simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0. - - - - - - - - -(2) \end{aligned}$$

are such that

$$a_1b_2 - a_2b_1 = 0.$$

In other words,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2},$$

which means that the x and y terms in one of the equations are proportional to the x and y terms in the other equation.

Two situations may arise which may be illustrated by the following examples:

EXAMPLES

1. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 10, \end{aligned}$$

   $\Delta_0 = 0$ but the second equation is simply a multiple of the first equation. That is, one of the equations is redundant and so there exists an **infinite number of solutions**; either of the variables may be chosen at random, with the remaining variable being expressible in terms of it.

2. For the set of equations

$$\begin{aligned} 3x - 2y &= 5, \\ 6x - 4y &= 7, \end{aligned}$$

$\Delta_0 = 0$ as before, but there is an inconsistency because, if the second equation is divided by 2, we obtain

$$3x - 2y = 3.5,$$

which is inconsistent with

$$3x - 2y = 5.$$

In this case **there are no solutions at all**.

Summary of the Special Cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.

7.1.4 EXERCISES

1. Write down the values of the following determinants:

   (a) $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$; (c) $\begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix}$;

   (d) $\begin{vmatrix} x & y \\ y & x \end{vmatrix}$; (e) $\begin{vmatrix} x & x \\ y & y \end{vmatrix}$; (f) $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$.

2. Use determinants (that is, 'Cramer's Rule') to solve the following sets of simultaneous linear equations:

   (a) $\begin{array}{rcrcr} 19x & + & 6y & = & 39, \\ 13x & - & 8y & = & -6. \end{array}$; (b) $\begin{array}{rcrcrcr} 3x & + & 4y & + & 6 & = & 0, \\ 5x & - & 3y & - & 19 & = & 0. \end{array}$;

   (c) $\begin{array}{rcrcr} 2x & + & 1 & = & 3y, \\ x & - & 5 & = & 7y. \end{array}$; (d) $\begin{array}{rcrcr} 4 & - & 2y & = & x, \\ 7 & + & 3y & = & 2x. \end{array}$;

   (e) $\begin{array}{rcrcr} 3i_1 & + & 2i_2 & = & 5, \\ i_1 & - & 3i_2 & = & 7. \end{array}$; (f) $\begin{array}{rcrcr} 2x & - & 4z & = & 5, \\ x & - & 2z & = & 1. \end{array}$.

3. By expanding out all of the determinants, verify the following results:

   (a) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$;

   (b) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + c_1 & b_1 \\ a_2 + c_2 & b_2 \end{vmatrix} - \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$;

   (c) $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 - kb_1 & b_1 \\ a_2 - kb_2 & b_2 \end{vmatrix}$ for any number, k.

7.1.5 ANSWERS TO EXERCISES

1. The values are:
   (a) 5; (b) 5; (c) -1;
   (d) $x^2 - y^2$; (e) 0; (f) $a^2 + b^2$.

2. (a)
$$\frac{x}{-276} = \frac{-y}{621} = \frac{1}{-230};$$
   hence, $x = 1.2$ and $y = 2.7$;
   (b)
$$\frac{x}{-58} = \frac{-y}{-87} = \frac{1}{-29};$$
   hence, $x = 2$ and $y = -3$;
   (c)
$$\frac{x}{22} = \frac{-y}{-11} = \frac{1}{-11};$$
   hence, $x = -2$ and $y = -1$;
   (d)
$$\frac{x}{-26} = \frac{-y}{1} = \frac{1}{-7};$$
   hence, $x = \frac{26}{7}$ and $y = \frac{1}{7}$;
   (e)
$$\frac{i_1}{-29} = \frac{-i_2}{-16} = \frac{1}{-11};$$
   hence, $i_1 = \frac{29}{11}$ and $i_2 = \frac{-16}{11}$;
   (f)
$$\frac{x}{-6} = \frac{-z}{3} = \frac{1}{0},$$
   which means that Cramer's rule breaks down. In fact, the second equation gives two contradictory statements $2x - 4z = 5$ and $2x - 4z = 2$.

3. By following the instructions, the results may be verified.

"JUST THE MATHS"

UNIT NUMBER

7.2

DETERMINANTS 2
(Consistency and third order determinants)

by

A.J.Hobson

UNIT 7.2 - DETERMINANTS 2

CONSISTENCY AND THIRD ORDER DETERMINANTS

7.2.1 CONSISTENCY FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

In a genuine scientific problem involving simultaneous equations, it is not necessarily true that there will be the same number of equations to solve as there are unknowns to be determined. We examine, here, an elementary situation in which there are three equations, but only two unknowns.

Consider the set of equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0, - - - - - - - - - (2) \\ a_3x + b_3y + c_3 &= 0, - - - - - - - - - -(3) \end{aligned}$$

where we shall assume that any pair of the three equations has a unique common solution which may be calculated by Cramer's Rule.

In order that the three equations shall be consistent, the common solution of any pair must also satisfy the remaining equation. In particular, the common solution of equations (2) and (3) must also satisfy equation (1).

By Cramer's Rule in equations (2) and (3),

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

This solution will satisfy equation (1) provided that

$$a_1 \frac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} - b_1 \frac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} + c_1 = 0.$$

In other words,

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is the determinant condition for the consistency of three simultaneous linear equations in two unknowns.

7.2.2 THE DEFINITION OF A THIRD ORDER DETERMINANT

In the consistency condition of the previous section, the expression on the left-hand-side is called a **"determinant of the third order"** and is denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

It has three **"rows"** (horizontally), three **"columns"** (vertically) and nine **"elements"** (the numbers inside the determinant).

The definition of a third order determinant may be stated in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notes:

(i) Other forms of the definition are also possible and will be encountered in Unit 7.3

(ii) The given formula for evaluating a third order determinant can be remembered by taking each element of the first row in turn and multiplying it by the so-called **"minor"** of the element, which is the second order determinant obtained by covering up the row and column in which the element appears; the results are then combined according to a $+$, $-$, $+$ pattern.

(iii) For the purpose of locating various parts of a determinant, the rows are counted from the top to the bottom and the columns are counted from the left to the right. Each row is read from the left to the right and each column is read from the top to the bottom. Thus, for example, the third element of the second column is b_3.

EXAMPLES

1. Evaluate the determinant
$$\Delta = \begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}.$$

   Solution
$$\Delta = -3\begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 5 & 3 \end{vmatrix} + 7\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix}.$$

   That is,
$$\Delta = -3(12-2) - 2(0+10) + 7(0-20) = -190.$$

2. Show that the simultaneous linear equations
$$\begin{aligned} 3x - y + 2 &= 0, \\ 2x + 5y - 1 &= 0, \\ 5x + 4y + 1 &= 0 \end{aligned}$$

   are consistent (assuming that any two of the three have a common solution), and obtain the common solution.

   Solution

   The condition for consistency is that the determinant of coefficients and constants must be zero. We have
$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & -1 \\ 5 & 4 & 1 \end{vmatrix} = 3\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix}$$
$$= 3(5+4) + (2+5) + 2(8-25) = 27 + 7 - 34 = 0.$$

   Thus, the equations are consistent and, to obtain their common solution, we may solve (say) the first two as follows:
$$\frac{x}{\begin{vmatrix} -1 & 2 \\ 5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}}.$$

That is,

$$\frac{x}{-9} = \frac{-y}{-7} = \frac{1}{17},$$

which gives

$$x = -\frac{9}{17} \text{ and } y = \frac{7}{17}.$$

Note:
It may be observed that the given set of simultaneous equations above is not an independent set because the third equation happens to be the sum of the other two. We say that the equations are **"linearly dependent"**; and this implies that the rows of the determinant of coefficients and constants are linearly dependent in the same way.
(In this case, Row 3 = Row 1 plus Row 2).

Furthermore, it may shown that the value of a determinant is zero if and only if its rows are linearly dependent. Hence, an alternative way of proving that a set of simultaneous linear equations is a consistent set is to show that they are linearly dependent in some way.

7.2.3 THE RULE OF SARRUS

From the given definition of a third order determinant, the complete **"expansion"** of the determinant may be given, in general, as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

But it may be observed that precisely the same terms may be obtained by first constructing a diagram which consists of the original determinant with the first two columns written out again to the right of this determinant. That is:

$$\left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right. .$$

Taking the sum of the possible products of the trios of numbers in the direction $\searrow$ and subtracting the sum of the possible products of the trios of numbers in the $\nearrow$ direction, we obtain the terms

$$(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_2c_3a_1 + c_3a_2b_1);$$

and it may be shown that these are exactly the same terms as those obtained by the original formula.

This **"Rule of Sarrus"** makes it possible to evaluate a third order determinant with an electronic calculator, almost without putting pen to paper, provided the calculator memory is used to store, then recall, the various products.

EXAMPLE

$$\begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} = \left|\begin{array}{ccc|cc} -3 & 2 & 7 & -3 & 2 \\ 0 & 4 & -2 & 0 & 4 \\ 5 & -1 & 3 & 5 & -1 \end{array}\right.$$

$$= ([-3].4.3 + 2.[-2].5 + 7.0.[-1]) - (5.4.7 + [-1].[-2].[-3] + 3.0.2)$$

$$= (-36 - 20 + 0) - (140 - 6 + 0) = -56 - 134 = -190$$

as in a previous example.

7.2.4 CRAMER'S RULE FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

In the same way that, under certain conditions, two simultaneous linear equations may be solved by determinants of the second order, it is possible to show that, under certain conditions, three simultaneous linear equations in three unknowns may be solved by determinants of the third order.

The proof of this result will not be included here , but we state it for reference.

The simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \end{aligned}$$

have a common solution, given in symmetrical form, by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

which is called the **"Key"** to the solution and requires that $\Delta_0 \neq 0$.

Again the rule itself is known as **"Cramer's Rule"**.

EXAMPLE

Using the Rule of Sarrus, obtain the common solution of the simultaneous linear equations

$$\begin{aligned} x + 4y - z + 2 &= 0, \\ -x - y + 2z - 9 &= 0, \\ 2x + y - 3z + 15 &= 0. \end{aligned}$$

Solution

The **"Key"** is

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

where

(i)

$$\Delta_0 = \begin{vmatrix} 1 & 4 & -1 \\ -1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} \begin{matrix} 1 & 4 \\ -1 & -1 \\ 2 & 1 \end{matrix}.$$

Hence,

$$\Delta_0 = (3 + 16 + 1) - (2 + 2 + 12) = 20 - 16 = 4,$$

which is non-zero, and so we may continue:

(ii)

$$\Delta_1 = \left|\begin{array}{rrr|rr} 4 & -1 & 2 & 4 & -1 \\ -1 & 2 & -9 & -1 & 2 \\ 1 & -3 & 15 & 1 & -3 \end{array}\right. .$$

Hence,

$$\Delta_1 = (120 + 9 + 6) - (4 + 108 + 15) = 135 - 127 = 8.$$

(iii)

$$\Delta_2 = \left|\begin{array}{rrr|rr} 1 & -1 & 2 & 1 & -1 \\ -1 & 2 & -9 & -1 & 2 \\ 2 & -3 & 15 & 2 & -3 \end{array}\right. .$$

Hence,

$$\Delta_2 = (30 + 18 + 6) - (8 + 27 + 15) = 54 - 50 = 4.$$

(iv)

$$\Delta_3 = \left|\begin{array}{rrr|rr} 1 & 4 & 2 & 1 & 4 \\ -1 & -1 & -9 & -1 & -1 \\ 2 & 1 & 15 & 2 & 1 \end{array}\right. .$$

Hence,

$$\Delta_3 = (-15 - 72 - 2) - (-4 - 9 - 60) = -89 + 73 = -16.$$

(v) The solutions are therefore

$$x = -\frac{\Delta_1}{\Delta_0} = -\frac{8}{4} = -2;$$

$$y = \frac{\Delta_2}{\Delta_0} = \frac{4}{4} = 1;$$

$$z = -\frac{\Delta_3}{\Delta_0} = -\frac{-16}{4} = 4.$$

Special Cases

If it should happen that $\Delta_0 = 0$ when solving a set of three simultaneous linear equations by Cramer's Rule, earlier work has demonstrated that the rows of Δ_0 must be linearly dependent. That is the three groups of x, y and z terms must be linearly dependent.

Different situations arise according to whether or not the constant terms can also be brought in to the linear dependence relationship and we illustrate with examples as follows:

EXAMPLES

1. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 4 &= 0, \end{aligned}$$

the third equation is the difference between the first two and hence it is redundant.

Any solution common to the first two equations will thus be an acceptable solution. In this case, there will be an infinite number of solutions since, for example, we may choose the variable z at random, solving for x and y to obtain

$$x = \frac{11 - 5z}{5} \quad \text{and} \quad y = \frac{5z - 3}{5}.$$

2. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 7 &= 0, \end{aligned}$$

the third equation is inconsistent with the difference between the first two equations. That is,

$$x - 3y + 4z - 7 = 0 \text{ is inconsistent with } x - 3y + 4z - 4 = 0.$$

In this case, there are no common solutions.

3. For the simultaneous linear equations

$$\begin{aligned} x - 2y + 3z - 1 &= 0, \\ 2x - 4y + 6z - 2 &= 0, \\ 3x - 6y + 9z - 3 &= 0, \end{aligned}$$

we have only one independent equation since the second and third equations are multiples of the first equation.

Again, there will be an infinite number of solutions which may be obtained by choosing two of the variables at random, then determining the corresponding value of the remaining variable.

Summary of the special cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.

7.2.5 EXERCISES

1. Show that the simultaneous linear equations

$$\begin{aligned} x + y + 2 &= 0, \\ 3x + 2y - 1 &= 0, \\ 2x + y - 3 &= 0, \end{aligned}$$

are consistent and determine their common solution.

2. Show that the simultaneous linear equations

$$\begin{aligned} 7x - 2y + 1 &= 0, \\ 3x + 2y - 4 &= 0, \\ x - 6y - 9 &= 0, \end{aligned}$$

are inconsistent.

3. Obtain the values of λ for which the simultaneous linear equations

$$\begin{aligned} 3x + 5y + (\lambda - 2) &= 0, \\ 2x + y - 5 &= 0, \\ (\lambda - 1)x + 2y - 10 &= 0, \end{aligned}$$

are consistent.

4. Use Cramer's Rule to solve, for x, y and z, the following simultaneous linear equations:

$$\begin{aligned} 5x + 3y - z + 10 &= 0, \\ -2x - y + 4z - 1 &= 0, \\ -x + 2y - 7z - 17 &= 0. \end{aligned}$$

5. Show that the simultaneous linear equations

$$\begin{aligned} x - y + 7z - 1 &= 0, \\ x + 2y - 3z + 5 &= 0, \\ 5x + 4y + 5z + 13 &= 0 \end{aligned}$$

are linearly dependent and obtain the common solution for which $z = -1$.

7.2.6 ANSWERS TO EXERCISES

1.

$$x = 5 \quad y = -7.$$

2.

$$\Delta_0 \neq 0.$$

3.

$$\lambda = 5 \quad \text{or} \quad \lambda = -23.$$

4.

$$x = -4 \quad y = 3 \quad z = -1.$$

5.

$$x = \frac{8}{3} \quad y = -\frac{16}{3} \quad z = -1.$$

"JUST THE MATHS"

UNIT NUMBER

7.3

DETERMINANTS 3
(Further evaluation of 3 x 3 determinants)

by

A.J.Hobson

UNIT 7.3 - DETERMINANTS 3

FURTHER EVALUATION OF THIRD ORDER DETERMINANTS

7.3.1 EXPANSION BY ANY ROW OR COLUMN

For the numerical evaluation of a third order determinant, the Rule of Sarrus is the easiest rule to apply; but we examine here some alternative versions of the original definition formula which will lead us to important standard properties of determinants.

Let us first re-state the original definition formula as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The question naturally arises as to whether the three elements of other rows (or even columns) may multiplied by their minors and the results combined in such a way as to give the same result as in the above formula. It turns out that any row or any column may be used in this way.

In order to illustrate this fact, we state again the more algebraic formula for a third order determinant, namely

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

ILLUSTRATION 1 - Expansion by the second row.

It may be observed that the expression

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

gives exactly the same result as in the original formula.

ILLUSTRATION 2 - Expansion by the third column.

It may be observed that the expression

$$c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)$$

gives exactly the same result as in the original formula.

Note:
Similar patterns of symbols give the expansions by the remaining rows and columns.

Summary
A third order determinant may be expanded (that is, evaluated) if we multiply each of the three elements in any row or (any column) by its minor then combined the results according the following pattern of so-called **"place-signs"**.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Note:
It is useful to have a special name for the **"signed-minor"** which any element of a determinant is mutiplied by, when expanding it by a row or a column. In fact every signed-minor is called a **"cofactor"**. This means that, wherever the place-sign is $+$, the minor and the cofactor are the same; but, wherever the place-sign is $-$, the cofactor is <u>numerically</u> equal to the minor but opposite in sign.

For instance,

(i) The minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$,

but the cofactor of b_1 is $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

(ii) The minor and cofactor of b_2 are both equal to $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

7.3.2 ROW AND COLUMN OPERATIONS ON DETERMINANTS

INTRODUCTION

Certain types of problem in scientific work can involve determinants for which some or all of the elements are variable quantities rather than fixed numerical quantities. In these cases, the methods so far encountered for expanding a determinant may not be appropriate.

Described below is a set of standard properties for determinants of any order but, where necessary, they will be explained using either 3×3 determinants or 2×2 determinants.

STANDARD PROPERTIES OF DETERMINANTS

In this section, the methods of expanding a determinant by any row or any column will be useful to have in mind.

1. If all of the elements in a row or a column have the value zero, then the value of the determinant is equal to zero.

   Proof:

   We simply expand the determinant by the row or column of zeros.

2. If all but one of the elements in a row or column are equal to zero, then the value of the determinant is the product of the non-zero element in that row or column with its cofactor.

   Proof:

   We simply expand the determinant by the row or column containing the single non-zero element; and we also notice that the determinant is effectively equivalent to a determinant of one order lower.

   For example,

$$\begin{vmatrix} 5 & 1 & 0 \\ -2 & 4 & 3 \\ 6 & 8 & 0 \end{vmatrix} = -3 \begin{vmatrix} 5 & 1 \\ 6 & 8 \end{vmatrix} = -3(40-6) = -102.$$

3. If a determinant contains two identical rows or two identical columns, then the value of the determinant is zero.

   Proof:

   If we expand the determinant by a row or column other than the two identical ones, it will turn out that all of the cofactors have value zero.

   For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0.$$

4. If two rows, or two columns, are interchanged the value of the determinant is unchanged numerically but it is reversed in sign.

   Proof:

   If we expand the determinant by a row or column other than the two which have been interchanged, then all of the cofactors will be changed in sign.

   For example,

$$\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = a_1\begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - a_2\begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} + a_3\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}.$$

5. If all of the elements in a row or column have a common factor, then this common factor may be removed from the determinant and placed outside.

   Proof:

   Expanding the determinant by the row or column which contains the common factor is equivalent to removing the common factor first, then expanding by the new row or column so created.

   For example,

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

   if we expand the left-hand-side by the second column.

   Note:

   Another way of stating this property is that, if all of the elements in any row or column of a determinant are multiplied by the same factor, then the value of the determinant is also multiplied by that factor.

6. If the elements of any row in a determinant are altered by adding to them (or subtracting from them) a common multiple of the corresponding elements in another row, then the value of the determinant is unaltered. A similar result applies to columns.

   ILLUSTRATION

   The validity of this result is easily shown in the case of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} = [(a_1 + kb_1)b_2 - (a_2 + kb_2)b_1] = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The above properties need not normally be used for the evaluation of determinants whose elements are simple numerical values; but, in the examples which follow, we include one such determinant in order to provide a simple introduction to the technique.

We shall use the symbols R_1, R_2 and R_3 to denote Row 1, Row 2 and Row 3; the symbols C_1, C_2 and C_3 will be used to denote Column 1, Column 2 and Column 3; and the symbol $\longrightarrow$ will stand for the word "becomes". The examples use what are called **"row operations"** and **"column operations"**.

EXAMPLES

1. Evaluate the determinant,

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix} \quad C_1 \longrightarrow C_1 \div 5;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 2 & 5 & 9 \\ 3 & 2 & 3 \end{vmatrix} \quad R_2 \longrightarrow R_2 - 2R_1;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 3 & 2 & 3 \end{vmatrix} \quad R_3 \longrightarrow R_3 - 3R_1;$$

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 0 & -7 & -18 \end{vmatrix}$$

$$= 5(18 - 35) = 5 \times -17 = -85.$$

2. Solve, for x, the equation

$$\begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0.$$

Solution

We could expand the determinant directly, but we would then obtain a cubic equation in x which may not be straightforward to solve.

A better method is to try to obtain factors of this cubic equation **before** expanding the determinant.

It may be observed in this example that the three expressions in each column add up to the same quantity, namely $x + 2$. Thus if we first add Row 2 to Row 1, then add Row 3 to the new Row 1, we shall obtain $x + 2$ as a factor of the first row.

We may write

$$0 = \begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} x+2 & x+2 & x+2 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x+2)$$

$$= (x+2)\begin{vmatrix} 1 & 1 & 1 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x+2)\begin{vmatrix} 1 & 0 & 0 \\ 5 & x-4 & -4 \\ -3 & -1 & x+1 \end{vmatrix}$$

$$= (x+2)[(x-4)(x+1) - 4] = (x-2)(x^2 - 3x - 8).$$

Hence,

$$x = -2 \text{ or } x = \frac{3 \pm \sqrt{9+32}}{2} = \frac{3 \pm \sqrt{41}}{2}.$$

3. Solve, for x, the equation

$$\begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} = 0,$$

Solution

We may observe that the sum of the corresponding pairs of elements in the first two rows is the same, namely $x - 4$. Hence we may proceed as follows:

$$0 = \begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2$$

$$= \begin{vmatrix} x-4 & x-4 & x-4 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x-4)$$

$$= (x-4)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x-4)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x & -1 \\ 7 & 1 & x \end{vmatrix}$$

$$= (x-4)(x^2+1).$$

In this case, the only real solution is $x = 4$, the others being complex numbers $x = \pm j$.

4. Solve, for x, the equation

$$\begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix},$$

Solution

We may observe that the 2 in Row 1 may be used to reduce to zero the 4 underneath it in Row 2.

Hence,

$$0 = \begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 - 2R_1$$

$$= \begin{vmatrix} x & 3 & 2 \\ 4-2x & x-2 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 \div (x-2)$$

$$= (x-2)\begin{vmatrix} x & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad C_1 \longrightarrow C_1 + 2C_2$$

$$= (x-2)\begin{vmatrix} x+6 & 3 & 2 \\ 0 & 1 & 0 \\ 4 & 1 & x-1 \end{vmatrix}$$

$$= (x-2)[(x+6)(x-1)-8] = (x-2)[x^2+5x-14] = (x-2)(x+7)(x-2).$$

Thus,

$$x = 2 \ \ \text{(repeated)} \ \ \text{and} \ \ x = -7.$$

Note:
It is not possible to cover, by examples, every type of problem which may occur. The secret is first to spend a few seconds examining whether or not the sum or difference of a group of rows or columns can give a common factor immediately. If not, the procedure is to look for ways of obtaining a row or column in which all but one of the elements is zero and hence, effectively, to reduce the order of the determinant.

7.3.3 EXERCISES

1. Use row and/or column operations to evaluate the following determinants:

   (a)

$$\begin{vmatrix} 100 & 101 & 102 \\ 101 & 102 & 103 \\ 102 & 103 & 104 \end{vmatrix};$$

   (b)

$$\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}.$$

2. Use row and/or column operations to evaluate, in terms of a and b, the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}.$$

3. Show that the equation

$$\begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix} = 0$$

   has one solution $x = -(a+b)$ and hence solve it completely.

4. Solve completely, for x, the following equations:

   (a)

$$\begin{vmatrix} x-3 & x+2 & x-1 \\ x+2 & x-4 & x \\ x-1 & x+4 & x-5 \end{vmatrix} = 0;$$

   (b)

$$\begin{vmatrix} x+1 & x+2 & 3 \\ 2 & x+3 & x+1 \\ x+3 & 1 & x+2 \end{vmatrix} = 0.$$

7.3.4 ANSWERS TO EXERCISES

1. (a)

$$0$$

   (b)

$$24$$

2.

$$ab$$

3.

$$x = -(a + b), \quad x = a, \quad x = b.$$

4. (a)

$$x = \frac{2}{3} \text{ only;}$$

   (b)

$$x = -3, \quad x = \pm\sqrt{3}.$$

"JUST THE MATHS"

UNIT NUMBER

7.4

DETERMINANTS 4
(Homogeneous linear equations)

by

A.J.Hobson

UNIT 7.4 - DETERMINANTS 4

HOMOGENEOUS LINEAR EQUATIONS

7.4.1 TRIVIAL AND NON-TRIVIAL SOLUTIONS

This Unit is concerned with a set of simultaneous linear equations in which all of the constant terms have value zero. Most of the discussion will involve three such **"homogeneous"** linear equations of the form

$$
\begin{aligned}
a_1x + b_1y + c_1z &= 0,\\
a_2x + b_2y + c_2z &= 0,\\
a_3x + b_3y + c_3z &= 0.
\end{aligned}
$$

These could have been discussed at the same time as Cramer's Rule but are worth considering as a completely separate case since, in scientific applications, they lend themselves conveniently to the methods of row and column operations.

Observations

1. In Cramer's Rule for the above set of equations, if the determinant, Δ_0 of the coefficients of x, y and z is non-zero, there will exist a unique solution, namely $x = 0$, $y = 0$, $z = 0$, since each of the determinants Δ_1, Δ_2 and Δ_3 will contain a column of zeros (that is, the constant terms of the three equations).

   But this solution is obvious from the given set of equations and we call it the **"trivial solution"**.

2. The question arises as to whether it is possible for the set of equations to have any **"non-trivial"** solutions.

3. We shall see that non-trivial solutions occur when the number of equations reduces to less than the number of variables being solved for; that is when the equations are not linearly independent.

   For example, if one of the equations is redundant, we could solve the remaining two in an infinite number of ways by choosing one of the variables at random. Also, if two of the equations are redundant, we could solve the remaining equation in an infinite number of ways by choosing two of the variables at random.

4. It is evident from previous work that the set of homogeneous linear equations will have non-trivial solutions provided that

$$\Delta_0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

5. Once it has been established that non-trivial solutions exist, it can be seen that any solution $x = \alpha$, $y = \beta$, $z = \gamma$ will imply other solutions of the form $x = \lambda\alpha$, $y = \lambda\beta$, $z = \lambda\gamma$, where λ is any non-zero number.

TYPE 1 - One of the three equations is redundant

The non-trivial solutions to a set which reduces to **two** linearly independent homogeneous linear equations in x, y and z may be stated in the form

$$x : y : z = \alpha : \beta : \gamma,$$

in which we mean that

$$\frac{x}{y} = \frac{\alpha}{\beta},\ \frac{y}{z} = \frac{\beta}{\gamma} \text{ and } \frac{x}{z} = \frac{\alpha}{\gamma}.$$

One method of obtaining these ratios is first to eliminate z between the two equations in order to obtain the ratio $x : y$, then to eliminate y between the two equations in order to find the ratio $x : z$; but a slightly simpler method is described in the first worked example to be discussed shortly.

TYPE 2 - Two of the three equations are redundant

This case arises when the three homogeneous linear equations are multiples of one another.

Again, any solution implies an infinite number of others in the same set of ratios, $x : y : z$. But it turns out that not **all** solutions are in the same set of ratios.

For example, if the only equation remaining is

$$ax + by + cz = 0,$$

we could choose any two of the variables at random and solve for the remaining variable.

In particular, we could substitute $y = 0$ to obtain $x : y : z = -\frac{c}{a} : 0 : 1$;

and we could also substitute $z = 0$ to obtain $x : y : z = -\frac{b}{a} : 1 : 0$

From these two, it is now possible to generate solutions with any value, β, of y and any value, γ, of z (as if we had chosen y and z at random) in order to solve for x.

In fact

$$x = -(\beta.\frac{b}{a} + \gamma.\frac{c}{a}), \quad y = \beta, \quad z = \gamma.$$

Note:
It may be shown that, for a set of homogeneous linear simultaneous equations, no types of solution exist other than those discussed above.

EXAMPLES

1. Show that the homogeneous linear equations

$$\begin{aligned} 2x + y - z &= 0, \\ x - 3y + 2z &= 0, \\ x + 4y - 3z &= 0 \end{aligned}$$

have solutions other than $x = 0$, $y = 0$, $z = 0$ and determine the ratios $x : y : z$ for these non-trivial solutions.

Solution

(a)

$$\Delta_0 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{vmatrix}\begin{matrix} 2 & 1 \\ 1 & -3 \\ 1 & 4 \end{matrix} = (18 + 2 - 4) - (3 + 16 - 3) = 0.$$

Thus, the equations are linearly dependent and, hence, have non-trivial solutions.

Note:
We could, alternatively, have noticed that the first equation is the sum of the second and third equations.

(b) It can always be arranged, in a set of ratios $\alpha : \beta : \gamma$, that any of the quantities which does not have to be equal to zero may be given the value 1. For example, $\frac{\alpha}{\gamma} : \frac{\beta}{\gamma} : 1$ is the same set of ratios as long as γ is not zero.

Let us now suppose that $z = 1$, giving

$$\begin{aligned} 2x + y - 1 &= 0, \\ x - 3y + 2 &= 0, \\ x + 4y - 3 &= 0 \end{aligned}$$

On solving any pair of these equations, we obtain $x = \frac{1}{7}$ and $y = \frac{5}{7}$, which means that

$$x : y : z = \frac{1}{7} : \frac{5}{7} : 1$$

That is,

$$x : y : z = 1 : 5 : 7$$

and any three numbers in these ratios form a solution.

2. Determine the values of λ for which the homogeneous linear equations

$$\begin{aligned} (1 - \lambda)x + y - 2z &= 0, \\ -x + (2 - \lambda)y + z &= 0, \\ y - (1 - \lambda)z &= 0 \end{aligned}$$

have non-trivial solutions.

Solution

First we solve the equation

$$0 = \begin{vmatrix} 1 - \lambda & 1 & -2 \\ -1 & 2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda) \end{vmatrix} \quad R_1 \longrightarrow R_1 - R_3$$

$$= \begin{vmatrix} 1 - \lambda & 0 & -1 + \lambda \\ -1 & 2 - \lambda & 1 \\ 0 & 1 & -1 - \lambda \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (1 - \lambda)$$

$$= (1-\lambda)\begin{vmatrix} 1 & 0 & -1 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \quad C_3 \quad \longrightarrow \quad C_3 + C_1$$

$$= (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix}$$

$$= -(1-\lambda)(2-\lambda)(1+\lambda)$$

Hence,

$$\lambda = 1, \quad -1 \ \text{ or } \ 2.$$

3. Determine the general solution of the homogeneous linear equation

$$3x - 7y + z = 0.$$

Solution

Substituting $y = 0$, we obtain $3x + z = 0$ and hence $x : y : z = -\frac{1}{3} : 0 : 1$.
Substituting $z = 0$, we obtain $3x - 7y = 0$ and hence $x : y : z = \frac{7}{3} : 1 : 0$
The general solution may thus be given by

$$x = \frac{7\beta}{3} - \frac{\gamma}{3}, \quad y = \beta, \quad z = \gamma$$

for arbitrary values of β and γ, though other equivalent versions are possible according to which of the three variables are chosen to have arbitrary values.

7.4.2 EXERCISES

1. Show that the homogeneous linear equations

$$\begin{aligned} x - 2y + 2z &= 0, \\ 2x - 2y - z &= 0, \\ 3x + y + z &= 0 \end{aligned}$$

have no solutions other than the trivial solution.

2. Show that the following sets of homogeneous linear equations have non-trivial solutions and express these solutions as a set of ratios for $x : y : z$

(a)

$$\begin{aligned} x - 2y + z &= 0, \\ x + y - 3z &= 0, \\ 3x - 3y - z &= 0; \end{aligned}$$

(b)

$$\begin{aligned} 3x + y - 2z &= 0, \\ 2x + 4y + 2z &= 0, \\ 4x + 3y - z &= 0. \end{aligned}$$

3. Determine the values of λ for which the homogeneous linear equations

$$\begin{aligned} \lambda x + 2y + 3z &= 0, \\ 2x + (\lambda + 3)y + 6z &= 0, \\ 3x + 4y + (\lambda + 6)z &= 0 \end{aligned}$$

have non-trivial solutions and solve them for the case when λ is an integer.

4. Determine the general solution to the homogeneous linear simultaneous equations

$$\begin{aligned} (\lambda + 1)x - 5y + 3z &= 0, \\ -2x + (\lambda - 8)y + 6z &= 0, \\ -3x - 15y + (\lambda + 11)z &= 0 \end{aligned}$$

in the case when $\lambda = -2$.

7.4.3 ANSWERS TO EXERCISES

1.

$$\Delta_0 \neq 0.$$

2. (a)

$$x : y : z = 5 : 4 : 3;$$

(b)

$$x : y : z = 1 : -1 : 1$$

3.

$$\lambda = 1,\ 0.83 \ \text{ or } \ -10.83$$

When $\lambda = 1$, $x : y : z = -1 : -1 : 1$

4.

$$x = -5\beta + 3\gamma, \quad y = \beta, \quad z = \gamma.$$

"JUST THE MATHS"

UNIT NUMBER

8.1

VECTORS 1
(Introduction to vector algebra)

by

A.J.Hobson

UNIT 8.1 - VECTORS 1 - INTRODUCTION TO VECTOR ALGEBRA

8.1.1 DEFINITIONS

1. A **"scalar"** quantity is one which has magnitude, but is not related to any direction in space.

   Examples: Mass, Speed, Area, Work.

2. A **"vector"** quantity is one which is specified by both a magnitude and a direction in space.

   Examples: Velocity, Weight, Acceleration.

3. A vector quantity with a fixed point of application is called a **"position vector"**.

4. A vector quantity which is restricted to a fixed line of action is called a **"line vector"**.

5. A vector quantity which is defined only by its magnitude and direction is called a **"free vector"**.

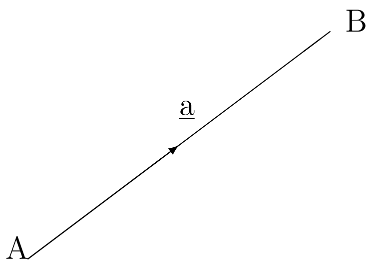
   Note:
   Unless otherwise stated, all vectors in the remainder of these units will be free vectors.

6. A vector quantity can be represented diagramatically by a directed straight line segment in space (with an arrow head) whose direction is that of the vector and whose length represents its magnitude according to a suitable scale.

7. The symbols $\underline{a}$, $\underline{b}$, $\underline{c}$, ... will be used to denote vectors with magnitudes a,b,c.... but it is sometimes more convenient to use a notation such as $\underline{AB}$ which means the vector represented by the line segment drawn from the point A to the point B.

   Notes:
   (i) The magnitude of the vector $\underline{AB}$, which is the length of the line AB can also be denoted by the symbol $|\underline{AB}|$.

   (ii) The magnitude of the vector $\underline{a}$, which is the number a, can also be denoted by the symbol $|\underline{a}|$.



8. A vector whose magnitude is 1 is called a **"unit vector"** and the symbol $\widehat{\underline{a}}$ denotes a unit vector in the same direction as $\underline{a}$. A vector whose magnitude is zero is called a **"zero vector"** and is denoted by **O** or $\underline{O}$. It has indeterminate direction.

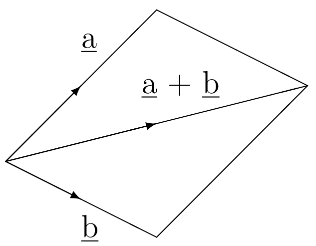
9. Two (free) vectors $\underline{a}$ and $\underline{b}$ are said to be **"equal"** if they have the same magnitude and direction.

   Note:
   This means that two directed straight line segments which are parallel and equal in length may be regarded as representing exactly the same vector.

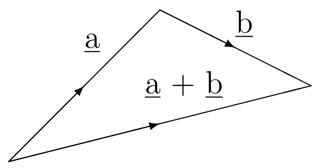
10. A vector whose magnitude is that of $\underline{a}$ but with opposite direction is denoted by $-\,\underline{a}$.

8.1.2 ADDITION AND SUBTRACTION OF VECTORS

Students may already know how the so-called **"resultant"** (or sum) of <u>particular</u> vectors, like <u>forces</u>, can be determined using either the **"Parallelogram Law"** or alternatively the **"Triangle Law"**. This previous knowledge is not essential here because we now <u>define</u> the sum of two arbitrary vectors diagramatically using either a parallelogram or a triangle. This will then lead also to a definition of subtraction for two vectors.



Parallelogram Law



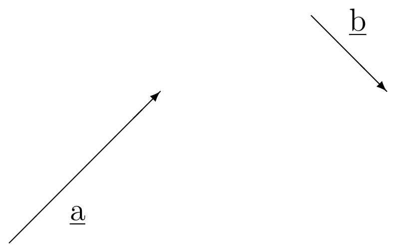
Triangle Law

Notes:
(i) The Triangle Law is more widely used than the Parallelogram Law because of its simplicity. We need to observe that $\underline{a}$ and $\underline{b}$ describe the triangle in the <u>same</u> sense while $\underline{a} + \underline{b}$ describes the triangle in the <u>opposite</u> sense.

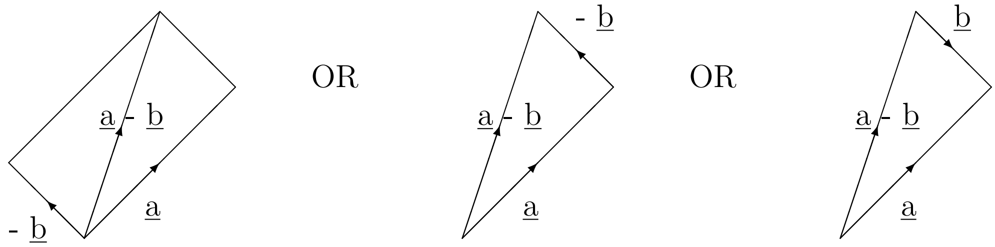
(ii) We define subtraction for vectors by considering that

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b}).$$

For example, to find $\underline{a} - \underline{b}$ for the vectors $\underline{a}$ and $\underline{b}$ below,



we may construct the following diagrams:



The third figure shows that, to find $\underline{a} - \underline{b}$, we require that $\underline{a}$ and $\underline{b}$ describe the triangle in opposite senses while $\underline{a} - \underline{b}$ describes the triangle in the same sense as $\underline{b}$

(iii) The sum of the three vectors describing the sides of a triangle in the same sense is always the zero vector.

8.1.3 MULTIPLICATION OF A VECTOR BY A SCALAR

If m is any positive real number, $m\underline{a}$ is defined to be a vector in the same direction as $\underline{a}$, but of m times its magnitude.

Similarly $-m\underline{a}$ is a vector in the opposite direction to $\underline{a}$, but of m times its magnitude.

Note:

$\underline{a} = a\underline{\widehat{a}}$ and hence

$$\frac{1}{a}.\underline{a} = \underline{\widehat{a}}.$$

That is, if any vector is multiplied by the reciprocal of its magnitude, we obtain a unit vector in the same direction. This process is called **"normalising the vector"**.

8.1.4 LAWS OF ALGEBRA OBEYED BY VECTORS

(i) The Commutative Law of Addition

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}.$$

(ii) The Associative Law of Addition

$$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c} = \underline{a} + \underline{b} + \underline{c}.$$

(iii) The Associative Law of Multiplication by a Scalar

$$m(n\underline{a}) = (mn)\underline{a} = mn\underline{a}.$$

(iv) The Distributive Laws for Multiplication by a Scalar

$$(m+n)\underline{a} = m\underline{a} + n\underline{a}$$

and

$$m(\underline{a} + \underline{b}) = m\underline{a} + m\underline{b}.$$

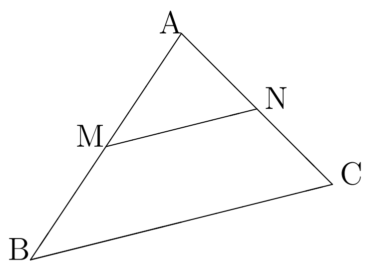
8.1.5 VECTOR PROOFS OF GEOMETRICAL RESULTS

The following examples illustrate how certain geometrical results which could be very cumbersome to prove using traditional geometrical methods can be much more easily proved using a vector method.

EXAMPLES

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of its length.

   Solution



By the Triangle Law,

$$\underline{BC} = \underline{BA} + \underline{AC}$$

and

$$\underline{MN} = \underline{MA} + \underline{AN} = \frac{1}{2}\underline{BA} + \frac{1}{2}\underline{AC}.$$

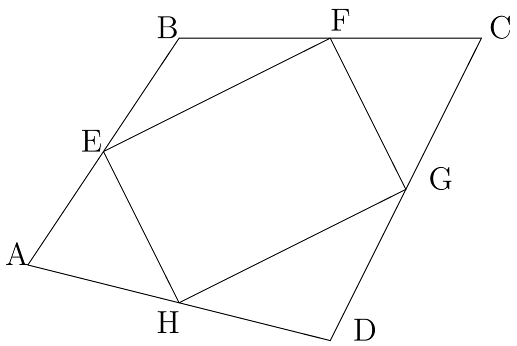
Hence,

$$\underline{MN} = \frac{1}{2}(\underline{BA} + \underline{AC}) = \frac{1}{2}\underline{BC},$$

which proves the result.

2. ABCD is a quadrilateral (four-sided figure) and E,F,G,H are the midpoints of AB, BC, CD and DA respectively. Show that EFGH is a parallelogram.

Solution



By the Triangle Law,

$$\underline{\mathrm{EF}} = \underline{\mathrm{EB}} + \underline{\mathrm{BF}} = \frac{1}{2}\underline{\mathrm{AB}} + \frac{1}{2}\underline{\mathrm{BC}} = \frac{1}{2}(\underline{\mathrm{AB}} + \underline{\mathrm{BC}}) = \frac{1}{2}\underline{\mathrm{AC}}$$

and also

$$\underline{\mathrm{HG}} = \underline{\mathrm{HD}} + \underline{\mathrm{DG}} = \frac{1}{2}\underline{\mathrm{AD}} + \frac{1}{2}\underline{\mathrm{DC}} = \frac{1}{2}(\underline{\mathrm{AD}} + \underline{\mathrm{DC}}) = \frac{1}{2}\underline{\mathrm{AC}}.$$

Hence,

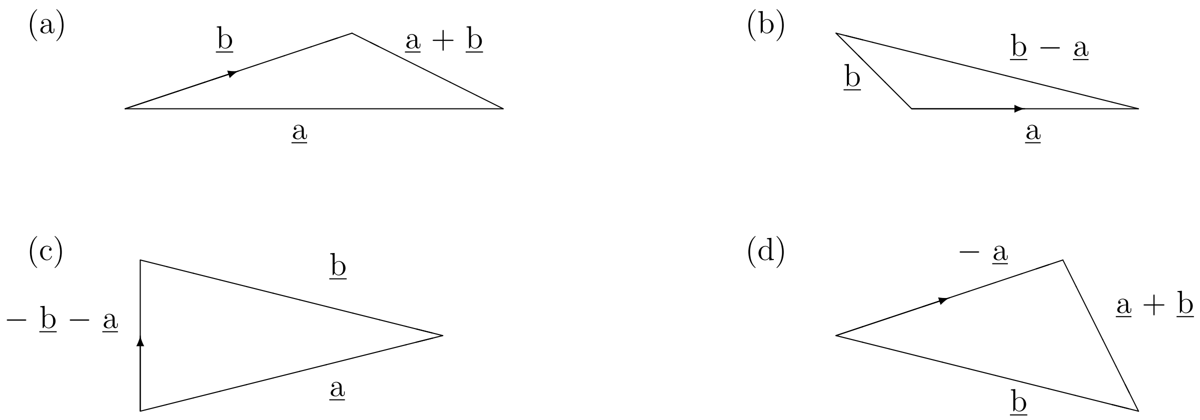
$$\underline{\mathrm{EF}} = \underline{\mathrm{HG}},$$

which proves the result.

8.1.6 EXERCISES

1. Which of the following are vectors and which are scalars ?
   (a) Kinetic Energy; (b) Volume; (c) Force;
   (d) Temperature; (e) Electric Field; (f) Thrust.

2. Fill in the missing arrows for the following vector diagrams:



3. ABCDE is a regular pentagon with centre O. Use the Triangle Law of Addition to show that
$$\underline{AB} + \underline{BC} + \underline{CD} + \underline{DE} + \underline{EA} = \mathbf{O}.$$

4. Draw to scale a diagram which illustrates the identity
$$4\underline{a} + 3(\underline{b} - \underline{a}) = \underline{a} + 3\underline{b}.$$

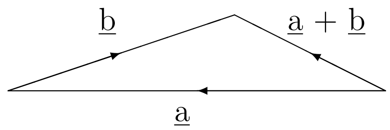
5. $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors and
$$\underline{p} = \underline{b} + \underline{c} - 2\underline{a}, \quad \underline{q} = \underline{c} + \underline{a} - 2\underline{b}, \quad \underline{r} = 3\underline{c} - 3\underline{b}.$$
   Show that the vector $3\underline{p} - 2\underline{q}$ is parallel to the vector $5\underline{p} - 6\underline{q} + \underline{r}$.

8.1.7 ANSWERS TO EXERCISES

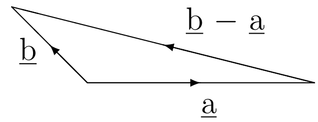
1. (a) Scalar; (b) Scalar; (c) Vector:
   (d) Scalar; (e) Vector; (f) Vector.

2. The completed diagrams are as follows:

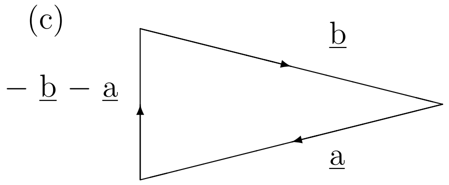
(a)



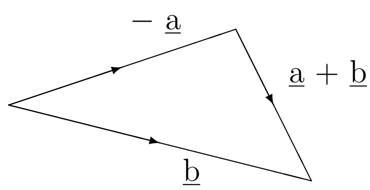
(b)



(c)

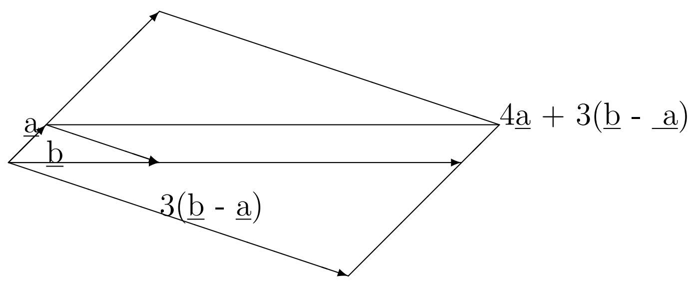


(d)



3. Join A,B,C,D and E up to the centre, O.

4. The diagram is



5. One vector is a scalar multiple of the other.

"JUST THE MATHS"

UNIT NUMBER

8.2

VECTORS 2
(Vectors in component form)

by

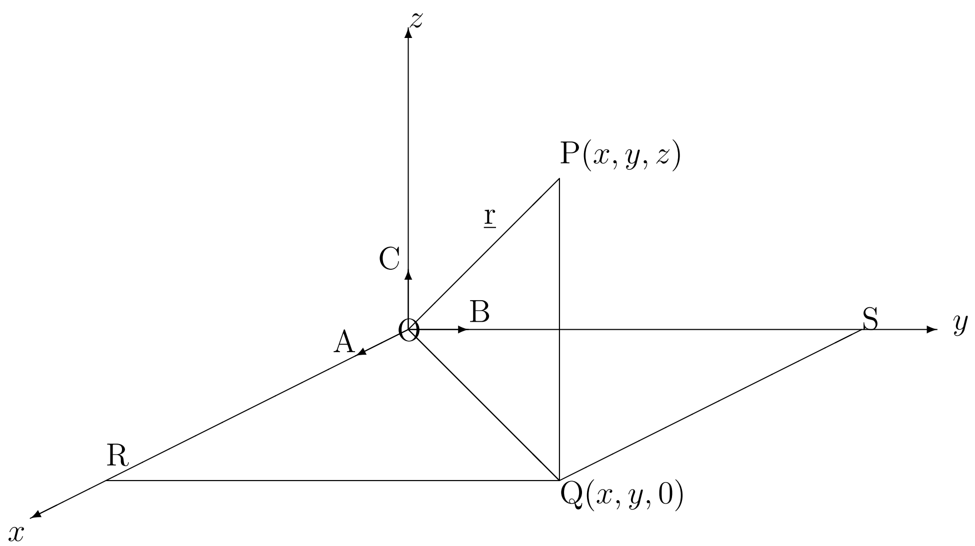
A.J.Hobson

UNIT 8.2 - VECTORS 2 - VECTORS IN COMPONENT FORM

8.2.1 THE COMPONENTS OF A VECTOR

The simplest way to define a vector in space is in terms of **unit vectors** placed along the axes Ox, Oy and Oz of a three-dimensional right-handed cartesian reference system. These unit vectors will be denoted respectively by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (omitting, for convenience, the "bars" underneath and the "hats" on the top).

Consider the following diagram:



In the diagram, $\underline{\text{OA}} = \mathbf{i}$, $\underline{\text{OB}} = \mathbf{j}$ and $\underline{\text{OC}} = \mathbf{k}$. P is the point with co-ordinates (x, y, z).

By the Triangle Law

$$\underline{\text{r}} = \underline{\text{OP}} = \underline{\text{OQ}} + \underline{\text{QP}} = \underline{\text{OR}} + \underline{\text{RQ}} + \underline{\text{QP}}.$$

That is,

$$\underline{\text{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Notes:

(i) The fact that we have considered a vector which emanates from the origin is not a special case since we are dealing with free vectors. Nevertheless $\underline{\text{OP}}$ is called the position vector of the point P.

(ii) The numbers x, y and z are called the **"components"** of $\underline{\text{OP}}$ (or of any other vector in space with the same magnitude and direction as $\underline{\text{OP}}$).

(iii) To multiply (or divide) a vector in component form by a scalar, we simply multiply (or divide) each of its components by that scalar.

8.2.2 THE MAGNITUDE OF A VECTOR IN COMPONENT FORM

Referring to the diagram in section 8.2.1, Pythagoras' Theorem gives

$$(\mathrm{OP})^2 = (\mathrm{OQ})^2 + (\mathrm{QP})^2 = (\mathrm{OR})^2 + (\mathrm{RQ})^2 + (\mathrm{QP})^2.$$

That is,

$$\mathrm{r} = |x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}.$$

EXAMPLE

Determine the magnitude of the vector

$$\underline{\mathrm{a}} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

and hence obtain a unit vector in the same direction.

Solution

$$|\underline{\mathrm{a}}| = \mathrm{a} = \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{30}.$$

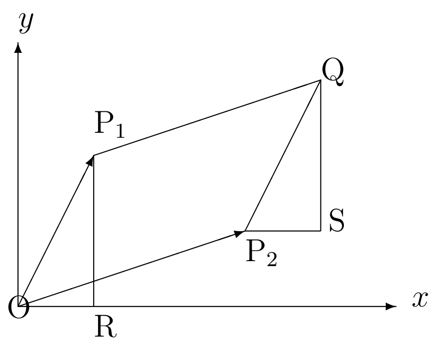
Hence, a unit vector in the same direction as $\underline{\mathrm{a}}$ is obtained by normalising $\underline{\mathrm{a}}$; that is, dividing it by its own magnitude.

The required unit vector is

$$\widehat{\underline{\mathrm{a}}} = \frac{1}{\mathrm{a}}.\underline{\mathrm{a}} = \frac{5\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{30}}.$$

8.2.3 THE SUM AND DIFFERENCE OF VECTORS IN COMPONENT FORM

We consider, first, a situation in **two** dimensions where two vectors are added together.



In the diagram, suppose P_1 has co-ordinates (x_1, y_1) and suppose P_2 has co-ordinates (x_2, y_2).

Then, since the triangle ORP_1 has exactly the same shape as the triangle P_2SQ, the co-ordinates of Q must be $(x_1 + x_2, y_1 + y_2)$.

But, by the Parallelogram Law, $\underline{OQ}$ is the sum of $\underline{OP_1}$ and $\underline{OP_2}$.

That is,

$$(x_1\mathbf{i} + y_1\mathbf{j}) + (x_2\mathbf{i} + y_2\mathbf{j}) = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j},$$

showing that the sum of two vectors may be found by adding together their separate components.

It can be shown that this result applies in three dimensions also and that, to find the **difference** of two vectors, we calculate the difference of their separate components.

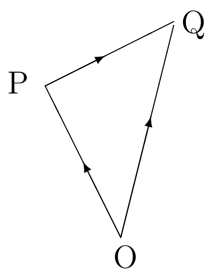
EXAMPLE

Two points P and Q in space have cartesian co-ordinates $(-3, 1, 4)$ and $(2, -2, 5)$ respectively. Determine the vector $\underline{PQ}$.

Solution

We are given that

$$\underline{OP} = -3\mathbf{i} + \mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \underline{OQ} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}.$$



By the triangle Law,

$$\underline{PQ} = \underline{OQ} - \underline{OP} = 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Note:
The vector $\underline{PQ}$ is, of course, the vector drawn from the point P to the point Q and it may seem puzzling that the result just obtained appears to be a vector drawn from the origin to the point $(5, -3, 1)$. However, we need to use again the fact that we are dealing with free vectors and the vector drawn from the origin to the point $(5, -3, 1)$ is parallel and equal in length to $\underline{PQ}$; in other words, it is the **same** as $\underline{PQ}$.

8.2.4 THE DIRECTION COSINES OF A VECTOR

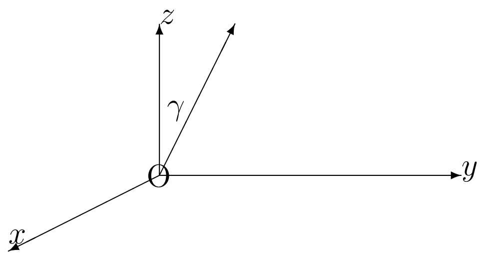
Suppose that

$$\underline{\mathrm{OP}} = \underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and suppose that $\underline{\mathrm{OP}}$ makes angles α, β and γ with Ox, Oy and Oz respectively.

Then,

$$\cos\alpha = \frac{x}{\mathrm{r}}, \quad \cos\beta = \frac{y}{\mathrm{r}} \text{ and } \cos\gamma = \frac{z}{\mathrm{r}}.$$



The three quantities $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **"direction cosines"** of $\underline{\mathrm{r}}$.

Any three numbers in the same ratio as the direction cosines are said to form a set of **"direction ratios"** for the vector $\underline{\mathrm{r}}$ and we note that $x : y : z$ is one possible set of direction ratios.

EXAMPLE

The direction cosines of the vector

$$6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

are

$$\frac{6}{\sqrt{41}}, \ \frac{2}{\sqrt{41}} \text{ and } \frac{-1}{\sqrt{41}},$$

since the vector has magnitude $\sqrt{36+4+1} = \sqrt{41}$.

A set of direction ratios for this vector are $6 : 2 : -1$.

8.2.5 EXERCISES

1. The position vectors of two points P and Q are, respectively,

$$\underline{\mathrm{r}_1} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{\mathrm{r}_2} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

   Determine the vector $\underline{\mathrm{PQ}}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and hence obtain

   (a) its magnitude;

   (b) its direction cosines.

2. Obtain a unit vector which is parallel to the vector $\underline{\mathrm{a}} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

3. If $\underline{\mathrm{a}} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$, $\underline{\mathrm{b}} = -2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\underline{\mathrm{c}} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, determine the following:

   (a)

$$2\underline{\mathrm{a}} - \underline{\mathrm{b}} + 3\underline{\mathrm{c}};$$

   (b)

$$|\underline{\mathrm{a}} + \underline{\mathrm{b}} + \underline{\mathrm{c}}|;$$

   (c)

$$|3\underline{\mathrm{a}} - 2\underline{\mathrm{b}} + 4\underline{\mathrm{c}}|;$$

   (d) a unit vector which is parallel to

$$3\underline{\mathrm{a}} - 2\underline{\mathrm{b}} + 4\underline{\mathrm{c}}.$$

4. Prove that the vectors $\underline{\mathrm{a}} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\underline{\mathrm{b}} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\underline{\mathrm{c}} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ can form the sides of a triangle.

   Determine also the lengths of the **"medians"** of this triangle (that is, the lines joining each vertex to the mid-point of the opposite side).

8.2.6 ANSWERS TO EXERCISES

1. $\underline{\text{PQ}} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$.

   (a) $|\underline{\text{PQ}}| = 7$;

   (b) The direction cosines are $\frac{2}{7}$, $-\frac{6}{7}$ and $\frac{3}{7}$.

2.
$$\pm\frac{3\mathbf{i} - \mathbf{j} + 5\mathbf{k}}{\sqrt{35}}.$$

3. (a)
$$11\mathbf{i} - 8\mathbf{k};$$

   (b)
$$\sqrt{93};$$

   (c)
$$\sqrt{398};$$

   (d)
$$\pm\frac{17\mathbf{i} - 3\mathbf{j} - 10\mathbf{k}}{\sqrt{398}}.$$

4. $\underline{a} = \underline{b} + \underline{c}$; therefore the vectors form a triangle.

   The medians have lengths equal to
$$5\sqrt{\frac{3}{2}},\ \sqrt{6} \text{ and } \frac{\sqrt{114}}{2}.$$

"JUST THE MATHS"

UNIT NUMBER

8.3

VECTORS 3
(Multiplication of one vector by another)

by

A.J.Hobson

UNIT 8.3 - VECTORS 3

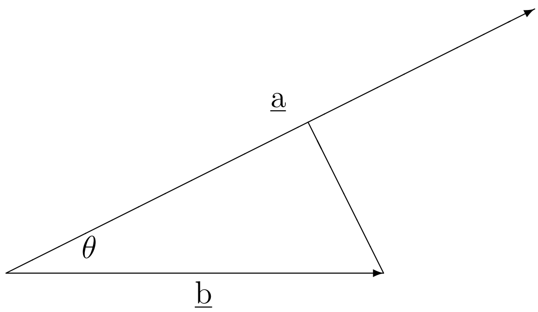
MULTIPLICATION OF ONE VECTOR BY ANOTHER

8.3.1 THE SCALAR PRODUCT (or "Dot" Product)

DEFINITION

The **"Scalar Product"** of two vectors $\underline{a}$ and $\underline{b}$ is defined as $\mathrm{ab}\cos\theta$, where θ is the angle between the directions of $\underline{a}$ and $\underline{b}$, drawn so that they have a common end-point and are directed away from that point. The Scalar Product is denoted by $\underline{a} \bullet \underline{b}$ so that

$$\underline{a} \bullet \underline{b} = \mathrm{ab}\cos\theta$$



$0 \leq \theta \leq \pi$

Scientific Application

If $\underline{b}$ were a force of magnitude b, then $\mathrm{b}\cos\theta$ would be its resolution (or component) along the vector $\underline{a}$. Hence, $\underline{a} \bullet \underline{b}$ would represent the work done by $\underline{b}$ in moving an object along the vector $\underline{a}$. Similarly, if $\underline{a}$ were a force of magnitude a, then $\mathrm{a}\cos\theta$ would be its resolution (or component) along the vector $\underline{b}$. Hence, $\underline{a} \bullet \underline{b}$ would represent the work done by $\underline{a}$ in moving an object along the vector $\underline{b}$.

8.3.2 DEDUCTIONS FROM THE DEFINITION OF DOT PRODUCT

(i) $\underline{a} \bullet \underline{a} = \mathrm{a}^2$.

Proof:

Clearly, the angle between $\underline{a}$ and itself is zero so that

$$\underline{a} \bullet \underline{a} = \mathrm{a.a}\cos 0 = \mathrm{a}^2.$$

(ii) $\underline{a} \bullet \underline{b}$ can be interpreted as the magnitude of one vector times the perpendicular projection of the other vector onto it.

Proof:

$\mathrm{b}\cos\theta$ is the perpendicular projection of $\underline{b}$ onto $\underline{a}$ and $\mathrm{a}\cos\theta$ is the perpendicular projection of $\underline{a}$ onto $\underline{b}$.

(iii) $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$.

Proof:

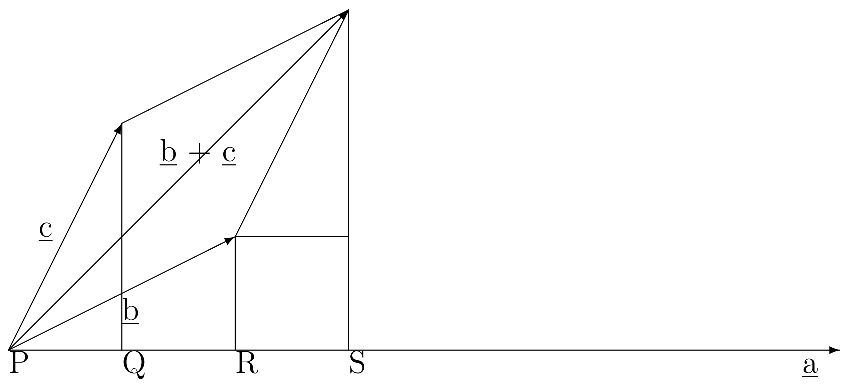
This follows since $ab\cos\theta = ba\cos\theta$.

(iv) Two non-zero vectors are perpendicular if and only if their Scalar Product is zero.

Proof:

$\underline{a}$ is perpendicular to $\underline{b}$ if and only if the angle $\theta = \frac{\pi}{2}$; that is, if and only if $\cos\theta = 0$ and hence, $ab\cos\theta = 0$.

(v) $\underline{a} \bullet (\underline{b} + \underline{c}) = \underline{a} \bullet \underline{b} + \underline{a} \bullet \underline{c}$.



The result follows from (ii) since the projections PR and PQ of $\underline{b}$ and $\underline{c}$ respectively onto $\underline{a}$ add up to the projection PS of $\underline{b} + \underline{c}$ onto $\underline{a}$.

Note:
We need to observe that RS is equal in length to PQ.

(vi) The Scalar Product of any two of the standard unit vectors **i**, **j** and **k** is given by the following multiplication table:

$\bullet$	**i**	**j**	**k**
i	1	0	0
j	0	1	0
k	0	0	1

That is, $\mathbf{i} \bullet \mathbf{i} = 1$, $\mathbf{j} \bullet \mathbf{j} = 1$ and $\mathbf{k} \bullet \mathbf{k} = 1$;

but,

$\mathbf{i} \bullet \mathbf{j} = 0$, $\mathbf{i} \bullet \mathbf{k} = 0$ and $\mathbf{j} \bullet \mathbf{k} = 0$.

8.3.3 THE STANDARD FORMULA FOR DOT PRODUCT

If

$$\underline{\mathrm{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \ \text{ and } \ \underline{\mathrm{b}} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{b}} = a_1b_1 + a_2b_2 + a_3b_3.$$

Proof:

This result follows easily from the multiplication table in (vi).

Note: The angle between two vectors

If θ is the angle between the two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$, then

$$\cos\theta = \frac{\underline{\mathrm{a}} \bullet \underline{\mathrm{b}}}{\mathrm{ab}}.$$

Proof:

This result is just a restatement of the original definition of a Scalar Product.

EXAMPLE

If

$$\underline{\mathrm{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = 3\mathbf{j} - 4\mathbf{k},$$

then,

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{b}} = 2 \times 0 + 2 \times 3 + (-1) \times (-4) = 10.$$

Hence,

$$\cos\theta = \frac{10}{\sqrt{2^2 + 2^2 + 1^2}\sqrt{3^2 + 4^2}} = \frac{10}{15} = \frac{2}{3}.$$

Thus,

$$\theta = 48.19^\circ \ \text{ or } \ 0.84 \text{ radians}.$$

8.3.4 THE VECTOR PRODUCT (or "Cross" Product)

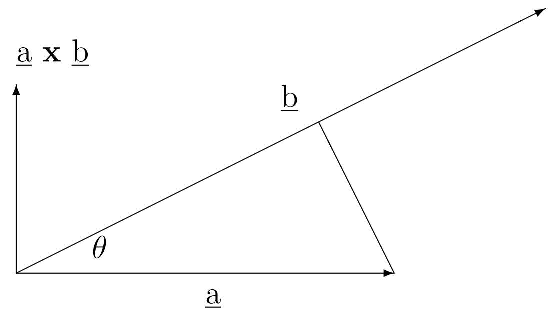
DEFINITION

If θ is the angle between two vectors $\underline{a}$ and $\underline{b}$, drawn so that they have a common end-point and are directed away from that point, then the **"Vector Product"** of $\underline{a}$ and $\underline{b}$ is defined to be a vector of magnitude

$$\text{ab}\sin\theta,$$

in a direction which is perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ and in a sense which obeys the **"right-hand-thread screw rule"** in turning from $\underline{a}$ to $\underline{b}$. The Vector Product is denoted by

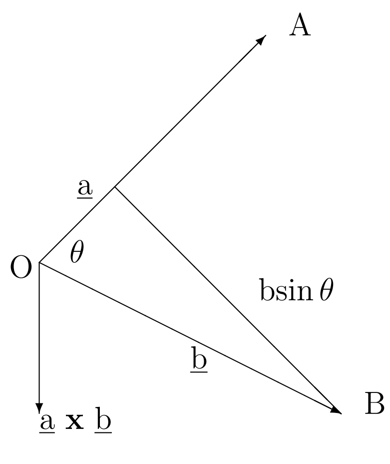
$$\underline{a}\ \mathbf{x}\ \underline{b}.$$



$0 \leq \theta \leq \pi$

Scientific Application

Consider the following diagram:



Suppose that the vector $\underline{OA} = \underline{a}$ represents a force acting at the point O and that the vector $\underline{OB} = \underline{b}$ is the position vector of the point B. Let the angle between the two vectors be θ.

Then the **"moment"** of the force $\underline{OA}$ about the point B is a vector whose magnitude is

$$\text{ab}\sin\theta$$

and whose direction is perpendicular to the plane of O, A and B in a sense which obeys the right-hand-thread screw rule in turning from $\underline{OA}$ to $\underline{OB}$. That is

$$\text{Moment} = \underline{a}\ \ \mathbf{x}\ \ \underline{b}.$$

Note:
The quantity $\text{b} \sin\theta$ is the perpendicular distance from the point B to the force $\underline{\text{OA}}$.

8.3.5 DEDUCTIONS FROM THE DEFINITION OF CROSS PRODUCT

(i)

$$\underline{\text{a}} \textbf{ x } \underline{\text{b}} = -(\underline{\text{b}} \textbf{ x } \underline{\text{a}}) = (-\underline{\text{b}}) \textbf{ x } \underline{\text{a}} = \underline{\text{b}} \textbf{ x } (-\underline{\text{a}}).$$

Proof:
This follows easily by considering the implications of the right-hand-thread screw rule.

(ii) Two vectors are parallel if and only if their Cross Product is a zero vector.

Proof:
Two vectors are parallel if and only if the angle, θ, between them is zero or π. In either case, $\sin\theta = 0$, which means that $\text{ab} \sin\theta = 0$; that is, $|\underline{\text{a}} \textbf{ x } \underline{\text{b}}| = 0$.

(iii) The Cross Product of a vector with itself is a zero vector.

Proof:
Clearly, the angle between a vector, $\underline{\text{a}}$, and itself is zero. Hence,

$$|\underline{\text{a}} \textbf{ x } \underline{\text{a}}| = \text{a.a.} \sin 0 = 0.$$

(iv)

$$\underline{\text{a}} \textbf{ x } (\underline{\text{b}} + \underline{\text{c}}) = \underline{\text{a}} \textbf{ x } \underline{\text{b}} + \underline{\text{a}} \textbf{ x } \underline{\text{c}}.$$

Proof:
This is best proved using the standard formula for a Cross Product in terms of components (see 8.3.6 below).

(v) The multiplication table for the Cross Products of the standard unit vectors **i**, **j** and **k** is as follows:

x	i	j	k
i	O	k	−j
j	− k	O	i
k	j	− i	O

That is,
i x i = **O**, **j x j** = **O**, **k x k** = **O**, **i x j** = **k**, **j x k** = **i**, **k x i** = **j**, **j x i** = − **k**, **k x j** = − **i** and **i x k** = − **j**.

8.3.6 THE STANDARD FORMULA FOR CROSS PRODUCT

If

$$\underline{\text{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \underline{\text{b}} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

then,

$$\underline{\text{a}} \textbf{ x } \underline{\text{b}} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This is usually abbreviated to

$$\underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

the symbol on the right hand side being called a **"determinant"** (see Unit 7.2).

EXAMPLES

1. If $\underline{\mathrm{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{\mathrm{b}} = 3\mathbf{j} - 4\mathbf{k}$, determine $\underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{b}}$.

   Solution

$$\underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{b}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 0 & 3 & -4 \end{vmatrix} = (-8+3)\mathbf{i} - (-8-0)\mathbf{j} + (6-0)\mathbf{k} = -5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}.$$

2. Show that, for any two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$,

$$(\underline{\mathrm{a}} + \underline{\mathrm{b}})\ \mathbf{x}\ (\underline{\mathrm{a}} - \underline{\mathrm{b}}) = 2(\underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{a}}).$$

   Solution

   The left hand side =

$$\underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{a}} - \underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{b}} + \underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{a}} - \underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{b}}.$$

   That is,

$$\mathbf{O} + \underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{a}} + \underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{a}} = 2(\underline{\mathrm{b}}\ \mathbf{x}\ \underline{\mathrm{a}}).$$

3. Determine the area of the triangle defined by the vectors

$$\underline{\mathrm{a}} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \underline{\mathrm{b}} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

   Solution

   If θ is the angle between the two vectors $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$, then the area of the triangle is $\frac{1}{2}\mathrm{ab}\sin\theta$ from elementary trigonometry. The area is therefore given by

$$\frac{1}{2}|\underline{\mathrm{a}}\ \mathbf{x}\ \underline{\mathrm{b}}|.$$

   That is,

$$\text{Area} = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{matrix}\right\| = \frac{1}{2}|4\mathbf{i} + \mathbf{j} - 5\mathbf{k}|.$$

   This gives

$$\text{Area} = \frac{1}{2}\sqrt{16+1+25} = \frac{1}{2}\sqrt{42} \simeq 3.24$$

8.3.7 EXERCISES

1. In the following cases, evaluate the Scalar Product $\underline{a} \bullet \underline{b}$ and hence determine the angle, θ between $\underline{a}$ and $\underline{b}$:

   (a) $\underline{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

   (b) $\underline{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{j} + \mathbf{k}$;

   (c) $\underline{a} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\underline{b} = 7\mathbf{i} - 2\mathbf{k}$.

2. Find out which of the following pairs of vectors are perpendicular and determine the cosine of the angle between those which are not:

   (a) $3\mathbf{j}$ and $2\mathbf{j} - 2\mathbf{k}$;

   (b) $\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$;

   (c) $2\mathbf{i} + 10\mathbf{k}$ and $7\mathbf{j}$;

   (d) $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

3. If $\underline{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\underline{b} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\underline{c} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, determine the length of the projection of $\underline{a} + \underline{c}$ onto $\underline{b}$.

4. If $\underline{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\underline{b} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, evaluate

$$(\underline{a} + \underline{b}) \bullet (\underline{a} - \underline{b}).$$

5. Determine the components of the vector $\underline{a}$ **x** $\underline{b}$ in the following cases:

   (a) $\underline{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;

   (b) $\underline{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\underline{b} = 3\mathbf{j} + \mathbf{k}$;

   (c) $\underline{a} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\underline{b} = 7\mathbf{i} - 2\mathbf{k}$.

6. If $\underline{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\underline{b} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, show that $\underline{a}$ **x** $\underline{b}$ is perpendicular to the vector $\underline{c} = 9\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

7. Given that $\underline{a}$ **x** $\underline{b}$ is perpendicular to each one of the vectors $\underline{a}$ and $\underline{b}$, determine a unit vector which is perpendicluar to each one of the vectors $\underline{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\underline{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
   Calculate also the sine of the angle, θ, between $\underline{a}$ and $\underline{b}$.

8. Determine the area of the triangle whose vertices are the points A(3, − 1,2), B(1, − 1, − 3) and C(4, − 3,1) in space. State your answer correct to two places of decimals.

8.3.8 ANSWERS TO EXERCISES

1. (a) Scalar Product $= -8$, $\cos\theta \simeq -0.381$ and $\theta \simeq 112.4°$;
   (b) Scalar Product $= 5$, $\cos\theta \simeq 0.645$ and $\theta \simeq 49.80°$;
   (c) Scalar Product $= -15$, $\cos\theta \simeq -0.485$ and $\theta \simeq 119.05°$
2. (a) Cosine $\simeq 0.707$;
   (b) The vectors are perpendicular;
   (c) The vectors are perpendicular;
   (d) Cosine $\simeq 0.190$
3. The length of the projection is $\frac{5}{3}$.
4. The value of the Dot Product is 24.
5. (a) The components are $-2, -7, -18$;
   (b) The components are $5, -1, 3$;
   (c) The components are $2, 26, 7$.
6. Show that $(\underline{a}\ \mathbf{x}\ \underline{b}) \bullet \underline{c} = 0$.
7. A unit vector is
$$\pm\frac{-3\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}}{\sqrt{155}}$$
and
$$\sin\theta = \frac{\sqrt{155}}{\sqrt{6}.\sqrt{26}} \simeq 0.997$$
8. The area is 6.42

"JUST THE MATHS"

UNIT NUMBER

8.4

VECTORS 4
(Triple products)

by

A.J.Hobson

UNIT 8.4 - VECTORS 4

TRIPLE PRODUCTS

INTRODUCTION

Once the ideas of scalar (dot) product and vector (cross) product for two vectors has been introduced, it is then possible to consider certain products of three or more vectors where, in some cases, there may be a mixture of scalar and vector products.

8.4.1 THE TRIPLE SCALAR PRODUCT

DEFINITION 1

Given three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, expressions such as

$$\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}), \quad \underline{b} \bullet (\underline{c} \mathbf{x} \underline{a}), \quad \underline{c} \bullet (\underline{a} \mathbf{x} \underline{b})$$

or

$$(\underline{a} \mathbf{x} \underline{b}) \bullet \underline{c}, \quad (\underline{b} \mathbf{x} \underline{c}) \bullet \underline{a}, \quad (\underline{c} \mathbf{x} \underline{a}) \bullet \underline{b}$$

are called **"triple scalar products"** because their results are all scalar quantities. Strictly speaking, the brackets are not necessary because there is no ambiguity without them; that is, it is not possible to form the vector product of a vector with the result of a scalar product.

In the work which follows, we shall take $\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c})$ as the typical triple scalar product.

The formula for a triple scalar product

Suppose that

$$\underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \text{ and } \underline{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

Then,

$$\underline{a} \bullet (\underline{b} \mathbf{x} \underline{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From the basic formula for scalar product, this becomes

$$\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c3 \end{vmatrix}.$$

Notes:

(i) From Unit 7.3, if two rows of a determinant are interchanged, the determinant remains unchanged in numerical value but is altered in sign.

Hence,

$$\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}}) = -\underline{\mathrm{a}} \bullet (\underline{\mathrm{c}} \mathbf{\ x\ } \underline{\mathrm{b}}) = \underline{\mathrm{c}} \bullet (\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{b}}) = -\underline{\mathrm{c}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{a}}) = \underline{\mathrm{b}} \bullet (\underline{\mathrm{c}} \mathbf{\ x\ } \underline{\mathrm{a}}) = -\underline{\mathrm{b}} \bullet (\underline{\mathrm{a}} \mathbf{\ x\ } \underline{\mathrm{c}}).$$

In other words, the **"cyclic permutations"** of $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$ are all equal in numerical value and in sign, while the remaining permutations are equal to $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$ in numerical value, but opposite in sign.

(ii) The triple scalar product, $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$, is often denoted by $[\underline{\mathrm{a}}, \underline{\mathrm{b}}, \underline{\mathrm{c}}]$.

EXAMPLE

Evaluate the triple scalar product, $\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}})$, in the case when

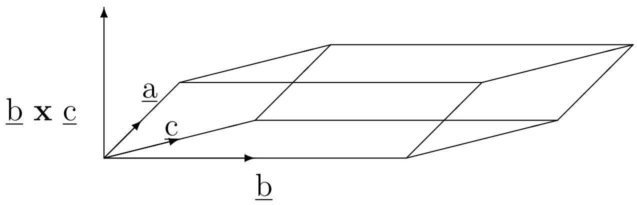
$$\underline{\mathrm{a}} = 2\mathbf{i} + \mathbf{k}, \quad \underline{\mathrm{b}} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ and } \underline{\mathrm{c}} = -\mathbf{i} + \mathbf{j}$$

Solution

$$\underline{\mathrm{a}} \bullet (\underline{\mathrm{b}} \mathbf{\ x\ } \underline{\mathrm{c}}) = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 2.(-2) - 0.(2) + 1.(2) = -2.$$

A geometrical application of the triple scalar product

Suppose that the three vectors $\underline{\mathrm{a}}$, $\underline{\mathrm{b}}$ and $\underline{\mathrm{c}}$ lie along three adjacent edges of a parallelepiped (correct pronunciation, *"parallel-epi-ped"*) as shown in the following diagram:



The area of the base of the parallelepiped, from the geometrical properties of vector products, is the **magnitude** of the vector, $\underline{b}$ **x** $\underline{c}$, which is perpendicular to the base.

The perpendicular height of the parallelepiped is the projection of the vector $\underline{a}$ onto the vector $\underline{b}$ **x** $\underline{c}$; that is,

$$\frac{\underline{a} \bullet (\underline{b} \mathbf{\ x\ } \underline{c})}{|\underline{b} \mathbf{\ x\ } \underline{c}|}.$$

Hence, since the volume, V, of the parallelepiped is equal to the area of the base times the perpendicular height, we conclude that

$$V = \underline{a} \bullet (\underline{b} \mathbf{\ x\ } \underline{c}),$$

at least numerically, since the triple scalar product could turn out to be negative.

Note:
The above geometrical application also provides a condition that three given vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$ lie in the same plane; that is, they are **"coplanar"**.

The condition is that

$$\underline{a} \bullet (\underline{b} \mathbf{\ x\ } \underline{c}) = 0,$$

since the three vectors would determine a parallelepiped whose volume is zero.

8.4.2 THE TRIPLE VECTOR PRODUCT

DEFINITION 2

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are any three vectors, then the expression

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c})$$

is called the **"triple vector product"** of $\underline{a}$ with $\underline{b}$ and $\underline{c}$.

Notes:

(i) The triple vector product is clearly a vector quantity.

(ii) The inclusion of the brackets in a triple vector product is important since it can be shown that, in general,

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c}) \neq (\underline{a} \textbf{ x } \underline{b}) \textbf{ x } \underline{c}.$$

For example, if the three vectors are considered as position vectors, with the origin as a common end-point, then $\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c})$ is perpendicular to both $\underline{a}$ and $\underline{b} \textbf{ x } \underline{c}$, the latter of which is already perpendicular to both $\underline{b}$ and $\underline{c}$. It therefore lies in the plane of $\underline{b}$ and $\underline{c}$.

Consequently, $(\underline{a} \textbf{ x } \underline{b}) \textbf{ x } \underline{c}$, which is the same as $-\underline{c} \textbf{ x } (\underline{a} \textbf{ x } \underline{b})$, will lie in the plane of $\underline{a}$ and $\underline{b}$.

Hence it will, in general, be different from $\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c})$.

The formula for a triple vector product

Suppose that

$$\underline{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \underline{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \quad \text{and} \quad \underline{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

Then,

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c}) = \underline{a} \textbf{ x } \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \end{vmatrix}.$$

The **i** component of this vector is equal to

$$a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) = b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3);$$

but, by adding and subtracting $a_1b_1c_1$, the right hand side can be rearranged in the form

$$(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1,$$

which is the **i** component of the vector $(\underline{\mathrm{a}} \bullet \underline{\mathrm{c}})\underline{\mathrm{b}} - (\underline{\mathrm{a}} \bullet \underline{\mathrm{b}})\underline{\mathrm{c}}$.

Similar expressions can be obtained for the **j** and **k** components and we may conclude that

$$\underline{\mathrm{a}} \ \mathbf{x} \ (\underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{c}}) = (\underline{\mathrm{a}} \bullet \underline{\mathrm{c}})\underline{\mathrm{b}} - (\underline{\mathrm{a}} \bullet \underline{\mathrm{b}})\underline{\mathrm{c}}.$$

EXAMPLE

Determine the triple vector product of $\underline{\mathrm{a}}$ with $\underline{\mathrm{b}}$ and $\underline{\mathrm{c}}$, where

$$\underline{\mathrm{a}} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \ \ \underline{\mathrm{b}} = -2\mathbf{i} + 3\mathbf{j} \ \text{ and } \ \underline{\mathrm{c}} = 3\mathbf{k}.$$

Solution

$$\underline{\mathrm{a}} \bullet \underline{\mathrm{c}} = -3 \ \text{ and } \ \underline{a} \bullet \underline{\mathrm{b}} = 4.$$

Hence,

$$\underline{\mathrm{a}} \ \mathbf{x} \ (\underline{\mathrm{b}} \ \mathbf{x} \ \underline{\mathrm{c}}) = -3\underline{\mathrm{b}} - 4\underline{\mathrm{c}} = 6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k}.$$

8.4.3 EXERCISES

1. Evaluate the triple scalar product, $\underline{a} \bullet (\underline{b} \textbf{ x } \underline{c})$, in the case when

$$\underline{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}, \quad \underline{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \underline{c} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}.$$

2. Determine the volume of the parallelepiped with adjacent edges defined by the vectors

$$\underline{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \underline{b} = 2\mathbf{i} - \mathbf{j} \quad \text{and} \quad \underline{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

3. Determine the triple vector product of $\underline{a}$ with $\underline{b}$ and $\underline{c}$ in the cases where

   (a)

$$\underline{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \underline{b} = 2\mathbf{i} + \mathbf{j} \quad \text{and} \quad \underline{c} = \mathbf{i} + \mathbf{j} + \mathbf{k};$$

   (b)

$$\underline{a} = 4\mathbf{i} - \mathbf{k}, \quad \underline{b} = 3\mathbf{i} + 5\mathbf{j} - \mathbf{k} \quad \text{and} \quad \underline{c} = \mathbf{i} - \mathbf{j} - \mathbf{k};$$

   (c)

$$\underline{a} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \underline{b} = 5\mathbf{i} \quad \text{and} \quad \underline{c} = -\mathbf{j} + 3\mathbf{k}.$$

4. Show that the following three vectors are coplanar:

$$\underline{a} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \underline{b} = 4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \underline{c} = -3\mathbf{i} + 12\mathbf{j} - 9\mathbf{k}.$$

5. Show that

$$[(\underline{a} + \underline{b}), (\underline{b} + \underline{c}), (\underline{c} + \underline{a})] = 2[\underline{a}, \underline{b}, \underline{c}].$$

6. Show that

$$(\underline{a} \textbf{ x } \underline{b}) \textbf{ x} (\underline{c} \textbf{ x } \underline{d}) = [\underline{a}, \underline{c}, \underline{d}]\underline{b} - [\underline{b}, \underline{c}, \underline{d}]\underline{a} = [\underline{a}, \underline{b}, \underline{d}]\underline{c} - [\underline{a}, \underline{b}, \underline{c}]\underline{d}.$$

7. Show that

$$\underline{a} \textbf{ x } (\underline{b} \textbf{ x } \underline{c}) + \underline{b} \textbf{ x } (\underline{c} \textbf{ x } \underline{a}) + \underline{c} \textbf{ x } (\underline{a} \textbf{ x } \underline{b}) = \mathbf{O}.$$

8. Show that

$$(\underline{a} \textbf{ x } \underline{b}) \bullet (\underline{c} \textbf{ x } \underline{d}) = \begin{vmatrix} \underline{a} \bullet \underline{c} & \underline{a} \bullet \underline{d} \\ \underline{b} \bullet \underline{c} & \underline{b} \bullet \underline{d} \end{vmatrix}.$$

8.4.4 ANSWERS TO EXERCISES

1.
$$20.$$

2.
$$8.$$

3. (a)
$$7\mathbf{i} + 4\mathbf{j} + \mathbf{k};$$

   (b)
$$2\mathbf{i} + 38\mathbf{j} + 8\mathbf{k};$$

   (c)
$$5\mathbf{i} + 20\mathbf{j} - 60\mathbf{k}.$$

4. Show that the triple scalar product is zero.

5. Remove all brackets and use the fact that a triple scalar product is zero when two of the vectors are the same.

6. Use the triple vector product formula.

7. Use the triple vector product formula.

8. Rearrange in the form
$$\underline{a} \bullet [\underline{b} \mathbf{\ x\ } (\underline{c} \mathbf{\ x\ } \underline{d})].$$

"JUST THE MATHS"

UNIT NUMBER

8.5

VECTORS 5
(Vector equations of straight lines)

by

A.J.Hobson

UNIT 8.5 - VECTORS 5

VECTOR EQUATIONS OF STRAIGHT LINES

8.5.1 INTRODUCTION

The concept of vector notation and vector products provides a convenient method of representing straight lines and planes in space by simple vector equations. Such vector equations may then, if necessary, be converted back to conventional cartesian or parametric equations.

We shall assume that the position vector of a variable point, $\mathrm{P}(x, y, z)$, is given by

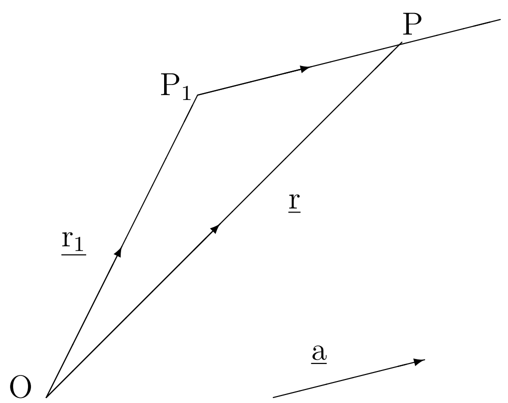
$$\underline{\mathrm{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

and that the position vectors of fixed points, such as $\mathrm{P}_1(x_1, y_1, z_1)$ and $\mathrm{P}_2(x_2, y_2, z_2)$, are given by

$$\underline{\mathrm{r}_1} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \quad \underline{\mathrm{r}_2} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}, \quad \text{etc.}$$

8.5.2 THE STRAIGHT LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

We consider, here, the straight line passing through the point, P_1, with position vector, $\underline{\mathrm{r}_1}$, and parallel to the vector, $\underline{\mathrm{a}} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.



From the diagram,

$$\underline{\mathrm{OP}} = \underline{\mathrm{OP_1}} + \underline{\mathrm{P_1P}}.$$

But,

$$\underline{\mathrm{P_1P}} = t\underline{\mathrm{a}},$$

for some number t.

Hence,

$$\underline{\mathrm{r}} = \underline{\mathrm{r_1}} + t\underline{\mathrm{a}},$$

which is the vector equation of the straight line.

The components of $\underline{\mathrm{a}}$ form a set of direction ratios for the straight line.

Notes:

(i) The vector equation of a straight line passing through the origin and parallel to a given vector $\underline{\mathrm{a}}$ will be of the form

$$\underline{r} = t\underline{\mathrm{a}}.$$

(ii) By equating **i**, **j** and **k** components on both sides, the vector equation of the straight line passing through $\mathrm{P_1}$ and parallel to $\underline{\mathrm{a}}$ leads to parametric equations

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t;$$

and, if these are solved for the parameter, t, we obtain

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3},$$

which is the standard cartesian form of the straight line.

EXAMPLES

1. Determine the vector equation of the straight line passing through the point with position vector $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and parallel to the vector, $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

   Express the vector equation of the straight line in standard cartesian form.

   Solution

   The vector equation of the straight line is

$$\underline{\mathrm{r}} = \mathbf{i} - 3\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

   or

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (1 + 2t)\mathbf{i} + (-3 + 3t)\mathbf{j} + (1 - 4t)\mathbf{k}.$$

   Eliminating t from each component, we obtain the cartesian form of the straight line,

$$\frac{x-1}{2} = \frac{y+3}{3} = \frac{z-1}{-4}.$$

2. The equations

$$\frac{3x+1}{2} = \frac{y-1}{2} = \frac{-z+5}{3}$$

   determine a straight line. Express them in vector form and obtain a set of direction ratios for the straight line.

   Solution

   Rewriting the equations so that the coefficients of x, y and z are unity, we have

$$\frac{x+\frac{1}{3}}{\frac{2}{3}} = \frac{y-1}{2} = \frac{z-5}{-3}.$$

   Hence, in vector form, the equation of the line is

$$\underline{\mathrm{r}} = -\frac{1}{3}\mathbf{i} + \mathbf{j} + 5\mathbf{k} + t\left(\frac{2}{3}\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\right).$$

   Thus, a set of direction ratios for the straight line are $\frac{2}{3} : 2 : -3$ or $2 : 6 : -9$.

3. Show that the two straight lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \;\; \text{and} \;\; \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

where

$$\underline{\mathrm{r}_1} = \mathbf{j}, \;\; \underline{\mathrm{a}_1} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},$$

and

$$\underline{\mathrm{r}_2} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \;\; \underline{\mathrm{a}_2} = -2\mathbf{i} - 2\mathbf{j},$$

have a common point and determine its co-ordinates.

Solution

Any point on the first line is such that

$$x = t, \;\; y = 1 + 2t, \;\; z = -t,$$

for some parameter value, t; and any point on the second line is such that

$$x = 1 - 2s, \;\; y = 1 - 2s, \;\; z = 1,$$

for some parameter value, s.

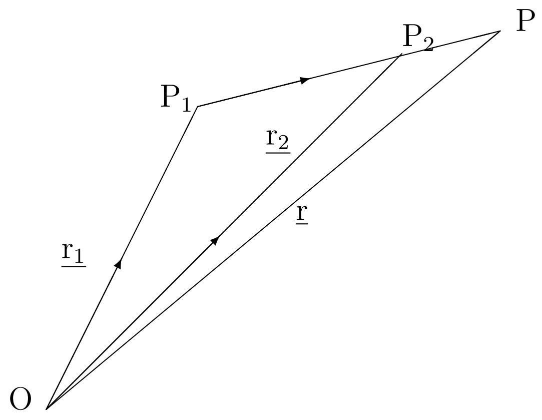
The lines have a common point if it is possible to find values of t and s such these are the same point.

In fact, $t = -1$ and $s = 1$ are suitable values and give the common point $(-1, -1, 1)$.

8.5.3 THE STRAIGHT LINE PASSING THROUGH TWO GIVEN POINTS

If a straight line passes through the two given points, P_1 and P_2, then it is certainly parallel to the vector,

$$\underline{\mathrm{a}} = \underline{\mathrm{P}_1\mathrm{P}_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$



Thus, the vector equation of the straight line is

$$\underline{\mathrm{r}} = \underline{\mathrm{r_1}} + t\underline{\mathrm{a}}$$

as before.

Notes:

(i) The parametric equations of the straight line passing through the points, P_1 and P_2, are

$$x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t, \quad z = z_1 + (z_2 - z_1)t;$$

and we notice that the "base-points" of the parametric representation (that is, P_1 and P_2) have parameter values $t = 0$ and $t = 1$ respectively.

(ii) The standard cartesian form of the straight line passing through P_1 and P_2 is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

EXAMPLE

Determine the vector equation of the straight line passing through the two points, $\mathrm{P}_1(3,-1,5)$ and $\mathrm{P}_2(-1,-4,2)$.

Solution

$$\underline{\mathrm{OP}_1} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

and

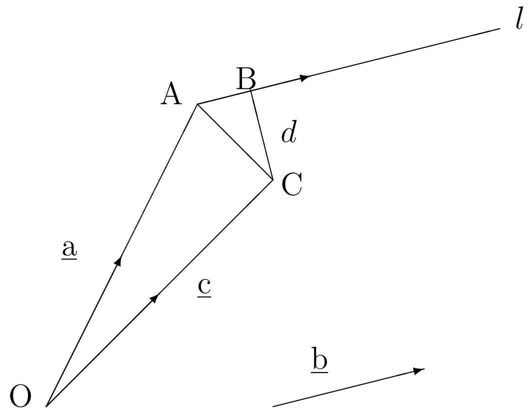
$$\underline{\mathrm{P}_1\mathrm{P}_2} = (-1-3)\mathbf{i} + (-4+1)\mathbf{j} + (2-5)\mathbf{k} = -4\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}.$$

Hence, the vector equation of the straight line is

$$\underline{\mathrm{r}} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} - t(4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}).$$

8.5.4 THE PERPENDICULAR DISTANCE OF A POINT FROM A STRAIGHT LINE

For a straight line, l, passing through a given point, A, with position vector, $\underline{\mathrm{a}}$ and parallel to a given vector, $\underline{\mathrm{b}}$, it may be necessary to determine the perpendicular distance, d, from this line, of a point, C, with position vector, $\underline{\mathrm{c}}$.



From the diagram, with Pythagoras' Theorem,

$$d^2 = (\mathrm{AC})^2 - (\mathrm{AB})^2.$$

But, $\underline{\mathrm{AC}} = \underline{\mathrm{c}} - \underline{\mathrm{a}}$, so that

$$(\mathrm{AC})^2 = (\underline{\mathrm{c}} - \underline{\mathrm{a}}) \bullet (\underline{\mathrm{c}} - \underline{\mathrm{a}}).$$

Also, the length, AB, is the projection of $\underline{\mathrm{AC}}$ onto the line, l, which is parallel to $\underline{\mathrm{b}}$.

Hence,

$$\mathrm{AB} = \frac{(\underline{\mathrm{c}} - \underline{\mathrm{a}}) \bullet \underline{\mathrm{b}}}{\mathrm{b}},$$

which gives the result

$$d^2 = (\underline{\mathrm{c}} - \underline{\mathrm{a}}) \bullet (\underline{\mathrm{c}} - \underline{\mathrm{a}}) - \left[\frac{(\underline{\mathrm{c}} - \underline{\mathrm{a}}) \bullet \underline{\mathrm{b}}}{\mathrm{b}}\right]^2.$$

From this result, d may be deduced.

EXAMPLE

Determine the perpendicular distance of the point $(3, -1, 7)$ from the straight line passing through the two points, $(2, 2, -1)$ and $(0, 3, 5)$.

Solution

In the standard formula, we have

$$\underline{\mathrm{a}} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{j},$$

$$\underline{b} = (0 - 2)\mathbf{i} + (3 - 2)\mathbf{j} + (5 - [-1])\mathbf{k} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k},$$

$$\mathrm{b} = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41},$$

$$\underline{\mathrm{c}} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k},$$

and

$$\underline{\mathrm{c}} - \underline{\mathrm{a}} = (3-2)\mathbf{i} + (-1-2)\mathbf{j} + (7-[-1])\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 8\mathbf{k}.$$

Hence, the perpendicular distance, d, is given by

$$d^2 = 1^2 + (-3)^2 + 8^2 - \frac{(1)(-2) + (-3)(1) + (8)(6)}{\sqrt{41}} = 74 - \frac{43}{\sqrt{41}}$$

which gives $d \simeq 8.20$.

8.5.5 THE SHORTEST DISTANCE BETWEEN TWO PARALLEL STRAIGHT LINES

The result of the previous section may also be used to determine the shortest distance between two parallel straight lines, because this will be the perpendicular distance from one of the lines of any point on the other line.

We may consider the perpendicular distance between

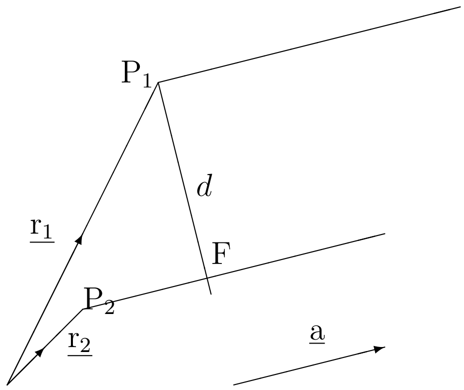
(a) the straight line passing through the fixed point with position vector $\underline{\mathrm{r}_1}$ and parallel to the fixed vector, $\underline{\mathrm{a}}$

and

(b) the straight line passing through the fixed point with position vector $\underline{\mathrm{r}_2}$, also parallel to the fixed vector, $\underline{\mathrm{a}}$.

These will have vector equations,

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{a}.$$



In the diagram, F is the foot of the perpendicular onto the second line from the point P_1 on the first line and the length of this perpendicular is d.

Hence,

$$d^2 = (\underline{r_2} - \underline{r_1}) \bullet (\underline{r_2} - \underline{r_1}) - \left[\frac{(\underline{r_2} - \underline{r_1}) \bullet \underline{a}}{a}\right]^2.$$

EXAMPLE

Determine the shortest distance between the straight line passing through the point with position vector $\underline{r_1} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$, parallel to the vector $\underline{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and the straight line passing through the point with position vector $\underline{r_2} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, parallel to $\underline{b}$.

Solution

From the formula,

$$d^2 = (-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \bullet (-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) - \left[\frac{(-6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}}\right]^2.$$

That is,

$$d^2 = (36 + 16 + 4) - \left[\frac{-6 + 4 - 2}{\sqrt{3}}\right]^2 = 56 - \frac{16}{3} = \frac{152}{3},$$

which gives

$$d \simeq 7.12$$

8.5.6 THE SHORTEST DISTANCE BETWEEN TWO SKEW STRAIGHT LINES

Two straight lines are said to be **"skew"** if they are not parallel and do not intersect each other. It may be shown that such a pair of lines will always have a common perpendicular (that is, a straight line segment which meets both, and is perpendicular to both). Its length will be the shortest distance between the two skew lines.

For the straight lines, whose vector equations are

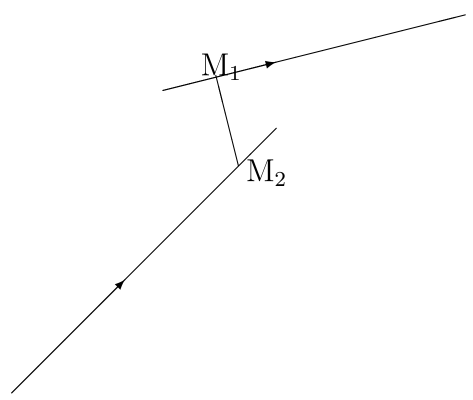
$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

suppose that the point, M_1, on the first line and the point, M_2, on the second line are the ends of the common perpendicular and have position vectors, $\underline{\mathrm{m}_1}$ and $\underline{\mathrm{m}_2}$, respectively.

Then,

$$\underline{\mathrm{m}_1} = \underline{\mathrm{r}_1} + t_1\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{m}_2} = \underline{\mathrm{r}_2} + t_2\underline{\mathrm{a}_2},$$

for some values, t_1 and t_2, of the parameter, t.



Firstly, we have

$$\underline{\mathrm{M_1M_2}} = \underline{\mathrm{m}_2} - \underline{\mathrm{m}_1} = (\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) + t_2\underline{\mathrm{a}_2} - t_1\underline{\mathrm{a}_1}.$$

Secondly, a vector which is certainly perpendicular to both of the skew lines is $\underline{\mathrm{a}_1}$ **x** $\underline{\mathrm{a}_2}$, so that a unit vector perpendicular to both of the skew lines is

$$\frac{\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}}{|\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}|}.$$

This implies that

$$(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) + t_2\underline{\mathrm{a}_2} - t_1\underline{\mathrm{a}_1} = \pm d\frac{\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}}{|\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}|},$$

where d is the shortest distance between the skew lines.

Finally, if we take the scalar (dot) product of both sides of this result with the vector $\underline{\mathrm{a}_1}$ **x** $\underline{\mathrm{a}_2}$, we obtain

$$(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) \ \bullet \ (\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}) = \pm d\frac{|\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}|^2}{|\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}|},$$

giving

$$d = \left| \frac{(\underline{\mathrm{r}_2} - \underline{\mathrm{r}_1}) \ \bullet \ (\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2})}{|\underline{\mathrm{a}_1} \ \mathbf{x} \ \underline{\mathrm{a}_2}|} \right|.$$

EXAMPLE

Determine the perpendicular distance between the two skew lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \ \text{ and } \ \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

where

$$\underline{\mathrm{r}_1} = 9\mathbf{j} + 2\mathbf{k}, \ \ \underline{\mathrm{a}_1} = 3\mathbf{i} - \mathbf{j} + \mathbf{k},$$

$$\underline{r_2} = -6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}, \quad \underline{a_2} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

Solution

$$\underline{r_2} - \underline{r_1} = -6\mathbf{i} - 14\mathbf{j} + 8\mathbf{k}$$

and

$$\underline{a_1} \ \mathbf{x} \ \underline{a_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix} = -6\mathbf{i} - 15\mathbf{j} + 3\mathbf{k},$$

so that

$$d = \frac{(-6)(-6) + (-14)(-15) + (8)(3)}{\sqrt{36 + 225 + 9}} = \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}.$$

8.5.7 EXERCISES

1. Determine the vector equation, and hence the parametric equations, of the straight line which passes through the point, $(5, -2, 1)$, and is parallel to the vector $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

2. The equations

$$\frac{-x+2}{7} = \frac{3y-1}{5} = \frac{2z+1}{3}$$

   determine a straight line. Determine the equation of the line in vector form, and state a set of direction ratios for this line.

3. Show that there is a point common to the two straight lines

$$\underline{r} = \underline{r_1} + t\underline{a_1} \quad \text{and} \quad \underline{r} = \underline{r_2} + t\underline{a_2},$$

   where

$$\underline{r_1} = 3\mathbf{j} + 2\mathbf{k}, \quad \underline{r_2} = -2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$

and

$$\underline{\mathrm{a}_1} = 2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}, \quad \underline{\mathrm{a}_2} = 9\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$$

Determine the co-ordinates of the common point.

4. Determine, in standard cartesian form, the equation of the straight line passing through the two points, $(-2, 4, 9)$ and $(2, -1, 6)$.

5. Determine the perpendicular distance of the point $(0, -2, 5)$ from the straight line which passes through the point $(1, -1, 3)$ and is parallel to the vector $3\mathbf{i} + \mathbf{j} + \mathbf{k}$.

6. Determine the shortest distance between the two parallel straight lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}},$$

where

$$\underline{\mathrm{r}_1} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \underline{\mathrm{r}_2} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

and

$$\underline{\mathrm{a}} = \mathbf{i} + 5\mathbf{j} + \mathbf{k}.$$

7. Determine the shortest distance between the two skew straight lines

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}_1} \quad \text{and} \quad \underline{\mathrm{r}} = \underline{\mathrm{r}_2} + t\underline{\mathrm{a}_2},$$

where

$$\underline{\mathrm{r}_1} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \underline{\mathrm{r}_2} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

and

$$\underline{\mathrm{a}_1} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \underline{\mathrm{a}_2} = 5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

8.5.8 ANSWERS TO EXERCISES

1.

$$\underline{\mathrm{r}} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

giving

$$x = 5 + t, \quad y = -2 - 3t, \quad z = 1 + t.$$

2. In vector form, the equation of the line is

$$\underline{\mathrm{r}} = 2\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{1}{2}\mathbf{k} + t\left(-7\mathbf{i} + \frac{5}{3}\mathbf{j} - \frac{3}{2}\mathbf{k}\right),$$

and set of direction ratios is

$$-42 : 10 : 9$$

3. The common point has co-ordinates $(1, 1, 1)$.

4.

$$\frac{x+2}{4} = \frac{y-4}{-5} = \frac{z-9}{-3}.$$

5.

$$d = \sqrt{\frac{62}{11}} \simeq 2.37$$

6.

$$d = \sqrt{\frac{98}{3}} \simeq 5.72$$

7.

$$d = \frac{5\sqrt{6}}{6} \simeq 2.04$$

"JUST THE MATHS"

UNIT NUMBER

8.6

VECTORS 6
(Vector equations of planes)

by

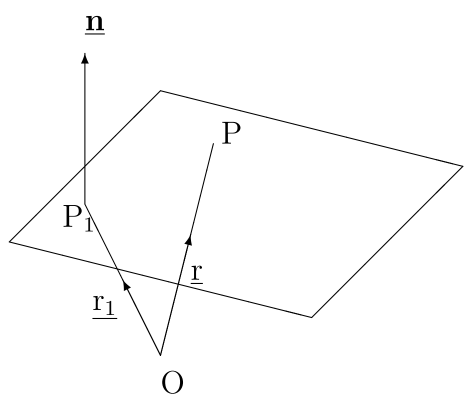
A.J.Hobson

UNIT 8.6 - VECTORS 6

VECTOR EQUATIONS OF PLANES

8.6.1 THE PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN VECTOR

A plane in space is completely specified if we know one point in it, together with a vector which is perpendicular to the plane.



In the diagram, the given point is P_1, with position vector, r_1, and the given vector is $\underline{n}$.

Hence, the vector, $\underline{P_1P}$, is perpendicular to $\underline{n}$, which leads to the equation

$$(\underline{r} - \underline{r_1}) \bullet \underline{n} = 0$$

or

$$\underline{r} \bullet \underline{n} = \underline{r_1} \bullet \underline{n} = d \text{ say.}$$

Notes:

(i) In the particular case when $\underline{n}$ is a unit vector, the constant, d, represents the perpendicular projection of $\underline{r_1}$ onto $\underline{n}$, which is therefore the perpendicular distance of the origin from the plane.

(ii) If $\underline{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\underline{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the cartesian form for the equation of the above plane will be

$$ax + by + cz = d.$$

That is, it is simply a linear equation in the variables x, y and z.

EXAMPLE

Determine the vector equation and, hence, the cartesian equation of the plane passing through the point with position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and perpendicular to the vector $\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.

Solution

The vector equation is

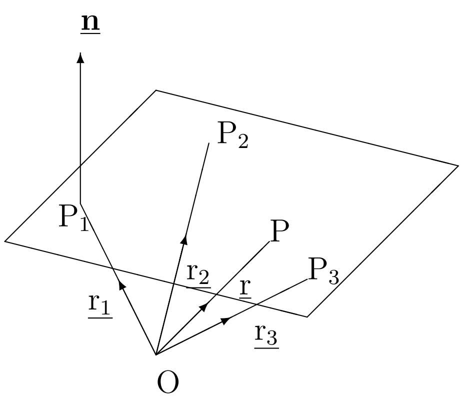
$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = (3)(1) + (-2)(-4) + (1)(-1) = 10$$

and, hence, the cartesian equation is

$$x - 4y - z = 10.$$

8.6.2 THE PLANE PASSING THROUGH THREE GIVEN POINTS

We consider a plane passing through the points, $\mathrm{P}_1(x_1, y_1, z_1)$, $\mathrm{P}_2(x_2, y_2, z_2)$ and $\mathrm{P}_3(x_3, y_3, z_3)$.



In the diagram, a suitable vector for $\underline{\mathrm{n}}$ is

$$\underline{\mathrm{P_1P_2}}\ \mathbf{x}\ \underline{\mathrm{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

and, hence, the equation,

$$(\underline{\mathrm{r}} - \underline{\mathrm{r}_1}) \bullet \underline{\mathrm{n}} = 0,$$

of the plane becomes

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

But, from the properties of determinants, this is equivalent to

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

which is the standard equation of the plane through the three given points.

EXAMPLE

Determine the cartesian equation of the plane passing through the three points, $(0, 2, -1)$, $(3, 0, 1)$ and $(-3, -2, 0)$.

Solution

The equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & -1 & 1 \\ 3 & 0 & 1 & 1 \\ -3 & -2 & 0 & 1 \end{vmatrix} = 0,$$

which, on expansion and simplification, gives

$$2x - 3y - 6z = 0,$$

showing that the plane also passes through the origin.

8.6.3 THE POINT OF INTERSECTION OF A STRAIGHT LINE AND A PLANE

First, we recall (from Unit 8.5) that the vector equation of a straight line passing through the fixed point, with position vector $\underline{\mathrm{r}_1}$, and parallel to the fixed vector $\underline{\mathrm{a}}$, is

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{a}}.$$

For the point of intersection of this line with the plane, whose vector equation is

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d,$$

the value of t must be such that

$$(\underline{\mathrm{r}_1} + t\underline{\mathrm{a}}) \bullet \underline{\mathrm{n}} = d,$$

which is an equation from which the appropriate value of t and, hence, the point of intersection may be found.

EXAMPLE

Determine the point of intersection of the plane, whose vector equation is

$$\underline{\mathrm{r}} \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7,$$

and the straight line passing through the point, $(4, -1, 3)$, which is parallel to the vector $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

Solution

We need to obtain the value of the parameter, t, such that

$$(4\mathbf{i} - \mathbf{j} + 3\mathbf{k} + t[2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}]) \bullet (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 7.$$

That is,

$$(4+2t)(1)+(-1-2t)(-3)+(3+5t)(-1)=7 \quad \text{or} \quad 4+3t=7,$$

which gives $t=1$; and, hence, the point of intersection is $(4+2,-1-2,3+5)=(6,-3,8)$.

8.6.4 THE LINE OF INTERSECTION OF TWO PLANES

Suppose we are given two non-parallel planes whose vector equations are

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}_1} = d_1 \quad \text{and} \quad \underline{\mathrm{r}} \bullet \underline{\mathrm{n}_2} = d_2.$$

Their line of intersection will be perpendicular to both $\underline{\mathrm{n}_1}$ and $\underline{\mathrm{n}_2}$, since these are the normals to the two planes.

The line of intersection will thus be parallel to $\underline{\mathrm{n}_1}$ **x** $\underline{\mathrm{n}_2}$, and all it remains to do, to obtain the vector equation of this line, is to determine any point on it.

For convenience, we may take the point (common to both planes) for which one of x, y or z is zero.

EXAMPLE

Determine the vector equation, and hence the cartesian equations (in standard form), of the line of intersection of the planes whose vector equations are

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}_1} = 2 \quad \text{and} \quad \underline{\mathrm{r}} \bullet \underline{\mathrm{n}_2} = 17,$$

where

$$\underline{\mathrm{n}_1} = \mathbf{i}+\mathbf{j}+\mathbf{k} \quad \text{and} \quad \underline{\mathrm{n}_2} = 4\mathbf{i}+\mathbf{j}+2\mathbf{k}.$$

Solution

Firstly,

$$\underline{\mathrm{n}_1} \ \mathbf{x} \ \underline{\mathrm{n}_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}.$$

Secondly, the cartesian equations of the two planes are

$$x + y + z = 2 \quad \text{and} \quad 4x + y + 2z = 17;$$

and, when $z = 0$, these become

$$x + y = 2 \quad \text{and} \quad 4x + y = 17,$$

which, as simultaneous linear equations, have a common solution of $x = 5$, $y = -3$.

Thirdly, therefore, a point on the line of intersection is $(5, -3, 0)$, which has position vector $5\mathbf{i} - 3\mathbf{j}$.

Hence, the vector equation of the line of intersection is

$$\underline{\mathrm{r}} = 5\mathbf{i} - 3\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}).$$

Finally, since $x = 5 + t$, $y = -3 + 2t$ and $z = -3t$ the line of intersection is represented, in standard cartesian form, by

$$\frac{x - 5}{1} = \frac{y + 3}{2} = \frac{z}{-3} \quad (= t).$$

8.6.5 THE PERPENDICULAR DISTANCE OF A POINT FROM A PLANE

Given the plane whose vector equation is $\underline{\mathrm{r}} \bullet \underline{\mathrm{n}} = d$ and the point, P_1, whose position vector is $\underline{\mathrm{r}_1}$, the straight line through the point, P_1, which is perpendicular to the plane has vector equation

$$\underline{\mathrm{r}} = \underline{\mathrm{r}_1} + t\underline{\mathrm{n}}.$$

This line meets the plane at the point, P_0, with position vector $\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}$, where

$$(\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) \bullet \underline{\mathrm{n}} = d.$$

That is,

$$(\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}}) + t_0\mathrm{n}^2 = d.$$

Hence,

$$t_0 = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}^2}$$

Finally, the vector $\underline{\mathrm{P}_0\mathrm{P}_1} = (\underline{\mathrm{r}_1} + t_0\underline{\mathrm{n}}) - \underline{\mathrm{r}_1} = t_0\underline{\mathrm{n}}$

and its magnitude, $t_0\mathrm{n}$, will be the perpendicular distance, p, of the point P_1 from the plane.

In other words,

$$p = \frac{d - (\underline{\mathrm{r}_1} \bullet \underline{\mathrm{n}})}{\mathrm{n}}.$$

Note:
In terms of cartesian co-ordinates, this formula is equivalent to

$$p = \frac{d - (ax_1 + by_1 + cz_1)}{\sqrt{a^2 + b^2 + c^2}},$$

where a, b and c are the **i**, **j** and **k** components of $\underline{\mathrm{n}}$ respectively.

EXAMPLE

Determine the perpendicular distance, p, of the point $(2, -3, 4)$ from the plane whose cartesian equation is $x + 2y + 2z = 13$.

Solution

From the cartesian formula

$$p = \frac{13 - [(1)(2) + (2)(-3) + (2)(4)]}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{9}{3} = 3.$$

8.6.6 EXERCISES

1. Determine the vector equation and hence the cartesian equation of the plane, passing through the point with position vector $\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, and perpendicular to the vector $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

2. Determine the cartesian equation of the plane passing through the three points $(1, -1, 2)$, $(3, -2, -1)$ and $(-1, 4, 0)$.

3. Determine the point of intersection of the plane, whose vector equation is

$$\underline{\mathrm{r}} \bullet (5\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = -3,$$

and the straight line passing through the point $(2, 1, -3)$, which is parallel to the vector $\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

4. Determine the vector equation, and hence the cartesian equations (in standard form), of the line of intersection of the planes, whose vector equations are

$$\underline{\mathrm{r}} \bullet \underline{\mathrm{n}_1} = 14 \text{ and } \underline{\mathrm{r}} \bullet \underline{\mathrm{n}_2} = -1,$$

where

$$\underline{\mathrm{n}_1} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k} \text{ and } \underline{\mathrm{n}_2} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

5. Determine, in surd form, the perpendicular distance of the point $(-5, -2, 8)$ from the plane whose cartesian equation is $2x - y + 3z = 17$.

8.6.7 ANSWERS TO EXERCISES

1.

$$\underline{\mathrm{r}} \bullet (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 10, \quad \text{or} \quad 2x + y - 3z = 10.$$

2.

$$17x + 10y + 8z = 23.$$

3.

$$(0, -1, 5).$$

4.

$$\underline{\mathrm{r}} = -2\mathbf{i} + 3\mathbf{j} + t(7\mathbf{i} + 10\mathbf{j} - 8\mathbf{k})$$

or

$$\frac{x+2}{7} = \frac{y-3}{10} = \frac{z}{-8} \quad (= t).$$

5.

$$\frac{1}{\sqrt{14}}.$$

"JUST THE MATHS"

UNIT NUMBER

9.1

MATRICES 1
(Definitions & elementary matrix algebra)

by

A.J.Hobson

UNIT 9.1 - MATRICES 1
DEFINITIONS AND ELEMENTARY MATRIX ALGEBRA

9.1.1 INTRODUCTION

(a) Presentation of Data
Sets of numerical information can often be presented as a rectangular **"array"** of numbers. For example, a football results table might look like this:

TEAM	PLAYED	WON	DRAWN	LOST
Blackburn	22	11	6	5
Burnley	22	9	6	7
Chelsea	21	7	8	6
Leicester	21	6	8	7
Stoke	21	6	6	9

If the headings are taken for granted, we write simply

$$\begin{bmatrix} 22 & 11 & 6 & 5 \\ 22 & 9 & 6 & 7 \\ 21 & 7 & 8 & 6 \\ 21 & 6 & 8 & 7 \\ 21 & 6 & 6 & 9 \end{bmatrix}$$

and this symbol is called a **"matrix"**.

Note:
The mould in which printers once cast **type** was called a matrix. In mathematics, the word "matrix" signifies that we have spaces into which **numbers** can be placed.

(b) Presentation of Algebraic Results
In two-dimensional geometry, the **"vector"**, $\underline{\mathrm{OP}_0}$, joining the origin to the point $\mathrm{P}_0(x_0, y_0)$ can be moved to the position $\underline{\mathrm{OP}_1}$ joining the origin to the point $\mathrm{P}_1(x_1, y_1)$ by means of a **"reflection"**, a **"rotation"**, a **"magnification"** or a combination of such operations. It can be shown that the relationship between the co-ordinates of P_0 and P_1 in any such operation is given by

$$\begin{aligned} x_1 &= ax_0 + by_0, \\ y_1 &= cx_0 + dy_0. \end{aligned}$$

ILLUSTRATIONS

1. The equations

$$\begin{aligned} x_1 &= -x_0, \\ y_1 &= \quad\quad y_0 \end{aligned}$$

represent a reflection in the y-axis.

2. The equations

$$\begin{aligned} x_1 &= kx_0, \\ y_1 &= \quad\quad ky_0 \end{aligned}$$

represent a magnification when $\mid k \mid > 1$ and a contraction when $\mid k \mid < 1$.

3. The equations

$$\begin{aligned} x_1 &= x_0 \cos\theta - y_0 \sin\theta, \\ y_1 &= x_0 \sin\theta + y_0 \cos\theta \end{aligned}$$

represent a rotation of $\underline{\mathrm{OP}}_0$ through an angle θ in a counter-clockwise direction.

Such an operation is called a **"linear transformation"** and is completely specified by the coefficients a, b, c, d in the correct positions. When referring to a linear transformation, it is therefore more convenient to write the matrix symbol

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Sometimes, we want to show that a transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

operates on the vector $\underline{\mathrm{OP}}_0$, transforming it into the vector $\underline{\mathrm{OP}}_1$. We write

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The two new symbols are still called matrices but can be given a special name as we see later.

9.1.2 DEFINITIONS

(i) A matrix is **a rectangular array of numbers** arranged in rows (horizontally), columns (vertically) and enclosed in brackets.

ILLUSTRATIONS:

$$\begin{bmatrix} 1 & 3 & 7 \\ 9 & 9 & 10 \end{bmatrix}, \begin{bmatrix} 10 & 16 & 17 \\ 3 & 5 & 11 \\ 4 & 0 & 10 \end{bmatrix}, [1 \quad 5 \quad 7].$$

Note:
The rows are counted from the top to the bottom of the matrix and the columns are counted from the left to the right of the matrix.

(ii) Any number within the array is called an **"element"** of the matrix. The term $\underline{ij - th \text{ element}}$ refers to the element lying in the i-th row and the j-th column of the matrix.

(iii) If a matrix has m rows and n columns, it is called a **"matrix of order** $m \times n$**"** or simply an "$m \times n$ **matrix"**. It clearly has mn elements.

(iv) A matrix of order $m \times m$ is called a **"square matrix"**.

Note:
A matrix of order 1×1 is considered to be the same as a single number. This assumption has particular significance in the definition of a matrix product (see later).

(v) A matrix of order $m \times 1$ is called a **"column vector"** and a matrix of order $1 \times n$ is called a **"row vector"**.

Note:
Matrices of order 2×1, 1×2, 3×1 and 1×3 have easy physical interpretations as vectors. But, in abstract linear algebra, there are vectors of **any** dimension; so the names "column vector" and "row vector" are retained no matter how many elements there are.

(vi) An arbitrary matrix whose elements and order do not have to be specified may be denoted by a single capital letter such as A,B,C, etc.

An arbitrary matrix of order $m \times n$ may be denoted fully by the symbol

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

where, in each double-subscript, the first number is the row number and the second number is the column number. In a situation where the matrix has already been denoted by a single capital letter, the subsequent use of the full notation should include the corresponding small letter on which to attach the double-subscripts.

If the matrix is square, the elements $a_{11}, a_{22}, a_{33}, ...a_{mm}$ are called the **"diagonal elements"** and their sum is called the **"trace"** of the matrix. This line of elements is called the **"leading diagonal"** of the matrix.

An abbreviated form of the full notation is $[a_{ij}]_{m \times n}$

EXAMPLE
A matrix $\mathrm{A} = [a_{ij}]_{2\times 3}$ is such that $a_{ij} = i^2 + 2j$. Write out A in full.

Solution

This is an artificially contrived example, but will serve to illustrate the use of the double-subscript notation. We obtain

$$\mathrm{A} = \begin{bmatrix} 3 & 5 & 7 \\ 6 & 8 & 10 \end{bmatrix}.$$

(vii) Given a matrix, A, of order $m \times n$, the matrix of order $n \times m$ obtained from A by writing the rows as columns is called the **"transpose"** of A and is denoted by A^{T} - some books use A' or $\tilde{\mathrm{A}}$.

ILLUSTRATION

If

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix},$$

then,

$$\mathrm{A}^{\mathrm{T}} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

(viii) A matrix, A, is said to be **"symmetric"** if $\mathrm{A} = \mathrm{A}^{\mathrm{T}}$.

ILLUSTRATION

$$\mathrm{A} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ is symmetric.}$$

(ix) A matrix, A, is said to be **"skew-symmetric"** if the elements of A^T are minus the corresponding elements of A itself. This will mean that the leading diagonal elements must be zero.

ILLUSTRATION

$$\mathrm{A} = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix} \text{ is skew} - \text{symmetric.}$$

(x) A matrix is said to be **"diagonal"** if the elements which are not on the leading diagonal have value zero while the elements on the leading diagonal are not all equal to zero.

9.1.3 THE ALGEBRA OF MATRICES (Part One)

An **"Algebra"** (coming from the Arabic word AL-JABR) refers to any mathematical system which uses the concepts of equality, addition, subtraction, multiplication and division. For example, the algebra of numbers is what we normally call **"arithmetic"**; but algebraical concepts can be applied to other mathematical systems of which matrices is one.

In meeting a new mathematical system for the first time, the concepts of equality, addition, subtraction, multiplication and division need to be properly defined, and that is the purpose of the present section. In some cases, the definitions are fairly obvious, but they need to be made without contradicting ideas already established in the system of numbers which matrices depend on.

1. Equality

Unlike a determinant, a matrix does **not** have a numerical value; so the use of equality here is not that which is made in elementary arithmetic. Rather we use it to mean "is the same as".

Two matrices are said to be equal if they have **the same order** and also **pairs of elements in corresponding positions are equal in value**.

In symbols, we could say that

$$[a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$$

provided

$$a_{ij} = b_{ij}.$$

For example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

implies that $x = 1, y = 2$ and $z = 3$.

We shall meet this type of conclusion later in a discussion on the solution, by matrices, of a set of simultaneous equations (see Unit 9.3).

2. Addition and Subtraction

The scientific applications of matrices require us to define the sum and difference of two matrices only when they have the same order.

The sum of two matrices of order $m \times n$ is formed by adding together the pairs of elements in corresponding positions. Similarly, the difference of two matrices of order $m \times n$ is formed by subtracting the pairs of elements in corresponding positions.

In symbols, we could say that

$$[a_{ij}]_{m\times n} \pm [b_{ij}]_{m\times n} = [c_{ij}]_{m\times n},$$

where

$$c_{ij} = a_{ij} \pm b_{ij}.$$

Note:
It may occur, in a calculation, that a matrix has to be subtracted from itself, in which case we would obtain another matrix of the same order but whose elements are all zero. This type of matrix is called a **"null matrix"** and may be denoted, for short, by $[0]_{m\times n}$ or just $[0]$ when the order is understood.

ILLUSTRATION
A grocer has two shops and, in each shop, he sells apples, oranges and bananas. The sales, in kilogrammes, of each fruit for the two shops on two separate days are represented by the matrices

$$\begin{bmatrix} 36 & 25 & 10 \\ 20 & 30 & 15 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 40 & 30 & 12 \\ 22 & 35 & 20 \end{bmatrix}$$

where the rows refer to the two shops and the columns refer to apples, oranges and bananas respectively.

The total sales, in kilogrammes, of each fruit for the two shops on both days together are represented by the matrix

$$\begin{bmatrix} 76 & 55 & 22 \\ 42 & 65 & 35 \end{bmatrix}.$$

The differences in sales of each fruit for the two shops between the second day and the first day are represented by the matrix

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & 5 & 5 \end{bmatrix}$$

3. Additive Identities and Additive Inverses

For the sake of completeness, we mention here that

(a) A null matrix, when added to any other matrix of the same order, leaves that other matrix identically the same as it was to start with. For this reason, a null matrix behaves as an **"additive identity"**.

(b) If any matrix A is added to the matrix $-$A (that is, the matrix obtained from A by reversing the signs of all the elements) the result obtained is the corresponding null matrix. For this reason $-$A can be called the **"additive inverse"** of A.

9.1.4 EXERCISES

1. State the order of each of the following matrices:
   (a) $\begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$; (c) $\begin{bmatrix} 3 & 5 & 4 \end{bmatrix}$.
2. Two of the following matrices are equal to each other. Which are they ?
   (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 1 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} \sin 60^\circ & \tan 30^\circ \\ \tan 45^\circ & \sin 90^\circ \end{bmatrix}$; (c) $\begin{bmatrix} \cos 60^\circ & \frac{\sqrt{3}}{3} \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}$.
3. Determine the values of x, y and z given that
   $$\begin{bmatrix} x+y & y+z \\ x+z & 3x-2y \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 6 & 0 \end{bmatrix}.$$
4. If A $= \begin{bmatrix} 1 & -3 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, B $= \begin{bmatrix} 4 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ and C $= \begin{bmatrix} -1 & 4 & -2 \\ 0 & 1 & 0 \end{bmatrix}$,
   determine the elements of the following matrices:
   (a) A + B; (b) (A + B) + C; (c) A + (B + C);
   (d) A - B - C; (e) $\mathrm{A}^\mathrm{T} + \mathrm{B}^\mathrm{T}$; (f) $(\mathrm{A}+\mathrm{B})^\mathrm{T}$.

5. Determine which pairs of the following matrices can be added and, for those which can, state the sum:

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, C = [4 \quad 2 \quad 1], D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, F = \begin{bmatrix} 2 & 3 \\ 5 & 0 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, J = [3 \quad 3 \quad 3].$$

9.1.5 ANSWERS TO EXERCISES

1. (a) 2×3; (b) 2×1; (c) 1×3.
2. (a) and (c) are equal to each other.
3. $x = 2, y = 3$ and $z = 4$.
4. (a) $\begin{bmatrix} 5 & 0 & 4 \\ 3 & 0 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} 4 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$; (c) $\begin{bmatrix} 4 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$;

   (d) $\begin{bmatrix} -2 & -10 & 2 \\ 5 & -1 & -1 \end{bmatrix}$; (e) $\begin{bmatrix} 5 & 3 \\ 0 & 0 \\ 4 & 3 \end{bmatrix}$; (f) $\begin{bmatrix} 5 & 3 \\ 0 & 0 \\ 4 & 3 \end{bmatrix}$.
5. $A + E = E + A = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $B + G = G + B = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$

   $C + J = J + C = [7 \quad 5 \quad 4]$, $D + F = F + D = \begin{bmatrix} 2 & 3 \\ 5 & 0 \end{bmatrix}$

   $G + H = H + G = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $B + H = H + B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

   That is, twelve possible additions but only six distinct results.

“JUST THE MATHS”

UNIT NUMBER

9.2

MATRICES 2
(Further matrix algebra)

by

A.J.Hobson

UNIT 9.2 - MATRICES 2 - THE ALGEBRA OF MATRICES (Part Two)

9.2.1 MULTIPLICATION BY A SINGLE NUMBER

If we were required to multiply a matrix of any order by a **positive whole number**, n, we would clearly regard the operation as equivalent to adding together n copies of the given matrix. Thus, in the result, every element of this given matrix would be multiplied by n; but it is logical to extend the idea to the multiplication of a matrix by **any** number, λ, not necessarily a positive whole number, the rule being to multiply every element of the matrix by λ.

In symbols we could say that

$$\lambda\,[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n}\,,$$

where

$$b_{ij} = \lambda a_{ij}.$$

Note:
The rule for multiplying a matrix by a single number can also be used in reverse to remove common factors from the elements of a matrix as illustrated as follows:

ILLUSTRATION

$$\begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

9.2.2 THE PRODUCT OF TWO MATRICES

The definition of a matrix product is more difficult to justify than the previous concepts, partly because it is by no means an obvious definition and partly because we cannot be sure exactly what originally led to the making of the definition.

Some hint is given by the matrix equation at the end of the introduction to Unit 9.1, where the product of a 2×2 matrix and a 2×1 matrix is another 2×1 matrix; but we must be prepared to meet other orders of matrix as well.

We shall introduce the definition with a semi-practical illustration, then make a formal statement of the definition itself.

ILLUSTRATION

A motor manufacturer, with three separate factories, makes two types of car, one called "standard" and the other called "luxury".
In order to manufacture each type of car, he needs a certain number of units of material and a certain number of units of labour each unit representing £300.

A table of data to represent this information could be

Type	Materials	Labour
Standard	12	15
Luxury	16	20

The manufacturer receives an order from another country to supply 400 standard cars and 900 luxury cars; but he distributes the export order amongst his three factories as follows:

Location	Standard	Luxury
Factory A	100	400
Factory B	200	200
Factory C	100	300

The number of units of material and the number of units of labour needed by each factory to complete the order may be given by another table, namely

Location	Materials	Labour
Factory A	$100 \times 12 + 400 \times 16$	$100 \times 15 + 400 \times 20$
Factory B	$200 \times 12 + 200 \times 16$	$200 \times 15 + 200 \times 20$
Factory C	$100 \times 12 + 300 \times 16$	$100 \times 15 + 300 \times 20$

If we now replace each table by the corresponding matrix, the calculations appear as the product of a 3×2 matrix and a 2×2 matrix. That is,

$$\underset{3\times 2}{\begin{bmatrix} 100 & 400 \\ 200 & 200 \\ 100 & 300 \end{bmatrix}} . \underset{2\times 2}{\begin{bmatrix} 12 & 15 \\ 16 & 20 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 100 \times 12 + 400 \times 16 & 100 \times 15 + 400 \times 20 \\ 200 \times 12 + 200 \times 16 & 200 \times 15 + 200 \times 20 \\ 100 \times 12 + 300 \times 16 & 100 \times 15 + 300 \times 20 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 7600 & 9500 \\ 5600 & 7000 \\ 6000 & 7500 \end{bmatrix}}$$

OBSERVATIONS

(i) The product matrix has 3 rows because the first matrix on the left has 3 rows.

(ii) The product matrix has 2 columns because the second matrix on the left has 2 columns.

(iii) The product cannot be worked out unless the number of columns in the first matrix matches the number of rows in the second matrix with no elements left over in the pairing-up process.

(iv) The elements of the product matrix are systematically obtained by multiplying (in pairs) the corresponding elements of each row in the first matrix with each column in the second matrix. To pair up the correct elements, we read each row of the first matrix from left to right and each column of the second matrix from top to bottom.

The Formal Definition of a Matrix Product
If A and B are matrices, then the product AB is defined (that is, it has a meaning) only when the number of columns in A is equal to the number of rows in B.

If A is of order $m \times n$ and B is of order $n \times p$, then AB is of order $m \times p$.

To obtain the element in the i-th row and j-th column of AB, we multiply corresponding elements of the i-th row of A and the j-th column of B, then add up the results.

ILLUSTRATION

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 6 \\ 3 & 2 & -1 & 1 \\ 1 & -2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & -1 & 12 \\ 1 & -10 & 1 & 15 \end{bmatrix}$$

Note:
Confusion could arise when multiplying a matrix of order 1×1 by another matrix. Apparently, the other matrix would need to have either a single row or a single column depending on the order of multiplication.

However, as stated in Unit 9.1, a matrix of order 1×1 is considered to be a special case, and is defined separately to be the same as a single number. Hence a matrix of any order can be multiplied by a matrix of order 1×1 even though this does not fit the formal rules for matrix multiplication in general.

9.2.3 THE NON-COMMUTATIVITY OF MATRIX PRODUCTS

In elementary arithmetic, if a and b are two numbers, then $ab = ba$ (that is, the product **"commutes"**). But this is not so for matrices A and B as we now show:

(a) If A is of order $m \times n$, then B must be of order $n \times m$ if both AB **and** BA are to be defined.

(b) AB and BA will have different orders unless $m = n$, in which case the two products will be square matrices of order $m \times m$.

(c) Even if A and B are **both** square matrices of order $m \times m$, it will not normally be the case that AB is the same as BA. A simple numerical example will illustrate this fact:

$$\begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 11 & 35 \end{bmatrix}; \quad \text{but} \quad \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 19 & 37 \end{bmatrix}.$$

Notes:
(i) If we simply wanted to show that $AB \neq BA$, we would need only to demonstrate that one pair of corresponding elements were unequal in value.

(ii) If such a basic rule of elementary arithmetic is false for matrices, we should, strictly speaking, be prepared to justify other basic rules of arithmetic. But it turns out that the non-commutativity of matrix products is the only one which causes problems.

For instance, it can be shown that, provided the matrices involved are compatible for addition or multiplication,

$A + B \equiv B + A$; the **"Commutative Law of Addition"**.

$A + (B + C) \equiv (A + B) + C$; the **"Associative Law of Addition"**.

$A(BC) \equiv (AB)C$; the **"Associative Law of Multiplication"**.

$A(B + C) \equiv AB + BC$ or $(A + B)C \equiv AC + BC$; the **"Distributive Laws"**.

(iii) In the matrix product, AB, we say either that B is **"pre-multiplied"** by A or that A is **"post-multiplied"** by B.

9.2.4 MULTIPLICATIVE IDENTITY MATRICES

In connection with matrix multiplication and, subsequently, the solution of simultaneous linear equations, an important type of matrix is a square matrix with the number 1 in each position of the leading diagonal, but zeros everywhere else. Such a matrix is denoted by I_n if there are n rows and (of course) n columns. If it is not necessary to specify the number of rows and columns, the notation I, without a subscript, is sufficient.

ILLUSTRATIONS

$$I_2 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad I_4 \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Any matrix of the type I_n multiplies another matrix (with an appropriate number of rows or columns) to leave it identically the same as it was to start with. For this reason, I_n is called a **"multiplicative identity matrix"**, although we normally call it just an "identity matrix" (unless it becomes necessary to distinguish it from the **additive** identity matrix referred to earlier). Another common name for it is a **"unit matrix"**.

For example, suppose that

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

Then, post-multiplying by I_2, it is easily checked that

$$\mathrm{AI}_2 = \mathrm{A}.$$

Similarly, pre-multiplying by I_3, it is easily checked that

$$\mathrm{I}_3\mathrm{A} = \mathrm{A}.$$

In general, if A is of order $m \times n$, then

$$\mathrm{AI}_n = \mathrm{I}_m\mathrm{A} = \mathrm{A}.$$

9.2.5 EXERCISES

1. If $\mathrm{A} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ and $\mathrm{C} = \begin{bmatrix} -1 & 4 & -2 \\ 0 & 1 & 0 \end{bmatrix}$,
   determine the elements of the following matrices:
   (a) A + 2B; (b) A + 2B - 3C; (c) $3\mathrm{A}^{\mathrm{T}} - \mathrm{B}^{\mathrm{T}}$.

2. Remove a common factor from each of the following matrices in order to express it as the product of a number and a matrix:
   (a) $\begin{bmatrix} 8 & -4 \\ -32 & 16 \end{bmatrix}$; (b) $\begin{bmatrix} -x^3 & -x^2 \\ x^2 & -4x^2 \end{bmatrix}$.

3. State the order of the product matrix in each of the following cases:
   (a) $\mathrm{A}_{1\times2}.\mathrm{B}_{2\times2}$; (b) $\mathrm{A}_{3\times1}.\mathrm{B}_{1\times2}$; (c) $\mathrm{A}_{4\times3}.\mathrm{B}_{3\times5}$.

4. For the matrices $\mathrm{A}_{2\times2}, \mathrm{B}_{2\times3}, \mathrm{C}_{3\times3}$ and $\mathrm{D}_{2\times4}$, which of the following products are defined.
   (a) A.B; (b) B.C; (c) C.D; (d) A.C; (e) A.D.

5. Determine the elements of the product matrix in each of the following:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 1 \end{bmatrix} . \begin{bmatrix} 12 & 24 \\ 24 & 36 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 4 & -3 \\ 1 & -1 & 0 \end{bmatrix} . \begin{bmatrix} 2 & 6 \\ 5 & -3 \\ -1 & 7 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 4 \end{bmatrix} . \begin{bmatrix} 7 & -3 \\ -1 & 4 \end{bmatrix}$; (e) $\begin{bmatrix} 4 & 2 & 1 \end{bmatrix} . \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$; (f) $\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} . \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$;

(g) $\begin{bmatrix} -1 & 2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; (h) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 5 & 8 & 9 \end{bmatrix} . \begin{bmatrix} -1 & 3 & -2 \\ -0.5 & -3 & 2.5 \\ 1 & 1 & -1 \end{bmatrix}$.

6. If $\mathrm{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, verify that $\mathrm{A}.\mathrm{A}^{\mathrm{T}} = \mathrm{I}_2$.

7. If $\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, verify that $(\mathrm{A}.\mathrm{B})^{\mathrm{T}} = \mathrm{B}^{\mathrm{T}}.\mathrm{A}^{\mathrm{T}}$ **not** $\mathrm{A}^{\mathrm{T}}.\mathrm{B}^{\mathrm{T}}$.

9.2.6 ANSWERS TO EXERCISES

1. (a) $\begin{bmatrix} 9 & 3 & 6 \\ 2 & 0 & 5 \end{bmatrix}$; (b) $\begin{bmatrix} 12 & -9 & 12 \\ 2 & -3 & 5 \end{bmatrix}$; (c) $\begin{bmatrix} -1 & 13 \\ -12 & 0 \\ 4 & 1 \end{bmatrix}$.

2. (a) $4.\begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$; (b) $-x^2.\begin{bmatrix} x & 1 \\ -1 & 4 \end{bmatrix}$.

3. (a) 1×2; (b) 3×2; (c) 4×5.

4. (a), (b) and (e) are defined.

5. (a) $\begin{bmatrix} 10 & 7 \\ 22 & 15 \end{bmatrix}$; (b) $\begin{bmatrix} 14 & 24 \\ 27 & 42 \end{bmatrix}$; (c) $\begin{bmatrix} 23 & -33 \\ -3 & 9 \end{bmatrix}$;

(d) $\begin{bmatrix} 5 & 5 \\ 21 & -9 \\ 3 & 13 \end{bmatrix}$; (e) [4] defined as 4; (f) $\begin{bmatrix} -4 & -2 & -1 \\ 12 & 6 & 3 \\ 8 & 4 & 2 \end{bmatrix}$;

(g) $\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$; (h) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

"JUST THE MATHS"

UNIT NUMBER

9.3

MATRICES 3
(Matrix inversion & simultaneous equations)

by

A.J.Hobson

UNIT 9.3 - MATRICES 3

MATRIX INVERSION AND SIMULTANEOUS LINEAR EQUATIONS

9.3.1 INTRODUCTION

In Matrix Algebra, there is no such thing as **division** in the usual sense of this word since we would effectively be dividing by a table of numbers with the headings of the table removed.

However, we may discuss an equivalent operation called **inversion** which is roughly similar to the process in elementary arithmetic where division by a value, a, is the same as multiplication by $\frac{1}{a}$.

For example, consider the solution of a single linear equation in one variable, x, namely

$$mx = k,$$

for which the solution is obviously

$$x = \frac{k}{m}.$$

An apparently over-detailed solution may be set out as follows:

(a) Pre-multiply both sides of the given equation by m^{-1}, giving

$$m^{-1}.(mx) = m^{-1}k.$$

(b) Use the Associative Law of Multiplication in elementary arithmetic to rearrange this as

$$(m^{-1}.m)x = m^{-1}k.$$

(c) Use the property that $m^{-1}.m = 1$ to conclude that

$$1.x = m^{-1}k.$$

(d) Use the fact that the number 1 is the multiplicative identity in elementary arithmetic to conclude that

$$x = m^{-1}k.$$

We shall see later how an almost identical sequence of steps, with matrices instead of numbers, can be used to solve a set of simultaneous linear equations with what is called the **"multiplicative inverse"** of a matrix.

Matrix inversion is a concept which is developed from the rules for matrix multiplication.

9.3.2 MATRIX REPRESENTATION OF SIMULTANEOUS LINEAR EQUATIONS

In this section, we consider the matrix equivalent of three simultaneous linear equations in three unknowns; this case is neither too trivial, nor too difficult to follow. Let the equations have the form:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3. \end{aligned}$$

Then, from the properties of matrix equality and matrix multiplication, these can be written as one matrix equation, namely

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix},$$

which will be written for short in the form

$$\mathrm{MX} = \mathrm{K}.$$

Note:

If we could find a matrix, say N, such that NM = I, we could pre-multiply the above statement by it to give

$$\mathrm{N(MX)} = \mathrm{NK}.$$

That is,

$$\mathrm{(NM)X} = \mathrm{NK}.$$

In other words,

$$\mathrm{IX} = \mathrm{NK}.$$

Hence,

$$\mathrm{X} = \mathrm{NK}.$$

Clearly N exhibits a similar behaviour to the number m^{-1} encountered earlier and, for this reason, a better notation for N would be M^{-1}. This is what we shall use in the work which follows.

9.3.3 THE DEFINITION OF A MULTIPLICATIVE INVERSE

The **"multiplicative inverse"** of a square matrix, M, is another matrix, denoted by M^{-1}, which has the property

$$M^{-1}.M = I.$$

Notes:

(i) It is certainly **possible** for the product of two matrices to be an identity matrix, as illustrated, for example, in the exercises on matrix products in Unit 9.2.

(ii) It is usually acceptable to call M^{-1} the "inverse" of M rather than the "multiplicative inverse", unless we wish to distinguish this kind of inverse from the additive inverse defined in Unit 9.1.

(iii) It can be shown that, when $M^{-1}.M = I$, it is also true that

$$M.M^{-1} = I,$$

even though matrices do not normally commute in multiplication.

(iv) It is easily shown that a square matrix cannot have more than one inverse; for, suppose a matrix A had two inverses, B and C.

Then,

$$C = CI = C(AB) = (CA)B = IB = B,$$

which means that C and B are the same matrix.

9.3.4 THE FORMULA FOR A MULTIPLICATIVE INVERSE

So far, we have established the fact that a set of simultaneous linear equations is expressible in the form

$$MX = K$$

and their solution is expressible in the form

$$X = M^{-1}K.$$

We therefore need to establish a method for determining the inverse, M^{-1}, in order to solve the equations for the values of the variables involved.

Development of this method will be dependent on the result known as "Cramer's Rule" for solving simultaneous equations by determinants. (see Units 7.2 and 7.3)

(a) The inverse of a 2 x 2 matrix

Taking

$$\mathrm{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

and

$$\mathrm{M}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

we require that

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} a_1P + b_1R &= 1, \\ a_2P + b_2R &= 0, \\ a_1Q + b_1S &= 0, \\ a_2Q + b_2S &= 1. \end{aligned}$$

It is easily verified that these equations are satisfied by the solution

$$P = \frac{b_2}{|\mathrm{M}|} \quad Q = -\frac{b_1}{|\mathrm{M}|} \quad R = -\frac{a_2}{|\mathrm{M}|} \quad S = \frac{a_1}{|\mathrm{M}|},$$

where

$$|\mathrm{M}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

and is called **"the determinant of the matrix** M**"**.

Summary
The formula for the inverse of a 2 x 2 matrix is given by

$$\mathrm{M}^{-1} = \frac{1}{|\mathrm{M}|} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix},$$

in which we interchange the diagonal elements of M and reverse the sign of the other two elements.

EXAMPLES

1. Write down the inverse of the matrix
$$\mathrm{M} = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}.$$

Solution
Since $\mid \mathrm{M} \mid = 41$, we have
$$\mathrm{M}^{-1} = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}.$$

Check
$$\mathrm{M}^{-1}.\mathrm{M} = \frac{1}{41}\begin{bmatrix} 7 & 3 \\ -2 & 5 \end{bmatrix}.\begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} 41 & 0 \\ 0 & 41 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Use matrices to solve the simultaneous linear equations

$$\begin{aligned} 3x + y &= 1, \\ x - 2y &= 5. \end{aligned}$$

Solution
The equations can be written in the form MX = K, where
$$\mathrm{M} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathrm{X} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathrm{K} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$
Initially, we **must** check that M is non-singluar by evaluating its determinant:
$$\mid \mathrm{M} \mid = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -\ 6 - 1 = -\ 7.$$
The inverse matrix is now given by
$$\mathrm{M}^{-1} = -\ \frac{1}{7}\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}.$$
Thus, the solution of the simultaneous equations is given by
$$\begin{bmatrix} x \\ y \end{bmatrix} = -\ \frac{1}{7}\begin{bmatrix} -2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = -\ \frac{1}{7}\begin{bmatrix} -7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$
That is,
$$x = 1 \text{ and } y = -2.$$

Note:
Example 2 should be regarded as a model solution to this type of problem but the student should not expect every exercise to yield whole number answers.

(b) The inverse of a 3 x 3 Matrix

The most convenient version of Cramer's rule to use here may be stated as follows:

The simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

have the solution

$$\frac{x}{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

METHOD
(i) We observe, first, that the last determinant in the Cramer's Rule formula contains the same arrangement of numbers as the matrix M; as in (a), we call it **"the determinant of the matrix** M**"** and we denote it by | M |.

In this determinant, we let the symbols A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3 denote the **"cofactors"** (or **"signed minors"**) of the elements a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2 and c_3 respectively.

(ii) We now observe that, for each of the elements k_1, k_2 and k_3 in the first three determinants of Cramer's Rule, the cofactor is the same as the cofactor of one of the elements in the final determinant | M |.

(iii) Expanding each of the first three determinants in Cramer's Rule along the column of k-values, the solutions for x, y and z can be written as follows:

$$x = \frac{1}{|\text{ M }|}\left(k_1A_1 + k_2A_2 + k_3A_3\right);$$

$$y = \frac{1}{\mid \mathrm{M} \mid}\left(k_1B_1 + k_2B_2 + k_3B_3\right);$$

$$z = \frac{1}{\mid \mathrm{M} \mid}\left(k_1C_1 + k_2C_2 + k_3C_3\right);$$

or, in matrix format,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\mid \mathrm{M} \mid}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

Comparing this statement with the statement

$$\mathrm{X} = \mathrm{M}^{-1}K,$$

we conclude that

$$\mathrm{M}^{-1} = \frac{1}{\mid \mathrm{M} \mid}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

Summary
Since it is known that Cramer's Rule is applicable to any number of equations in the same number of unknowns, similar working would occur for larger or smaller systems of equations. In general, the inverse of a square matrix is **the transpose of the matrix of cofactors times the reciprocal of the determinant of the matrix**.

Notes:
(i) If it should happen that $\mid \mathrm{M} \mid= 0$, then the matrix M does not have an inverse and is said to be **"singular"**. In all other cases, it is said to be **"non-singular"**.

(ii) The transpose of the matrix of cofactors is called the **"adjoint"** of M, denoted by AdjM. There is always an adjoint though not always an inverse; but, when the inverse does exist, we can say that

$$\mathrm{M}^{-1} = \frac{1}{\mid \mathrm{M} \mid}\mathrm{AdjM}.$$

(iii) The inverse of a matrix of order 2×2 fits the above scheme also. The cofactor of each element will be a single number associated with a **"place-sign"** according to the following pattern:

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}.$$

Hence, if

$$\mathrm{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix},$$

then,

$$\mathrm{M}^{-1} = \frac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

As before, we see that the matrix part of the result can be obtained by interchanging the diagonal elements of M and reversing the signs of the other two elements.

EXAMPLE

Use matrices to solve the simultaneous linear equations

$$\begin{aligned} 3x + y - z &= 1, \\ x - 2y + z &= 0, \\ 2x + 2y + z &= 13. \end{aligned}$$

Solution

The equations can be written in the form MX = K, where

$$\mathrm{M} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathrm{K} = \begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix}.$$

Initially, we **must** check that M is non-singluar by evaluating its determinant:

$$\mid \mathrm{M} \mid = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 3(-2-2) - 1(1-2) + (-1)(2+4) = -17.$$

If C denotes the matrix of cofactors, then

$$\mathrm{C} = \begin{bmatrix} \boxed{-4} & 1 & \boxed{6} \\ -3 & \boxed{5} & -4 \\ \boxed{-1} & -4 & \boxed{-7} \end{bmatrix}.$$

Notes:
(i) The framed elements indicate those for which the place sign is positive and hence no changes of sign are required to convert the minor of the corresponding elements of M into their cofactor. It is a good idea to work these out first.

(ii) The remaining four elements are those for which the place sign is negative and so the cofactor is minus the value of the minor in these cases. If these, too, are worked out together, there is less scope for making mistakes.

(iii) It is important to remember that, in finding the elements of C, we **do not multiply the cofactors of the elements in** M **by the elements themselves**.

The transpose of C now provides the adjoint matrix and hence the inverse as follows:

$$\mathrm{M}^{-1} = \frac{1}{\mid \mathrm{M} \mid}\mathrm{Adj}\mathrm{M} = \frac{1}{-17}\mathrm{C}^{\mathrm{T}} = \frac{1}{-17}\begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}.$$

Thus, the solution of the simultaneous equations is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -4 & -3 & -1 \\ 1 & 5 & -4 \\ 6 & -4 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 13 \end{bmatrix} = \frac{1}{-17}\begin{bmatrix} -17 \\ -51 \\ -85 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

That is,

$$x = 1 \quad y = 3 \quad \text{and} \quad z = 5.$$

Note:
Once again, the example should be regarded as a model solution to this type of problem but the student should not expect every exercise to yield whole number answers.

9.3.5 EXERCISES

1. Determine, where possible, the multiplicative inverses of the following matrices:
(a) $\begin{bmatrix} 1 & 5 \\ 7 & 9 \end{bmatrix}$; (b) $\begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$; (c) $\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$;
(d) $\begin{bmatrix} 1 & 4 & -2 \\ -3 & 2 & -7 \\ 1 & 0 & 5 \end{bmatrix}$; (e) $\begin{bmatrix} 6 & 3 & 1 \\ 0 & -5 & 2 \\ 1 & 1 & 4 \end{bmatrix}$; (f) $\begin{bmatrix} 11 & 3 & 5 \\ 6 & 2 & 2 \\ 10 & 3 & 4 \end{bmatrix}$.

2. Use a matrix inverse method to solve the following sets of simultaneous linear equations:

(a)

$$\begin{aligned} 4x - 3y &= 9, \\ 3x - 2y &= 7; \end{aligned}$$

(b)

$$\begin{aligned} x + 2y &= -2, \\ 5x - 4y &= 3; \end{aligned}$$

(c)

$$\begin{aligned} 3x + 2y + z &= 3, \\ 5x + 4y + 3z &= 3, \\ 6x + y + z &= 5; \end{aligned}$$

(d)

$$\begin{aligned} x + 2y + 2z &= 0, \\ 2x + 5y + 4z &= 2, \\ x - y - 6z &= 4; \end{aligned}$$

(e)

$$\begin{aligned} 3x + 2y - 2z &= 16, \\ 4x + 3y + 3z &= 2' \\ 2x - y + z &= -1; \end{aligned}$$

(f)

$$\begin{aligned} 2(i_3 - i_2) + 5(i_3 - i_1) &= 4, \\ (i_2 - i_3) + 2i_2 + (i_2 - i_1) &= 0, \\ 5(i_1 - i_3) + 2(i_1 - i_2) + i_2 &= 1. \end{aligned}$$

3. If A $= \begin{bmatrix} 4 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$ and C $= \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & -4 \\ 0 & 5 & -2 \end{bmatrix}$,
determine a matrix, B, such that A.B = C.

9.3.6 ANSWERS TO EXERCISES

1. (a) $-\frac{1}{26}\begin{bmatrix} 9 & -5 \\ -7 & 1 \end{bmatrix}$; (b) No Inverse, (c) $-\frac{1}{4}\begin{bmatrix} 2 & -2 \\ -1 & -1 \end{bmatrix}$;
(d) $\frac{1}{46}\begin{bmatrix} 10 & -20 & -24 \\ 8 & 7 & 13 \\ -2 & 4 & 14 \end{bmatrix}$; (e) $-\frac{1}{121}\begin{bmatrix} -22 & -11 & 11 \\ 2 & 23 & -12 \\ 5 & -3 & -30 \end{bmatrix}$; (f) There is no inverse.
2. (a) $x = 3$ and $y = 1$; (b) $x = \frac{1}{7}$ and $y = -\frac{13}{14}$; (c) $x = 1$, $y = 1$ and $z = -2$;
(d) $x = -\frac{3}{2}$, $y = 2$ and $z = -\frac{5}{4}$; (e) $x = 2$, $y = \frac{3}{2}$ and $z = -\frac{7}{2}$;
(f) $i_1 = \frac{11}{6}$, $i_2 = 1$ and $i_3 = \frac{13}{6}$.
3.

$$B = \frac{1}{41}\begin{bmatrix} 11 & -2 & -1 \\ -13 & 21 & -10 \\ -3 & 8 & 4 \end{bmatrix} . \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & -4 \\ 0 & 5 & -2 \end{bmatrix} = \frac{1}{41}\begin{bmatrix} -17 & 15 & 21 \\ 76 & -55 & -77 \\ 27 & 22 & -43 \end{bmatrix}.$$

“JUST THE MATHS”

UNIT NUMBER

9.4

MATRICES 4
(Row operations)

by

A.J.Hobson

UNIT 9.4 - MATRICES 4 - ROW OPERATIONS

9.4.1 MATRIX INVERSES BY ROW OPERATIONS

In this section, we shall examine an alternative method for finding the multiplicative inverse of a matrix but the techniques introduced will lead on to other aspects of solving simultaneous linear equations not discussed in earlier units.

DEFINITION
An **"elementary row operation"** on a matrix is any one of the following three possibilities:

(a) The interchange of two rows;
(b) The multiplication of the elements in any row by a non-zero number;
(c) The addition of the elements in any row to the corresponding elements in another row.

Notes:
(i) Elementary row operations are essentially the same kind of operations as those used in the solution of a set of simultaneous linear equations by the method of elimination; but here we shall be considering sets of coefficients in the form of matrices rather than complete sets of equations.

(ii) Elementary row operations of types (b) and (c) imply that the elements in any row may be **subtracted** from the corresponding elements in another row and, more generally, multiples of the elements in any row may be added to or subtracted from the corresponding elements in another row.

RESULT 1.
To perform an elementary row operation on a matrix **algebraically**, we may pre-multiply the matrix by an identity matrix on which the same elementary row operation has been already performed. For example, in the matrix

$$\left[\begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right],$$

suppose we wished to subtract twice the third row from the second row. It is easy enough to carry this out by inspection, but could also be regarded as the succession of two elementary row operations as follows:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right].\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right].\left[\begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ 2a_3 & 2b_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 - 2a_3 & b_2 - 2b_3 \\ a_3 & b_3 \end{bmatrix} .$$

DEFINITION

An **"elementary matrix"** is a matrix obtained from an identity matrix by performing upon it one elementary row operation.

RESULT 2.

If a certain sequence of elementary row operations converts a given square matrix, M, into the corresponding identity matrix, then the same sequence of elementary row operations in the same order will convert the identity matrix into M^{-1}.

Proof:

Suppose that

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.M = \mathrm{I}$$

where $E_1, E_2, E_3, E_4, ..., E_{n-1}, En$ are elementary matrices.

Then, by post-multiplying both sides with M^{-1}, we obtain

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.\mathrm{M}.\mathrm{M}^{-1} = \mathrm{I}.\mathrm{M}^{-1}.$$

In other words,

$$E_n.E_{n-1}....E_4.E_3.E_2.E_1.\mathrm{I} = \mathrm{M}^{-1},$$

which proves the result.

EXAMPLES

1. Use elementary row operations to determine the inverse of the matrix

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

Solution

First we write down the given matrix side-by-side with the corresponding identity matrix in the following format:

$$\begin{bmatrix} 3 & 7 & \vdots & 1 & 0 \\ 2 & 5 & \vdots & 0 & 1 \end{bmatrix}.$$

Secondly, we try to arrange that the first element in the first column of this arrangement is 1; and this can be carried out by subtracting the second row from the first row. (Instruction: $R_1 \to R_1 - R_2$).

$$\begin{bmatrix} 1 & 2 & \vdots & 1 & -1 \\ 2 & 5 & \vdots & 0 & 1 \end{bmatrix}.$$

Thirdly, we try to convert the first column of the display into the first column of the identity matrix; and this can be carried out by subtracting twice the first row from the second row.
(Instruction: $R_2 \to R_2 - 2R_1$).

$$\begin{bmatrix} 1 & 2 & \vdots & 1 & -1 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix}.$$

Lastly, we try to convert the second column of the display into the second column of the identity matrix: and this can be carried out by subtracting twice the second row from the first row.
(Instruction: $R_1 \to R_1 - 2R_2$).

$$\begin{bmatrix} 1 & 0 & \vdots & 5 & -7 \\ 0 & 1 & \vdots & -2 & 3 \end{bmatrix}.$$

The inverse matrix is therefore

$$\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix},$$

as we would have obtained by the cofactor method.

Notes:

(i) The technique used for a 2×2 matrix applies to square matrices of all orders with appropriate modifications.

(ii) The idea of the method is to obtain, in chronological order, the columns of the identity matrix from the columns of the given matrix. We do this by using elementary row operations to convert each diagonal element in the given matrix to 1 and then using multiples of 1 to reduce the remaining elements in the same column to zero.

(iii) Once any row has been used to reduce elements to zero, that row must not be used again as the operator; otherwise the zeros obtained may change to other values.

2. Use elementary row operations to show that the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

has no inverse.

Solution

We set up the scheme in the following format:

$$\begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 2 & 6 & \vdots & 0 & 1 \end{bmatrix}.$$

Then, we proceed according to the instructions indicated:

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & -1 & 1 \end{bmatrix}.$$

There is no way now of continuing to convert the given matrix into the identity matrix; hence, there is no inverse.

3. Use elementary row operations to determine the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 6 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$$

Solution

We set up the scheme in the following format:

$$\begin{bmatrix} 4 & 1 & 6 & \vdots & 1 & 0 & 0 \\ 2 & 1 & 3 & \vdots & 0 & 1 & 0 \\ 3 & 2 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Then, we proceed according to the instructions indicated:

$R_1 \rightarrow R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 1 & 0 & 1 \\ 2 & 1 & 3 & \vdots & 0 & 1 & 0 \\ 3 & 2 & 5 & \vdots & 0 & 0 & 1 \end{bmatrix};$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & -2 & 1 & 2 \\ 0 & 5 & 2 & -3 & 0 & 4 \end{array}\right];$$

$R_2 \to 2R_2 - R_3$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & -3 & 0 & 4 \end{array}\right];$$

$R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 5R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 2 & -10 & 4 \end{array}\right];$$

$R_3 \to R_3 \times \frac{1}{2}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -5 & 2 \end{array}\right];$$

$R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 7 & -3 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 5 & 2 \end{array}\right].$$

The required inverse matrix is therefore

$$\begin{bmatrix} -1 & 7 & -3 \\ -1 & 2 & 0 \\ 1 & -5 & 2 \end{bmatrix}.$$

9.4.2 GAUSSIAN ELIMINATION - THE ELEMENTARY VERSION

Elementary row operations can also be conveniently used in another method of solving simultaneous linear equations which relates closely again to the elimination method sometimes encountered in courses which do not include matrices. The method will be introduced through the case of three equations in three unkowns, but may be applied in other cases as well.

Suppose a set of simultaneous linear equations in the variables x, y and z appeared in the special form:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ b_2y + c_2z &= k_2, \\ c_3z &= k_3. \end{aligned}$$

Then it is very simple to solve the equations by first obtaining z from the third equation, substituting its value into the second equation in order to find y, then substituting for both y and z in the first equation in order to find x.

The method of Gaussian Elimination reduces any set of linear equations to this triangular form by adding or subtracting suitable multiples of pairs of the equations; but the method is more conveniently laid out in a tabular form using only the coefficients of the variables and the constant terms. We illustrate with an example.

EXAMPLE
Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution
For the simplest arithmetic, we try to arrange that the first coefficient in the first equation is 1. In the current example, we could interchange the first two equations.

$$\begin{array}{cccc|c} \boxed{1} & -2 & -1 & & 2 \\ 2 & 1 & 1 & & 3 \\ 3 & -1 & 1 & & 8 \end{array}$$

This format is known as an **"augmented matrix"**. The matrix of coefficients has been augmented by the matrix of constant terms.

Using the notation of the previous section, we apply the instructions $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$ giving a new table, namely

$$\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & \boxed{5} & 3 & -1 \\ 0 & 5 & 4 & 2 \end{array}$$

Next, we apply the instruction $R_3 \rightarrow R_3 - R_2$ giving

$$\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & 1 & 3 \end{array}$$

The numbers enclosed in the boxes are called the **"pivot elements"** and are used to reduce to zero the elements below them in the same column.

The final table above provides a new set of equations, equivalent to the original, namely

$$\begin{aligned} x - 2y - z &= 2, \\ 5y + 3z &= -1, \\ z &= 3. \end{aligned}$$

Hence, $\boxed{z = 3, \quad y = -2, \quad x = 1}$.

INSERTING A CHECK COLUMN

In the above example, the numbers were fairly simple, giving little scope for careless mistakes. However, with large numbers of equations, often involving awkward decimal quantities, the margin for error is greatly increased.

As a check on the arithmetic at each stage, we deliberately introduce some **additional** arithmetic which has to remain consistent with the calculations already being carried out. The method is to add together the numbers in each row in order to produce an extra column; each row operation is then performed on the extended rows with the result that, in the new table, the final column should still be the sum of the numbers to the left of it.

The working for our previous example would be as follows:

$$\left.\begin{array}{ccc|c|c} \boxed{1} & -2 & -1 & 2 & 0 \\ 2 & 1 & 1 & 3 & 7 \\ 3 & -1 & 1 & 8 & 11 \end{array}\right.$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\left.\begin{array}{ccc|c|c} 1 & -2 & -1 & 2 & 0 \\ 0 & \boxed{5} & 3 & -1 & 7 \\ 0 & 5 & 4 & 2 & 11 \end{array}\right.$$

$R_3 \to R_3 - R_2$

$$\left.\begin{array}{ccc|c|c} 1 & -2 & -1 & 2 & 0 \\ 0 & 5 & 3 & -1 & 7 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right.$$

9.4.4 EXERCISES

1. Use elementary row operations to determine (where possible) the inverses of the following matrices:

   (a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$;

   (b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$;

   (c) $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$;

   (d) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -7 \\ 3 & 11 & 3 \end{bmatrix}$;

   (e) $\begin{bmatrix} 0 & 3 & -2 \\ 1 & -1 & 5 \\ 1 & 5 & 1 \end{bmatrix}$;

   (f) $\begin{bmatrix} 2 & -3 & 4 \\ -1 & 3 & -4 \\ 1 & 0 & 2 \end{bmatrix}$;

   (g) $\begin{bmatrix} -8 & -7 & -6 & -5 \\ -4 & -3 & -2 & -1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$.

2. Use Gaussian Elimination (with a check column) to solve the following sets of simultaneous linear equations:

   (a)

$$\begin{aligned} x + 3y &= -8, \\ 5x - 2y &= 11; \end{aligned}$$

   (b)

$$\begin{aligned} x + 2y &= -2, \\ 5x - 4y &= 3; \end{aligned}$$

   (c)

$$\begin{aligned} 2x - y + z &= 7, \\ 3x + y - 5z &= 13, \\ x + y + z &= 5; \end{aligned}$$

   (d)

$$\begin{aligned} 3x + 2y - 2z &= 16, \\ 4x + 3y + 3z &= 2, \\ 2x - y + z &= -1. \end{aligned}$$

9.4.5 ANSWERS TO EXERCISES

1. (a) $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$;

   (b) $\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$;

   (c) $\begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{bmatrix}$;

   (d) $\begin{bmatrix} 77 & -17 & -14 \\ -27 & 6 & 5 \\ 22 & -5 & -4 \end{bmatrix}$;

   (e) There is no inverse;

   (f) $\begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$;

   (g) There is no inverse.

2. (a) $x = 1$ and $y = -3$;

   (b) $x = -\frac{1}{7}$ and $y = -\frac{13}{14}$;

   (c) $x = 4$, $y = 1$ and $z = 0$;

   (d) $x = 2$, $y = \frac{3}{2}$ and $z = -\frac{7}{2}$.

“JUST THE MATHS”

UNIT NUMBER

9.5

MATRICES 5
(Consistency and rank)

by

A.J.Hobson

UNIT 9.5 - MATRICES 5 - CONSISTENCY AND RANK

9.5.1 THE CONSISTENCY OF SIMULTANEOUS LINEAR EQUATIONS

Introduction

Methods of solving simultaneous linear equations in earlier Units have already shown that some sets of equations cannot be solved to give a unique solution. The Gaussian Elimination method described in Unit 9.4 is able to detect such situations as illustrated in the work below:

ILLUSTRATION 1.
Suppose, firstly, that we were required to investigate the solution of the simultaneous linear equations

$$\begin{aligned} 3x - y + z &= 1, \\ 2x + 2y - 5z &= 0, \\ 5x + y - 4z &= 7. \end{aligned}$$

The Gaussian Elimination solution, with check column, proceeds as follows

$$\begin{array}{lll|l|l} 3 & -1 & 1 & 1 & 4 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 7 & 9 \end{array}$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{array}{lll|l|l} 1 & -3 & 6 & 1 & 5 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 7 & 9 \end{array}$$

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 5R_1$

$$\begin{array}{lll|l|l} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 16 & -34 & 2 & -16 \end{array}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{array}{lll|l|l} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 0 & 0 & 6 & 6 \end{array}$$

The third line seems to imply that $0.z = 6$; that is, $0 = 6$ which is impossible.

Hence the equations have no solution and are said to be **"inconsistent"**.

ILLUSTRATION 2.
Secondly, let us consider the simultaneous linear equations

$$\begin{aligned} 3x - y + z &= 1, \\ 2x + 2y - 5z &= 0, \\ 5x + y - 4z &= 1, \end{aligned}$$

which differ from the first illustration in one number only, the constant term of the third equation; but the conclusion will be very different.

This time, the Gaussian Elimination method gives the following sequence of steps:

$$\begin{array}{ccc|c|c} 3 & -1 & 1 & 1 & 4 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 1 & 3 \end{array}$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 2 & 2 & -5 & 0 & -1 \\ 5 & 1 & -4 & 1 & 3 \end{array}$$

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 5R_1$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 16 & -34 & -4 & -22 \end{array}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{array}{ccc|c|c} 1 & -3 & 6 & 1 & 5 \\ 0 & 8 & -17 & -2 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

The third line here implies that the third equation is redundant since any set of x, y and z values would satisfy it. Hence, the original set of equations is equivalent to two equations in three unknowns, namely

$$\begin{aligned} x - 3y + 6z &= 1, \\ 8y - 17z &= -2. \end{aligned}$$

To state a suitable form of the conclusion, we could use the fact that any one of the three variables may be chosen at random, the other two being expressed in terms of it. For instance, if we choose z at random, then

$$x = \frac{3z+2}{8} \quad \text{and} \quad y = \frac{17z-2}{8}.$$

However, there is a neater way of arriving at the conclusion.

Neater form of solution

Suppose that $x = x_0$, $y = y_0$, $z = z_0$ is any **known** solution to the equations. It could be determined, for instance, by starting with $z = 0$; in this case, $z = 0$, $y = -\frac{1}{4}$ and $x = \frac{1}{4}$.

Let us now substitute

$$\begin{aligned} x &= x_1 + x_0, \\ y &= y_1 + y_0, \\ z &= z_1 + z_0, \end{aligned}$$

from which we obtain

$$\begin{aligned} (x_1 + x_0) - 3(y_1 + y_0) + 6(z_1 + z_0) &= 1, \\ 8(y_1 + y_0) - 17(z_1 + z_0) &= -2. \end{aligned}$$

But, because (x_0, y_0, z_0) is a known solution, this reduces to

$$\begin{aligned} x_1 - 3y_1 + 6z_1 &= 0, \\ 8y_1 - 17z_1 &= 0. \end{aligned}$$

This is a set of **"homogeneous linear equations"** and, although clearly satisfied by $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, we regard this as a **"trivial solution"** and ignore it.

Of more use to us is the fact that an infinite number of non-trivial solutions can be found for each of which the variables x_1, y_1 and z_1 are in a certain set of ratios. In the present case, from the second equation, we have

$y_1 = \frac{17}{8}z_1$ which means that $y_1 : z_1 = 17 : 8$.

Substituting into the first equation gives

$x_1 - \frac{51}{8}z_1 + 6z_1 = 0$ which means that $x_1 = \frac{3}{8}z_1$; that is, $x_1 : z_1 = 3 : 8$.

Combining these conclusions, we can say that

$$x_1 : y_1 : z_1 = 3 : 17 : 8$$

and any three numbers in these ratios will serve as a set of values for x_1, y_1 and z_1.

The neater form of solution can, in general, be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix},$$

where α may be any non-zero number.

In our present example,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ 17 \\ 8 \end{bmatrix}.$$

Notes:
(i) An alternative way of determining the ratios $x_1 : y_1 : z_1$ is to use the fact that, since at least one of the variables is going to be non-zero, we may begin by letting it equal 1.

In the above illustration, suppose we let $z_1 = 1$; then

$$y_1 = \frac{17}{8} \text{ and } x_1 = \frac{3}{8}.$$

Hence,

$$x_1 : y_1 : z_1 = \frac{3}{8} : \frac{17}{8} : 1,$$

which can be rewritten

$$x_1 : y_1 : z_1 = 3 : 17 : 8,$$

as before.

(ii) Should it happen that a set of simultaneous linear equations reduces to **only one** equation (that is, each equation is just a multiple of the first) then a similar procedure can be applied as in the following illustration:

ILLUSTRATION 3.

Using trial and error, the equation

$$3x - 2y + 5z = 6$$

has a particular solution $x_0 = 1, y_0 = 1, z_0 = 1$, so that the general solution is given by

$$x = x_0 + x_1,\ y = y_0 + y_1,\ z = z_0 + z_1,$$

where

$$3x_1 - 2y_1 + 5z_1 = 0.$$

We could choose $x_1 = \alpha$ and $y_1 = \beta$ at random for this equation to give $z_1 = \frac{2\beta - 3\alpha}{5}$. That is,

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}.$$

9.5.2 THE ROW-ECHELON FORM OF A MATRIX

In the Gaussian Elimination method to solve a set of simltaneous linear equations,

$$MX = K,$$

we begin with the augmented matrix M|K and use elementary row operations to obtain more zeros at the beginning of each row than at the beginning of the previous row.

If desired, the first non-zero element in each row could be reduced to 1 by simply dividing that row throughout by the value of the first non-zero element. This leads to the following definition:

DEFINITION
The **"row echelon form"** of a matrix is that for which the first non-zero element in each row is 1 and occurs to the right of the first non-zero element in the previous row.

Note:
When using the term "row echelon form" in future, we shall not insist that the first non-zero element in each row has **actually** been reduced to 1.

9.5.3 THE RANK OF A MATRIX

The two illustrations of Gaussian Elimination discussed in section 9.5.1 may be used to imply that special conclusions are reached when, in the final row echelon form, either a complete row of M has reduced to zero (Illustration 1.) or a complete row of M|K has reduced to zero (Illustration 2.) This leads to another definition as follows:

DEFINITION
The **"rank"** of a matrix is the number or rows which do not reduce to a complete row of zeros when the matrix has been converted to row echelon form.

ILLUSTRATIONS

1. In our previous Illustration 1, M had rank 2 but M|K had rank 3. The equations were inconsistent.
2. In our previous Illustration 2, M had rank 2 and M|K also had rank 2. The equations had an infinite number of solutions.
3. In the examples of Unit 9.4 both M and M|K had rank 3. There was a unique solution to the simultaneous equations.

A general summary of these observations may be set out as follows:

1. The equations MX = K are inconsistent if rank M $<$ rank M|K.
2. The equations MX = K have an infinite number of solutions if rank M = rank M|K $< n$ where n is the number of equations.
3. The equations MX = K have a unique solution if rank M = rank M|K $= n$ where n is the number of equations.

9.5.4 EXERCISES

1. Use Gaussian Elimination to show that the following sets of simultaneous equations are inconsistent:

   (a)
$$\begin{aligned} x - y + 2z &= 2, \\ 3x + y - z &= 3, \\ 5x - y + 3z &= 4; \end{aligned}$$

   (b)
$$\begin{aligned} x - y + 2z &= 1, \\ -x + 3y - z &= -1, \\ 3x - 7y + 4z &= 5. \end{aligned}$$

2. Determine the rank of the following matrices:
(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 5 \\ 3 & 4 & 11 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} a & 1 & 2 \\ -1 & -3 & 8 \\ 1 & 12 & -3 \\ 4 & -3 & 7 \end{bmatrix}$.

3. Determine the general solution of the following equations by reducing the augmented matrix to row echelon form:
$$\begin{aligned} x + 3y - z &= 6, \\ 8x + 9y + 4z &= 21, \\ 2x + y + 2z &= 3. \end{aligned}$$

4. State the value of t for which the matrix

$$\mathrm{M} = \begin{bmatrix} 2 & 1 & -3 \\ 4 & t & -6 \\ 3t & 3 & -9 \end{bmatrix}$$

has rank 1 and determine the general solution of the system of equations

$$\mathrm{MX} = \mathrm{K},$$

where

$$\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathrm{K} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

5. For the system of simultaneous linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1, \\ 2x_1 - x_2 + tx_3 &= 2, \\ -x_1 + 2x_2 + x_3 &= s, \end{aligned}$$

determine for which values of s and t there exists

(a) no solution;

(b) a unique solution;

(c) an infinite number of solutions.

Solve the equations for the two cases $s = 1, t = 1$ and $s = -1, t = 7$.

6. Determine the values of t for which the matrix

$$\mathrm{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & t \\ 3t & 2 & -2 \end{bmatrix}$$

has rank 2.

For each of these two values of t, solve the system of equations

$$\mathrm{MX} = \mathrm{K},$$

where $\mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathrm{K} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

9.5.5 ANSWERS TO EXERCISES

1. The rank of the matrix of x, y and z coefficients is less than the rank of the augmented matrix.

2. (a) 2; (b) 3.

3. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{9}{5} \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -7 \\ 4 \\ 5 \end{bmatrix}$.

4.
$$t = 2$$
and
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}.$$

5. (a) $t = 7$ and $s \neq -1$; (b) $t \neq 7$; (c) $t = 7$ and $s = -1$.
If $s = 1$ and $t = -1$, then $x_1 = \frac{7}{4}$, $x_2 = \frac{5}{4}$ and $x_3 = \frac{1}{4}$.
If $s = -1$ and $t = 7$, then
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}.$$

6. The rank is 2 when $t = -1$ or when $t = \frac{2}{3}$.
If $t = -1$, the equations are inconsistent.
If $t = \frac{2}{3}$, the equations have general solution
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -5 \\ 8 \\ 3 \end{bmatrix}.$$

"JUST THE MATHS"

UNIT NUMBER

9.6

MATRICES 6
(Eigenvalues and eigenvectors)

by

A.J.Hobson

UNIT 9.6 - MATRICES 6

EIGENVALUES AND EIGENVECTORS

9.6.1 THE STATEMENT OF THE PROBLEM

Suppose A is any square matrix, and let X be a column vector with the same number of rows as there are columns in A. For example, if A is of order $m \times m$, then X must be of order $m \times 1$ and AX will also be of order $m \times 1$.

We ask the question:

"Is it ever possible that AX can be just a scalar multiple of X ?"

We exclude the case when the elements of X are all zero since, in a practical application, these elements will usally be the components of an actual vector quantity; and, if they are all zero, the direction of the vector will be indeterminate.

ILLUSTRATIONS

1.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix}.$$

2.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The formal statement of the problem

For a given square matrix, A, we investigate the existence of column vectors, X, such that

$$\mathrm{AX} = \lambda \mathrm{X}$$

for some scalar quantity λ. Each such column vector is called an **"eigenvector"** of the matrix, A; and each corresponding value of λ is called an **"eigenvalue"** of the matrix, A.

Notes:

(i) The German word "eigen" (meaning "hidden") gives rise to the above names, but other alternatives are "latent values and latent vectors" or "characteristic values and characteristic vectors".

(ii) In the discussion which follows, A will be, mostly, a matrix of order 3×3, but the ideas involved will apply to square matrices of other orders also (see Example 1 and Exercises 9.6.3, question 1).

9.6.2 THE SOLUTION OF THE PROBLEM

Assuming that

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } \mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the matrix equation, $\mathrm{AX} = \lambda\mathrm{X}$, may be written out fully in the following form:

$$\begin{aligned} a_1x + b_1y + c_1z &= \lambda x, \\ a_2x + b_2y + c_2z &= \lambda y, \\ a_3x + b_3y + c_3z &= \lambda z; \end{aligned}$$

or, on rearrangement,

$$\begin{aligned} (a_1 - \lambda)x + b_1y + c_1z &= 0, \\ a_2x + (b_2 - \lambda)y + c_2z &= 0, \\ a_3x + b_3y + (c_3 - \lambda)z &= 0, \end{aligned}$$

which is a set of homogeneous linear equations in x, y and z and may be written, for short, in the form

$$(\mathrm{A} - \lambda\mathrm{I})\mathrm{X} = [0],$$

where I denotes the identity matrix of order 3×3.

From the results of Unit 7.4, the condition that the three homogeneous linear equations have a solution other than $x = 0$, $y = 0$, $z = 0$ is

$$|\mathrm{A} - \lambda \mathrm{I}| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{vmatrix} = 0.$$

This equation will give, on expansion, a cubic equation in λ called the **"characteristic equation"** of A. Its left-hand side is called the **"characteristic polynomial"** of A.

The characteristic equation of a 3×3 matrix, being a cubic equation, will (in general) have three solutions.

EXAMPLES

1. Determine the eigenvalues and eigenvectors of the matrix

$$\mathrm{A} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}.$$

Solution

(a) The eigenvalues

The characteristic equation is given by

$$0 = |\mathrm{A} - \lambda \mathrm{I}| = \begin{vmatrix} 2 - \lambda & 4 \\ 5 & 3 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda - 14 = (\lambda + 2)(\lambda - 7).$$

The eigenvalues are therefore $\lambda = -2$ and $\lambda = 7$.

(b) The eigenvectors

Case 1. $\lambda = -2$

We require to solve the equation $x + y = 0$,

giving $x : y = -1 : 1$ and a corresponding eigenvector

$$\mathrm{X} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where α is any **non-zero** scalar.

Case 2. $\lambda = 7$

We require to solve the equation $5x - 4y = 0$,

giving $x : y = 4 : 5$ and a corresponding eigenvector

$$X = \beta \begin{bmatrix} 4 \\ 5 \end{bmatrix},$$

where β is any **non-zero** scalar.

2. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Solution

(a) The eigenvalues

The characteristic equation is given by

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}.$$

Direct expansion of the determinant gives the equation

$$-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0,$$

which will factorise into

$$(1+\lambda)^2(8-\lambda) = 0.$$

Note:
Students who have studied row and column operations for determinants (see Unit 7.3) may obtain this by simplifying the determinant first. (One way is to subtract the third column from the first column and then add the third row to the first row).

The eigenvalues are therefore $\lambda = -1$(repeated) and $\lambda = 8$.

(b) The eigenvectors

Case 1. $\lambda = 8$

We require to solve the homogeneous equations

$$\begin{aligned} -5x + 2y + 4z &= 0, \\ 2x - 8y + 2z &= 0, \\ 4x + 2y - 5z &= 0. \end{aligned}$$

Eliminating x from the second and third equations gives $18y - 9z = 0$.

Eliminating y from the second and third equations gives $18x - 18z = 0$.

Since z appears twice in the two statements, we may try letting $z = 1$ to give $y = \frac{1}{2}$ and $x = 1$.

Hence,

$$x : y : z = 1 : \frac{1}{2} : 1 = 2 : 1 : 2$$

The eigenvectors corresponding to $\lambda = 8$ are thus given by

$$\mathrm{X} = \alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where α is any **non-zero** scalar.

Case 2. $\lambda = -1$

This time, we require to solve the homogeneous equations

$$\begin{aligned} 4x + 2y + 4z &= 0, \\ 2x + y + 2z &= 0, \\ 4x + 2y + 4z &= 0, \end{aligned}$$

which, of course, are all the same equation, $2x + y + 2z = 0$.

Hence, two of the variables may be chosen at random (say $y = \beta$ and $z = \gamma$), then the third variable may be expressed in terms of them; (in this case $x = -\frac{1}{2}\beta - \gamma$).

However, a neater technique is to obtain, first, a pair of independent particular solutions by setting one pair of the variables at 1 and 0, in both orders. For example $y = 1$ and $z = 0$ gives $x = -\frac{1}{2}$ while $y = 0$ and $z = 1$ gives $x = -1$.

The general solution (in which y and z are randomly chosen as β and γ respectively) is then given by

$$\mathrm{X} = \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where β and γ are **not both equal to zero** at the same time.

Notes:

(i) Similar results in Case 2 could be obtained by choosing **different** pairs of the three variables at random.

(ii) Other special cases arise if the three homogeneous equations reduce to a single equation in which one or even two of the variables is absent.

For example, if they reduced to $y = 0$, the corresponding eigenvectors could be given by

$$\mathrm{X} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

allowing $x = \alpha$ and $z = \gamma$ to be chosen at random, assuming that α and γ are not both zero simultaneously.

Alternatively, if the homogeneous equations reduced to $3x + 5z = 0$, then the corresponding eigenvectors could be given by

$$\mathrm{X} = \alpha \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{5} \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

allowing $x = \alpha$ and $y = \beta$ to be chosen at random, assuming that α and β are not both zero simultaneously.

9.6.3 EXERCISES

Determine the eigenvalues and eigenvectors of the following matrices:

1.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

2.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

3.

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

4.

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

5.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

6.

$$\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

7.

$$\begin{bmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

8.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

9.

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 0 & 2 \end{bmatrix}.$$

10.

$$\begin{bmatrix} 4 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 3 \end{bmatrix}.$$

9.6.4 ANSWERS TO EXERCISES

The eigenvectors will be written as X^T instead of X, in order to avoid unnecessary waste of space.

1. $\lambda = 0$, giving $X^T = \alpha[-3 \quad 1]$,
   where α is any non-zero scalar;

   $\lambda = 7$, giving $X^T = \beta[1 \quad 2]$,
   where β is any non-zero scalar.

2. $\lambda = 1$, giving $X^T = \alpha[1 \quad -1 \quad 0]$,
   where α is any non-zero scalar;

   $\lambda = 2$, giving $X^T = \beta[-2 \quad 1 \quad 2]$,
   where β is any non-zero scalar;

   $\lambda = 3$, giving $X^T = \gamma[-1 \quad 1 \quad 2]$,
   where γ is any non-zero scalar.

3. $\lambda = 1$, giving $X^T = \alpha[3 \quad 2 \quad 1]$,
   where α is any non-zero scalar;

   $\lambda = -1$, giving $X^T = \beta[1 \quad 0 \quad 1]$,
   where β is any non-zero scalar;

   $\lambda = 2$, giving $X^T = \gamma[1 \quad 3 \quad 1]$,
   where γ is any non-zero scalar.

4. $\lambda = 1$, giving $X^T = \alpha[1 \quad 1 \quad -1]$,
   where α is any non-zero scalar;

   $\lambda = 2$ (repeated), giving $X^T = \beta[2 \quad 1 \quad 0]$,
   where β is any non-zero scalar.

5. $\lambda = 1$ (repeated), giving $X^T = \alpha[1 \quad 1 \quad 1]$,
   where α is any non-zero scalar.

6. $\lambda = 0$, giving $X^T = \alpha[-4 \quad 1 \quad 0]$,
   where α is any non-zero scalar;

   $\lambda = 2$, giving $X^T = \beta[-2 \quad 1 \quad 0]$,
   where β is any non-zero scalar;

   $\lambda = -1$, giving $X^T = \gamma[28 \quad -8 \quad 3]$,
   where γ is any non-zero scalar.

7. $\lambda = 1$, giving $\mathrm{X}^{\mathrm{T}} = \alpha[-12 \;\; 4 \;\; 1]$,
   where α is any non-zero scalar;

   $\lambda = 0$ (repeated), giving $\mathrm{X}^{\mathrm{T}} = \beta[-3 \;\; 1 \;\; 0] + \gamma[-4 \;\; 0 \;\; 1]$,
   where β and γ are any scalar numbers which are not simultaneously zero.

8. $\lambda = 5$, giving $\mathrm{X}^{\mathrm{T}} = \alpha[1 \;\; 1 \;\; 1]$,
   where α is any non-zero scalar;

   $\lambda = 1$ (repeated), giving $\mathrm{X}^{\mathrm{T}} = \beta[-2 \;\; 1 \;\; 0] + \gamma[-1 \;\; 0 \;\; 1]$,
   where β and γ are any scalar numbers which are not simultaneously zero.

9. $\lambda = 2$ (repeated), giving $\mathrm{X}^{\mathrm{T}} = \beta[0 \;\; 1 \;\; 0] + \gamma[0 \;\; 0 \;\; 1]$
   where β and γ are any scalar numbers which are not simultaneously zero.

10. $\lambda = 2$, giving $\mathrm{X}^{\mathrm{T}} = \alpha[1 \;\; 1 \;\; 1]$,
    where α is any non-zero number;

    $\lambda = 3$, giving $\mathrm{X}^{\mathrm{T}} = \alpha[1 \;\; \frac{1}{2} \;\; 0] + \gamma[0 \;\; 0 \;\; 1]$,
    where α and γ are any scalar numbers which are not simultaneously zero.

"JUST THE MATHS"

UNIT NUMBER

9.7

MATRICES 7
(Linearly independent eigenvectors)
&
(Normalised eigenvectors)

by

A.J.Hobson

UNIT 9.7 - MATRICES 7

LINEARLY INDEPENDENT AND NORMALISED EIGENVECTORS

9.7.1 LINEARLY INDEPENDENT EIGENVECTORS

It is often useful to know if an $n \times n$ matrix, A, possesses a full set of n eigenvectors $X_1, X_2, X_3, \ldots, X_n$, which are **"linearly independent"**.

That is, they are **not** connected by any relationship of the form

$$a_1X_1 + a_2X_2 + a_3X_3 + \ldots \equiv 0,$$

where $a_1, a_2, a_3, \ldots$ are constants.

If the eigenvalues of A are distinct, it turns out that the eigenvectors are linearly independent; but, if any of the eigenvalues are repeated, further investigation may be necessary.

ILLUSTRATIONS

1. In Unit 9.6, it was shown that the matrix

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 8$ and $\lambda = -1$(repeated), with corresponding eigenvectors,

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},$$

where α is any **non-zero** scalar and

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where β and γ are not both equal to zero at the same time.

The matrix, A, possesses a set of three linearly independent eigenvectors which may, conveniently, be chosen as

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

It is reasonably obvious that these are linearly independent, but a formal check would be to show that the matrix,

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

has rank 3.

2. It may be shown that the matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix},$$

has eigenvalues $\lambda = 2$ (repeated) and $\lambda = 1$, with corresponding eigenvectors,

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \beta \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

where α and β are any non-zero numbers.

In this case, it is not possible to obtain a full set of three linearly independent eigenvectors.

9.7.2 NORMALISED EIGENVECTORS

It is sometimes convenient to use a set of **"normalised"** eigenvectors, which means that, for each eigenvector, the sum of the squares of its elements is equal to 1.

An eigenvector may be normalised if we multiply it by (plus or minus) the reciprocal of the square root of the sum of the squares of its elements.

ILLUSTRATIONS

1. A set of linearly independent normalised eigenvectors for the matrix,

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix},$$

is

$$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

2. A set of linearly independent normalised eigenvectors for the matrix,

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix},$$

is

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

9.7.3 EXERCISES

1. Determine the eigenvalues and a set of linearly independent normalised eigenvectors for the following 2×2 matrices:

$$\text{(a)} \quad \begin{bmatrix} 17 & -6 \\ 45 & -16 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} 5 & -2 \\ 7 & -4 \end{bmatrix}, \quad \text{(c)} \quad \begin{bmatrix} 16 & -8 \\ 24 & -12 \end{bmatrix}.$$

2. Determine the eigenvalues and a set of linearly independent normalised eigenvectors for the following 3×3 matrices:

(a)

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix};$$

(b)

$$\begin{bmatrix} 7 & 0 & 4 \\ 7 & 0 & 4 \\ 0 & 0 & 11 \end{bmatrix};$$

(c)

$$\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix};$$

(d)

$$\begin{bmatrix} -2 & 0 & -14 \\ -7 & 5 & -14 \\ 0 & 0 & 5 \end{bmatrix};$$

(e)

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}.$$

9.7.4 ANSWERS TO EXERCISES

1. (a) The eigenvalues are 2 and -1.
   A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{29}}\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   (b) The eigenvalues are 3 and -2.
   A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{53}}\begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

   (c) The eigenvalues are 4 and 0.
   A set if linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

2. (a) The eigenvalues are 1, 0 and -2.
A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix},\quad \frac{1}{\sqrt{230}}\begin{bmatrix}10\\3\\-11\end{bmatrix}\quad \text{and}\quad \frac{1}{\sqrt{74}}\begin{bmatrix}4\\3\\-7\end{bmatrix}.$$

(b) The eigenvalues are 11, 7 and 0.
A set of linearly independent normalised eigenvectors are

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},\quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\quad \text{and}\quad \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

(c) The eigenvalues are 2 (repeated) and -2.
A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix},\quad \text{and}\quad \frac{1}{\sqrt{66}}\begin{bmatrix}4\\1\\7\end{bmatrix}.$$

(d) The eigenvalues are 5 (repeated) and -2.
A set of linearly independent normalised eigenvectors is

$$\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix},\quad \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\0\\1\end{bmatrix}\quad \text{and}\quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

(e) The eigenvalues are 4 (repeated) and 3.
A set of linearly independent normalised eigenvectors is

$$\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \begin{bmatrix}0\\0\\1\end{bmatrix}\quad \text{and}\quad \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}.$$

"JUST THE MATHS"

UNIT NUMBER

9.8

MATRICES 8
(Characteristic properties)
&
(Similarity transformations)

by

A.J.Hobson

UNIT 9.8 - MATRICES 8

CHARACTERISTIC PROPERTIES AND SIMILARITY TRANSFORMATIONS

9.8.1 PROPERTIES OF EIGENVALUES AND EIGENVECTORS

We list, here, a number of standard properties, together with either their proofs or an illustration of their proofs.

(i) The eigenvalues of a matrix are the same as those of its transpose.

Proof:

Given a square matrix, A, the eigenvalues of A^{T} are the solutions of the equation

$$|\mathrm{A}^{\mathrm{T}} - \lambda \mathrm{I}| = 0.$$

But, since I is a symmetric matrix, this is equivalent to

$$|(\mathrm{A} - \lambda \mathrm{I})^{\mathrm{T}}| = 0.$$

The result follows since a determinant is unchanged in value when it is transposed.

(ii) The eigenvalues of the multiplicative inverse of a matrix are the reciprocals of the eigenvalues of the matrix itself.

Proof:

If λ is any eigenvalue of a square matrix, A, then

$$\mathrm{AX} = \lambda \mathrm{X},$$

for some column vector, X.

Premultiplying this relationship by A^{-1}, we obtain

$$\mathrm{A}^{-1}\mathrm{AX} = \mathrm{A}^{-1}(\lambda \mathrm{X}) = \lambda(\mathrm{A}^{-1}\mathrm{X}).$$

Thus,

$$\mathrm{A}^{-1}\mathrm{X} = \frac{1}{\lambda}\mathrm{X}.$$

(iii) The eigenvectors of a matrix and its multiplicative inverse are the same.

Proof:

This follows from the proof of **(ii)**, since

$$\mathrm{A}^{-1}\mathrm{X} = \frac{1}{\lambda}\mathrm{X}$$

implies that X is an eigenvector of A^{-1}.

(iv) If a matrix is multiplied by a single number, the eigenvalues are multiplied by that number, but the eigenvectors remain the same.

Proof:

If A is multiplied by α, we may write the equation $\mathrm{AX} = \lambda\mathrm{X}$ in the form $\alpha\mathrm{AX} = \alpha\lambda\mathrm{X}$.

Thus, $\alpha\mathrm{A}$ has eigenvalues, $\alpha\lambda$, and eigenvectors, X.

(v) If $\lambda_1, \quad \lambda_2, \quad \lambda_3, \ . \ .$ are the eigenvalues of the matrix A and n is a positive integer, then $\lambda_1^n, \quad \lambda_2^n, \quad \lambda_3^n, \ . \ .$ are the eigenvalues of A^n.

Proof:

If λ denotes any one of the eigenvalues of the matrix, A, then $\mathrm{AX} = \lambda\mathrm{X}$.

Premultiplying both sides by A, we obtain $\mathrm{A}^2\mathrm{X} = \mathrm{A}\lambda\mathrm{X} = \lambda\mathrm{A}X = \lambda^2\mathrm{X}$

Hence, λ^2 is an eigenvlaue of A^2.

Similarly, $\mathrm{A}^3\mathrm{X} = \lambda^3\mathrm{X}$, and so on.

(vi) If $\lambda_1, \ \lambda_2, \ \lambda_3, \ \ . \ . \ .$ are the eigenvalues of the $n \times n$ matrix A, I is the $n \times n$ multiplicative identity matrix and k is a single number, then the eigenvalues of the matrix $\mathrm{A} + k\mathrm{I}$ are $\lambda_1 + k, \ \lambda_2 + k, \ \lambda_3 + k, \ . \ . \ .$.

Proof:

If λ is any eigenvalue of A, then $AX = \lambda X$.

Hence,

$$(A + kI)X = AX + kX = \lambda X + kX = (\lambda + k)X.$$

(vii) A matrix is singular ($|A| = 0$) if and only if at least one eigenvalue is equal to zero.

Proof:

(a) If X is an eigenvector corresponding to an eigenvalue, $\lambda = 0$, then $AX = \lambda X = [0]$.

From the theory of homogeneous linear equations (see Unit 7.4), it follows that $|A| = 0$.

(b) Conversely, if $|A| = 0$, the homogeneous system $AX = [0]$ has a solution for X other than $X = [0]$. Hence, at least one eigenvalue must be zero.

(viii) If A is an orthogonal matrix ($AA^T = I$), then every eigenvalue is either $+1$ or -1.

Proof:

The statement $AA^T = I$ can be written $A^{-1} = A^T$ so that, by **(i)** and **(ii)**, the eigenvalues of A are equal to their own reciprocals.

That is, they must have values $+1$ or -1.

(ix) If the elements of a matrix below the leading diagonal or the elements above the leading diagonal are all equal zero, then the eigenvalues are equal to the diagonal elements.

ILLUSTRATION

An **"upper-triangular matrix"**, A, of order 3×3, has the form

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{bmatrix}.$$

The characteristic equation is given by

$$0 = |\mathrm{A} - \lambda\mathrm{I}| = \begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ 0 & b_2 - \lambda & c_2 \\ 0 & 0 & c_3 - \lambda \end{vmatrix} = (a_1 - \lambda)(b_2 - \lambda)(c_3 - \lambda).$$

Hence, $\lambda = a_1,\ b_2$ or c_3.

A similar proof holds for a **"lower-triangular matrix"**.

Note:
A special case of both a lower-triangular matrix and an upper-triangular matrix is a diagonal matrix.

(x) The sum of the eigenvalues of a matrix is equal to the trace of the matrix (the sum of the diagonal elements) and the product of the eigenvalues is equal to the determinant of the matrix.

ILLUSTRATION

We consider the case of a 2×2 matrix, A, given by

$$\mathrm{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

The characteristic equation is

$$0 = \begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = \lambda^2 - (a_1 + b_2)\lambda + (a_1 b_2 - a_2 b_1).$$

But, for any quadratic equation, $a\lambda^2 + b\lambda + c = 0$, the sum of the solutions is equal to $-b/a$ and the product of the solutions is equal to c/a.

In this case, therefore, the sum of the solutions is $a_1 + b_2$, while the product of the solutions is $a_1 b_2 - a_2 b_1$.

9.8.2 SIMILAR MATRICES

In the previous section, a matrix and its transpose illustrated how two matrices can have the same eigenvalues. In this section, we deal with a more general case of this occurrence.

DEFINITION

Two matrices, A and B, are said to be **"similar"** if

$$B = P^{-1}AP,$$

for some non-singular matrix, P.

Notes:

(i) P is certainly square, so that A and B must also be square and of the same order as P.

(ii) The relationship $B = P^{-1}AP$ is regarded as a **"transformation"** of the matrix, A, into the matrix, B.

(iii) A relationship of the form $B = QAQ^{-1}$ may also be regarded as a similarity transformation on A, since Q is the multiplicative inverse of Q^{-1}.

THEOREM

Two similar matrices, A and B, have the same eigenvalues. Furthermore, if the similarity transformation from A to B is $B = P^{-1}AP$, then the eigenvectors, X and Y, of A and B respectively are related by the equation

$$Y = P^{-1}X.$$

Proof:

The eigenvalues, λ, and the eigenvectors, X, of A satisfy the relationship $AX = \lambda X$.

Hence,

$$P^{-1}AX = \lambda P^{-1}X.$$

Secondly, using the fact that $PP^{-1} = I$, we have

$$P^{-1}APP^{-1}X = \lambda P^{-1}X,$$

which may be written

$$(P^{-1}AP)(P^{-1}X) = \lambda(P^{-1}X)$$

or

$$BY = \lambda Y,$$

where $B = P^{-1}AP$ and $Y = P^{-1}X$.

This shows that the eigenvalues of A are also the eigenvalues of B and that the eigenvectors of B are of the form $P^{-1}X$.

Reminders

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right];$$

and, in general, for a square matrix M,

$$M^{-1} = \frac{1}{|M|} \times \text{ the transpose of the cofactor matrix.}$$

9.8.3 EXERCISES

1. For the matrix,

$$A = \left[\begin{array}{ccc} 1 & -1 & 0 \\ 2 & 4 & 0 \\ -2 & -1 & 1 \end{array}\right],$$

   and its multiplicative inverse, A^{-1}, determine the eigenvalues and a set of corresponding linearly independent normalised eigenvectors.

2. State the eigenvalues for the upper-triangular matrix

$$\begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

and, hence, obtain a set of linearly independent normalised eigenvectors for the matrix.

3. State the eigenvalues of the lower-triangular matrix

$$\begin{bmatrix} 6 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & -10 \end{bmatrix}$$

and, hence, obtain a set of linearly independent normalised eigenvectors for the matrix.

4. Determine the eigenvalues and a set of corresponding linearly independent eigenvectors for the matrix $B = P^{-1}AP$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}.$$

5. Determine the eigenvalues and a set of corresponding linearly independent eigenvectors for the matrix $B = P^{-1}AP$, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 0 \\ -2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -2 & 4 \\ 5 & 1 & 6 \end{bmatrix}.$$

6. Show that the matrix

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

is orthogonal and verify that its eigenvalues are either 1 or -1.

9.8.4 ANSWERS TO EXERCISES

1. The eigenvalues of A are 3, 2 and 1 with corresponding normalised eigenvectors,

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of A^{-1} are $\frac{1}{3}$, $\frac{1}{2}$ and 1, with corresponding normalised eigenvectors the same as for A.

2. The eigenvalues are 3, 2 and -1, with corresponding linearly independent normalised eigenvectors,

$$\frac{1}{\sqrt{17}}\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}.$$

3. The eigenvalues are 6, 0 and -10, with corresponding linearly independent normalised eigenvectors,

$$\frac{1}{\sqrt{1305}}\begin{bmatrix} 32 \\ 16 \\ 5 \end{bmatrix}, \quad \frac{1}{\sqrt{101}}\begin{bmatrix} 0 \\ 10 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

4. The eigenvalues of B are the same as those of A, namely 0 and 7.

$$\text{Also,} \quad P^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}.$$

Hence, a set of linearly independent eigenvectors for B is

$$\begin{bmatrix} -13 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -12 \\ 7 \end{bmatrix}.$$

5. The eigenvalues of B are the same as those of A which, from question 1, are 3, 2 and 1. Also,

$$P^{-1} = -\frac{1}{22}\begin{bmatrix} -16 & 9 & 2 \\ 20 & -3 & -8 \\ 10 & -7 & -4 \end{bmatrix},$$

so that a set of linearly independent eigenvectors for B are

$$\begin{bmatrix} -17 \\ 13 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} -27 \\ 31 \\ 21 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}.$$

6. $CC^T = I$ and the eigenvalues are 1 (repeated) and -1.

"JUST THE MATHS"

UNIT NUMBER

9.9

MATRICES 9
(Modal & spectral matrices)

by

A.J.Hobson

UNIT 9.9 - MATRICES 9

MODAL AND SPECTRAL MATRICES

9.9.1 ASSUMPTIONS AND DEFINITIONS

For convenience, we shall make, here, the following assumptions:

(a) The n eigenvalues, $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$, of an $n \times n$ matrix, A, are arranged in order of decreasing value.

(b) Corresponding to $\lambda_1,\ \lambda_2,\ \lambda_3, \ldots, \lambda_n$, respectively, A possesses a full set of eigenvectors $X_1,\ X_2,\ X_3, \ldots, X_n$, which are linearly independent.

If two eigenvalues coincide, the order of writing down the corresponding pair of eigenvectors will be immaterial.

DEFINITION 1

The square matrix obtained by using as its columns any set of linearly independent eigenvectors of a matrix A is called a **"modal matrix"** of A, and may be denoted by M.

Notes:

(i) There are infinitely many modal matrices for a given matrix, A, since any multiple of an eigenvector is also an eigenvector.

(ii) It is sometimes convenient to use a set of normalised eigenvectors.

When using normalised eigenvectors, the modal matrix may be denoted by N and, for an $n \times n$ matrix, A, there are 2^n possibilities for N since each of the n columns has two possibilities.

DEFINITION 2

If $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix, A, then the diagonal matrix,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ 0 & 0 & \lambda_3 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \lambda_n \end{bmatrix}$$

is called the **"spectral matrix"** of A, and may be denoted by S.

EXAMPLE

For the matrix,

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

determine a modal matrix, a modal matrix of normalised eigenvectors and the spectral matrix.

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0,$$

which may be shown to give

$$-(1+\lambda)(1-\lambda)(2-\lambda) = 0.$$

Hence, the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 1$ and $\lambda_3 = -1$ in order of decreasing value.

Case 1. $\lambda = 2$

We solve the simultaneous equations

$$\begin{aligned} -x + y - 2z &= 0, \\ -x + 0y + z &= 0, \\ 0x + y - 3z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 3 : 1$

Case 2. $\lambda = 1$

We solve the simultaneous equations

$$\begin{aligned} 0x + y - 2z &= 0, \\ -x + y + z &= 0, \\ 0x + y - 2z &= 0, \end{aligned}$$

which give $x : y : z = 3 : 2 : 1$

Case 3. $\lambda = -1$

We solve the simultaneous equations

$$\begin{aligned} 2x + y - 2z &= 0, \\ -x + 3y + z &= 0, \\ 0x + y + 0z &= 0, \end{aligned}$$

which give $x : y : z = 1 : 0 : 1$

A modal matrix for A may therefore be given by

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

A modal matrix of normalised eigenvectors may be given by

$$\mathrm{N} = \begin{bmatrix} \frac{1}{\sqrt{11}} & \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{11}} & \frac{2}{\sqrt{14}} & 0 \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The spectral matrix is given by

$$\mathrm{S} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

9.9.2 DIAGONALISATION OF A MATRIX

Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, it is clear that a matrix, A, and its spectral matrix, S, have the same eigenvalues.

From the Theorem in Unit 9.8, therefore, it seems reasonable that A and S could be similar matrices; and this is the content of the following result which will be illustrated rather than proven.

THEOREM

The matrix, A, is similar to its spectral matrix, S, the similarity transformation being

$$\mathrm{M}^{-1}\mathrm{AM} = \mathrm{S},$$

where M is a modal matrix for A.

ILLUSTRATION:

Suppose that $\mathrm{X}_1,\ \mathrm{X}_2$ and X_3 are linearly independent eigenvectors of a 3×3 matrix, A, corresponding to eigenvalues $\lambda_1,\ \lambda_2$ and λ_3, respectively.

Then,

$$\mathrm{AX}_1 = \lambda_1\mathrm{X}_1, \quad \mathrm{AX}_2 = \lambda_2\mathrm{X}_2, \quad \text{and} \quad \mathrm{AX}_3 = \lambda_3\mathrm{X}_3.$$

Also,

$$\mathrm{M} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}.$$

If M is premultiplied by A, we obtain a 3×3 matrix whose columns are $\mathrm{AX}_1,\ \mathrm{AX}_2,$ and AX_3.

That is,

$$\mathrm{AM} = [\,\mathrm{AX}_1 \quad \mathrm{AX}_2 \quad \mathrm{AX}_3\,] = [\,\lambda_1\mathrm{X}_1 \quad \lambda_2\mathrm{X}_2 \quad \lambda_3\mathrm{X}_3\,]$$

or

$$\mathrm{AM} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} . \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \mathrm{MS}.$$

We conclude that

$$\mathrm{M}^{-1}\mathrm{AM} = \mathrm{S}.$$

Notes:

(i) M^{-1} exists only because $\mathrm{X}_1,\ \mathrm{X}_2$ and X_3 are linearly independent.

(ii) The similarity transformation in the above theorem reduces the matrix, A, to **"diagonal form"** or **"canonical form"** and the process is often referred to as the **"diagonalisation"** of the matrix, A.

EXAMPLE

Verify the above Theorem for the matrix

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Solution

From an earlier example, a modal matrix for A may be given by

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The spectral matrix is given by

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It may be shown that

$$M^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}$$

and, hence,

$$M^{-1}AM = \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}.\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.\begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -2 & 2 & 2 \\ 3 & 0 & -3 \\ -1 & -2 & 7 \end{bmatrix}.\begin{bmatrix} 2 & 3 & -1 \\ 6 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = S.$$

9.9.3 EXERCISES

1. Determine a modal matrix, M, of linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} -1 & 6 & -12 \\ 0 & -13 & 30 \\ 0 & -9 & 20 \end{bmatrix}.$$

   Verify that $M^{-1}AM = S$, where S is the spectral matrix of A.

2. Determine a modal matrix, M, of linearly independent eigenvectors for the matrix

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

   Verify that $M^{-1}AM = S$, where S is the spectral matrix of B.

3. Determine a modal matrix, N, of linearly independent normalised eigenvectors for the matrix

$$C = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}.$$

   Verify that $N^{-1}AN = S$, where S is the spectral matrix of C.

4. Show that the following matrices are not similar to a diagonal matrix:

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}, \qquad \text{(b)} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

9.9.4 ANSWERS TO EXERCISES

1. The eigenvalues are 5,2 and -1, which gives

$$\mathrm{M} = \begin{bmatrix} -1 & 0 & 1 \\ 5 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}.$$

2. The eigenvalues are 2, 1 and -1, which gives

$$\mathrm{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

3. The eigenvalues are 4, 2 and 1, which gives

$$\mathrm{N} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{5}} & 1 \end{bmatrix}.$$

4. (a) The eigenvalues are 2 (repeated) and 1 but there are only two linearly independent eigenvectors, namely

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(b) There is only one eigenvalue, 1 (repeated), and only one linearly independent eigenvector, namely

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

"JUST THE MATHS"

UNIT NUMBER

9.10

MATRICES 10
(Symmetric matrices & quadratic forms)

by

A.J.Hobson

UNIT 9.10 - MATRICES 10

SYMMETRIC MATRICES AND QUADRATIC FORMS

9.10.1 SYMMETRIC MATRICES

The definition of a symmetric matrix was introduced in Unit 9.1 and matrices of this type have certain special properties with regard to eigenvalues and eigenvectors. We list them as follows:

(i) All of the eigenvalues of a symmetric matrix are real and, hence, so are the eigenvectors.

(ii) A symmetric matrix of order $n \times n$ always has n linearly independent eigenvectors.

(iii) For a symmetric matrix, suppose that X_i and X_j are linearly independent eigenvectors associated with <u>different</u> eigenvectors; then

$$\mathrm{X}_i\mathrm{X}_j^{\mathrm{T}} \equiv x_i x_j + y_i y_j + z_i z_j = 0.$$

We say that X_i and X_j are **"mutually orthogonal"**.

If a symmetric matrix has any repeated eigenvalues, it is still possible to determine a full set of mutually orthogonal eigenvectors, but not <u>every</u> full set of eigenvectors will have the orthogonality property.

(iv) A symmetric matrix always has a modal matrix whose columns are mutually orthogonal. When the eigenvalues are distinct, this is true for <u>every</u> modal matrix.

(v) A modal matrix, N, of normalised eigenvectors is an orthogonal matrix.

ILLUSTRATIONS

1. If N is of order 3×3, we have

$$\mathrm{N}^{\mathrm{T}}.N = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} . \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. It was shown in Unit 9.6 that the matrix

$$\mathrm{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has eigenvalues $\lambda = 8$, and $\lambda = -1$ (repeated), with associated eigenvectors

$$\alpha \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} -\frac{1}{2}\beta - \gamma \\ \beta \\ \gamma \end{bmatrix}.$$

A set of **linearly independent** eigenvectors may therefore be given by

$$\mathrm{X}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathrm{X}_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathrm{X}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, X_1 is orthogonal to X_2 and X_3, but X_2 and X_3 are not orthogonal to each other. However, we may find β and γ such that

$$\beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ is orthogonal to } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We simply require that

$$\frac{1}{2}\beta + 2\gamma = 0$$

or

$$\beta + 4\gamma = 0;$$

and this will be so, for example, when $\beta = 4$ and $\gamma = -1$.

A new set of linearly independent mutually orthogonal eigenvectors can thus be given by

$$\mathrm{X}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathrm{X}_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathrm{X}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

9.10.2 QUADRATIC FORMS

An algebraic expression of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2yzx + 2hxy$$

is called a **"quadratic form"**.

In matrix notation, it may be written as

$$[\, x \quad y \quad z \,] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv \mathrm{X}^{\mathrm{T}}\mathrm{AX},$$

and we note that the matrix A is symmetric.

In the scientific applications of quadratic forms, it is desirable to know whether such a form is

(a) always positive,

(b) always negative,

(c) both positive and negative.

It may be shown that, if we change to new variables, (u, v, w), using a linear transformation

$$\mathrm{X} = \mathrm{PU},$$

where P is some non-singular matrix, then the new quadratic form has the same properties as the original, concerning its sign.

We now show that a good choice for P is a modal matrix, N, of normalised, linearly independent, mutually orthogonal eigenvectors for A.

Putting X = NU, the expression X^TAX becomes U^TN^TANX.

But, since N is orthogonal when A is symmetric, $N^T = N^{-1}$ and, hence, N^TAN is the spectral matrix, S, for A.

The new quadratic form is therefore

$$U^TSU \equiv [\,u \quad v \quad w\,].\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.\begin{bmatrix} u \\ v \\ w \end{bmatrix} \equiv \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2.$$

Clearly, if all of the eigenvalues are positive, then the new quadratic form is always positive; and, if all of the eigenvalues are negative, then the new quadratic form is always negative.

The new quadratic form is called the **"canonical form under similarity"** of the original quadratic form.

9.10.3 EXERCISES

1. For the following symmetric matrices, determine a set of three linearly independent and mutually orthogonal eigenvectors:

$$\text{(a)} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 10 & 6 \\ 0 & 6 & 5 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. Repeat the previous question for the following symmetric matrices:

$$\text{(a)} \quad \begin{bmatrix} 3 & 3 & 3\sqrt{2} \\ 3 & 3 & 3\sqrt{2} \\ 3\sqrt{2} & 3\sqrt{2} & 6 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{bmatrix}.$$

3. Using the results of question 1, show that the following quadratic forms are always positive:

(a)

$$2x^2 + 10y^2 + 5z^2 + 12yz;$$

(b)

$$2x^2 + 5y^2 + 3z^2 + 4xy.$$

4. Using the results of question 2(b), obtain the matrix, P, of the orthogonal transformation, X = PU, which transforms the quadratic function

$$2x^2 + y^2 - 2z^2 + 4xz$$

into the quadratic function

$$2u^2 + 2v^2 - 3w^2.$$

State whether or the not the original quadratic form is always positive.

9.10.4 ANSWERS TO EXERCISES

1. (a) The eigenvalues are 14, 2 and 1 and a set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}.$$

(b) The eigenvalues are 6, 3 and 1 and a set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

2. (a) The eigenvalues are 12 and 0 (repeated) and set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}.$$

(b) The eigenvalues are 2 and -3 (repeated) and a set of linearly independent mutually orthogonal eigenvectors is

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

3. The eigenvalues are all positive and hence the quadratic forms are always positive.
4.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{bmatrix}.$$

The original quadratic form may take both positive and negative values since the associated eigenvalues are not all positive.

"JUST THE MATHS"

UNIT NUMBER

10.1

DIFFERENTIATION 1
(Functions and limits)

by

A.J.Hobson

UNIT 10.1 - DIFFERENTIATION 1

10.1.1 FUNCTIONAL NOTATION

Introduction

If a variable quantity, y, depends for its values on another variable quantity, x, we say that **"y is a function of x"** and we write, in general:

$$y = f(x),$$

which is pronounced "y equals f of x".

Notes:

(i) y is called the **"dependent variable"** and x is called the **"independent variable"**. The choice of value for x will be arbitrary, within certain possible constraints, but, once it **is** chosen, the value of y is then fixed.

(ii) The use of the letter f in $f(x)$ is logical because it stands for the word "function"; but once the format of the notation is understood, we can use other letters as appropriate. For example:
the statement $P = P(T)$ could be used to indicate that a pressure, P, is a function of absolute temperature, T;
the statement $i = i(t)$ could be used to indicate that an electric current, i, is a function of time t;
the original statement could have been written $y = y(x)$ without using f at all.

The general format of functional notation may be described as follows:

DEPENDENT VARIABLE =
DEPENDENT VARIABLE(INDEPENDENT VARIABLE)

10.1.2 NUMERICAL EVALUATION OF FUNCTIONS

If α is a number, then $f(\alpha)$ denotes the value of the function $f(x)$ when $x = \alpha$ is substituted into it.

For example, if

$$f(x) \equiv 4\sin 3x,$$

then,

$$f\left(\frac{\pi}{4}\right) = 4\sin\frac{3\pi}{4} = 4 \times \frac{1}{\sqrt{2}} \cong 2.828$$

10.1.3 FUNCTIONS OF A LINEAR FUNCTION

The notation

$$f(ax+b),$$

where a and b are constants, implies a **known** function, $f(x)$, in which x has been replaced by the linear function $ax+b$.

For example, if

$$f(x) \equiv 3x^2 - 7x + 4,$$

then,

$$f(5x-1) \equiv 3(5x-1)^2 - 7(5x-1) + 4;$$

but, in the applications of this kind of notation, it usually best to leave the expression in the bracketed form rather than to expand out the brackets and so lose any obvious connection between $f(x)$ and $f(ax+b)$.

10.1.4 COMPOSITE FUNCTIONS (or Functions of a Function) IN GENERAL

The symbol

$$f[g(x)]$$

implies a **known** function, $f(x)$, in which x has been replaced by **another known** function, $g(x)$.

For example, if

$$f(x) \equiv x^2 + 2x - 5$$

and

$$g(x) \equiv \sin x,$$

then,

$$f[g(x)] \equiv \sin^2 x + 2\sin x - 5;$$

but we can observe also that

$$g[f(x)] \equiv \sin(x^2 + 2x - 5),$$

which is not identical to the first result. Hence, in general,

$$f[g(x)] \not\equiv g[f(x)].$$

There are some exceptions to this, however, as in the case when

$$f(x) \equiv e^x \quad \text{and} \quad g(x) \equiv \log_e x,$$

whereupon we obtain
$$f[g(x)] \equiv e^{\log_e x} \equiv x$$
and
$$g[f(x)] \equiv \log_e (e^x) \equiv x.$$
The functions $\log_e x$ and e^x are said to be **"inverses"** of each other.

10.1.5 INDETERMINATE FORMS

Certain fractional expressions involving functions can become problematic if the values of the variable being substituted into them reduce them to either of the forms
$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

Both of these forms are meaningless or **"indeterminate"** and need to be dealt with using a concept called **"limiting values"**.

(a) The Indeterminate Form $\frac{0}{0}$

Suppose the fractional expression
$$\frac{f(x)}{g(x)}$$
is such that both $f(x)$ **and** $g(x)$ take the value zero when $x = \alpha$; that is, $f(\alpha) = 0$ and $g(\alpha) = 0$. It is impossible, therefore, to evaluate the fraction when $x = \alpha$; but we may consider its values as x becomes increasingly close to α with out actually reaching it. The standard terminology is to say that **"x tends to α"**, written $x \to \alpha$, for short.

We note that, by the **Factor Theorem**, discussed in Unit 1.8, $(x - \alpha)$ must be a factor of both numerator and denominator; and it turns out that, by cancelling this common factor (which is allowed if x is not going to reach α) we can assign a value to $\frac{f(x)}{g(x)}$ called a limiting value. It still will not be the value of this fraction **at** $x = \alpha$, but represents the value it approaches as x **tends** to α. The result is denoted by
$$\lim_{x\to\alpha} \frac{f(x)}{g(x)}.$$

EXAMPLE
Calculate
$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3}.$$

Solution

First we factorise the denominator, knowing already that one of its factors must be $x - 1$ because it takes the value zero when $x = 1$.

The result is therefore

$$\lim_{x\to 1} \frac{x-1}{(x-1)(x+3)}$$

$$= \lim_{x\to 1} \frac{1}{x+3}.$$

What we now need to establish is the fixed value which this new fraction approaches as x becomes increasingly close to 1. But since there are no longer any problems with indeterminate forms, we do in fact simply substitute $x = 1$, obtaining the number $\frac{1}{4}$.

Hence,

$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3} = \frac{1}{4}.$$

(b) The Indeterminate Form $\frac{\infty}{\infty}$

This kind of indeterminate form is usually encountered when the value of x itself becomes infinite, either positively or negatively. The object is to evaluate either

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

or

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)}.$$

EXAMPLE
Calculate

$$\lim_{x\to\infty} \frac{2x^2+3x-1}{7x^2-2x+5}.$$

Solution

There is no factorising to do in this type of exercise; we simply divide the numerator and the denominator by the highest power of x appearing, then allow x to become infinite.

The result is therefore

$$\lim_{x\to\infty} \frac{2+\frac{3}{x}-\frac{1}{x^2}}{7-\frac{2}{x}+\frac{5}{x^2}} = \frac{2}{7}.$$

Note:
In the case of the ratio of two polynomials of equal degree, the limiting value as $x \to \pm\infty$ will always be the ratio of the leading coefficients of x. The same principle can be applied to the ratio of two polynomials of unequal degree if we insert zero coefficients in appropriate places to consider them as being of equal degree. The results then obtained will be either zero or infinity.

ILLUSTRATION

$$\lim_{x\to\infty} \frac{5x+11}{3x^2-4x+1} = \lim_{x\to\infty} \frac{0x^2+5x+11}{3x^2-4x+1} = \frac{0}{3} = 0.$$

A Useful Standard Limit

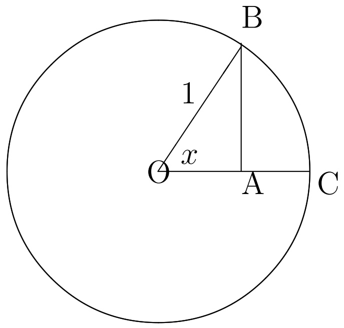
In Unit 3.3, it is shown that, for very small values of x in radians, $\sin x \simeq x$.

This suggests that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

though we may not actually use the result from Unit 3.3 to **prove** the validity of this new limiting value. We shall see later that "$\sin x \simeq x$ for small x" is developed from a calculus technique which **assumes** that $\frac{\sin x}{x} \to 1$ as $x \to 0$; so, we would be using the result to prove itself !

An alternative, non-rigorous, proof is to consider the following diagram in which the angle x is situated at the centre of a circle with radius 1:



In the diagram, the length of line AB $= \sin x$ and the length of arc BC $= x$. Furthermore, as x decreases almost to zero, the two lengths become closer and closer to each other in value. That is,

$$\sin x \to x \quad \text{as} \quad x \to 0$$

or

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

10.1.6 EVEN AND ODD FUNCTIONS

It is easy to see that any **even** power of x will be unchanged in value if x is replaced by $-x$. In a similar way, any **odd** power of x will be unchanged in numerical value, though altered in sign, if x is replaced by $-x$. These two powers of x are examples of an **"even function"** and an **"odd function"** respectively; but the true definition includes a wider range of functions as follows:

DEFINITION
A function $f(x)$ is said to be **"even"** if it satisfies the identity

$$f(-x) \equiv f(x).$$

ILLUSTRATIONS: x^2, $2x^6 - 4x^2 + 5$, $\cos x$.

DEFINITION
A function $f(x)$ is aid to be **"odd"** if it satisfies the identity

$$f(-x) \equiv -f(x).$$

ILLUSTRATIONS x^3, $x^5 - 3x^3 + 2x$, $\sin x$.

Note:
It is not necessary for every function to be either even or odd. For example, the function $x + 3$ is neither even nor odd.

EXAMPLE

Express an arbitrary function, $f(x)$, as the sum of an even function and an odd function.

Solution

We may write

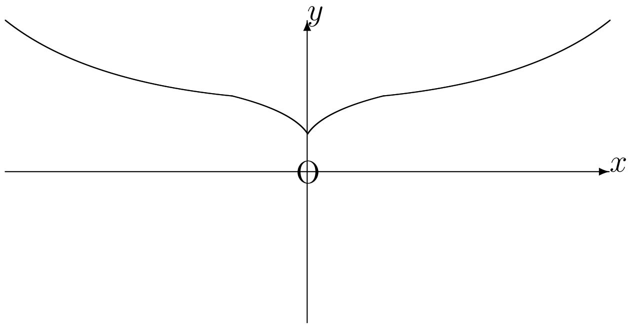
$$f(x) \equiv \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2},$$

in which the first term on the right hand side is unchanged if x is replaced by $-x$ and the second term on the right hand side is reversed in sign if x is replaced by $-x$.

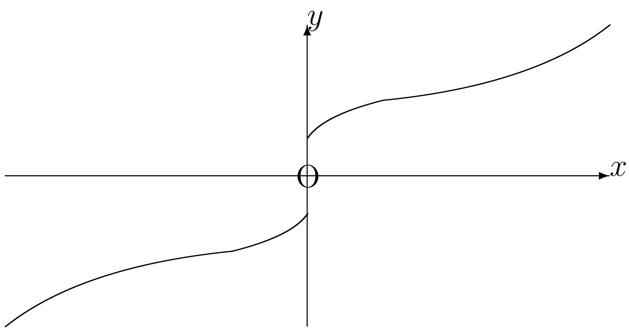
We have thus expressed $f(x)$ as the sum of an even function and an odd function.

(ii) GRAPHS OF EVEN AND ODD FUNCTIONS

(i) The graph of the relationship $y = f(x)$, where $f(x)$ is **even**, will be symmetrical about the y-axis since, for every point (x, y) on the graph, there is also the point $(-x, y)$.



(ii) The graph of the relationship $y = f(x)$, where $f(x)$ is **odd**, will be symmetrical with respect to the origin since, for every point (x, y) on the graph, there is also the point $(-x, -y)$. However, odd functions are more easily recognised by noticing that the part of the graph for $x < 0$ can be obtained from the part for $x > 0$ by reflecting it first in the x-axis and then in the y-axis.

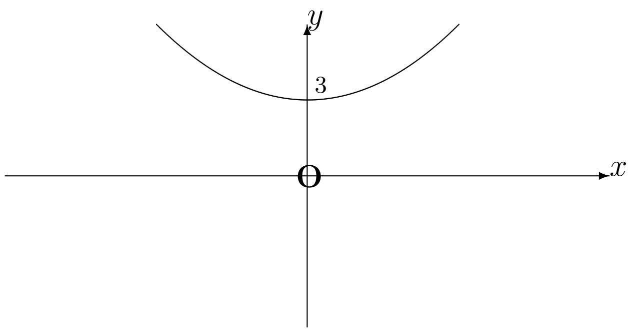


EXAMPLE

Sketch the graph, from $x = -3$ to $x = 3$, of the even function, $f(x)$, defined in the interval $0 < x < 3$ by the formula

$$f(x) \equiv 3 + x^3.$$

Solution



ALGEBRAIC PROPERTIES OF ODD AND EVEN FUNCTIONS

1. The product of an even function and an odd function is an odd function.

Proof:

If $f(x)$ is even and $g(x)$ is odd, then

$$f(-x).g(-x) \equiv f(x).[-g(x)] \equiv -f(x).g(x).$$

2. The product of an even function and an even function is an even function.

Proof:

If $f(x)$ and $g(x)$ are both even functions, then

$$f(-x).g(-x) \equiv f(x).g(x).$$

3. The product of an odd function and an odd function is and even function.

Proof:

If $f(x)$ and $g(x)$ are both odd functions, then

$$f(-x).g(-x) \equiv [-f(x)].[-g(x)] \equiv f(x).g(x).$$

EXAMPLE

Classify the function $f(x) \equiv \sin^4 x.\tan x$ as even, odd or neither even nor odd.

Solution

$$f(-x) \equiv \sin^4(-x).\tan(-x) \equiv \sin^4 x.[-\tan x] \equiv -\sin^4 x.\tan x.$$

The function, $f(x)$, is therefore odd.

10.1.7 EXERCISES

1. If
$$f(x) \equiv 3 + \cos^2\left(\frac{x}{2}\right)$$
determine the values of $f(0), f(\pi), f(\frac{\pi}{2})$.

2. If
$$f(x) \equiv 2x \quad \text{and} \quad g(x) \equiv x^2,$$
verify that
$$f[g(x)] \not\equiv g[f(x)].$$

3. If
$$f(x) \equiv x^2 - 4x + 6,$$
verify that
$$f(2-x) \equiv f(2+x).$$

4. Determine simple functions $f(x)$ and $g(x)$ such that the following functions can be identified with $f[g(x)]$:

   (a)
   $$3(x^2+2)^3;$$

   (b)
   $$(x^2+1)^{-\frac{1}{2}};$$

   (c)
   $$\cos^2 x.$$

5. Evaluate the following limits:

   (a)
$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 8x + 7};$$

   (b)
$$\lim_{x \to 2} \frac{x - 2}{x^2 - (x + 2)};$$

   (c)
$$\lim_{x \to \infty} \frac{3x^2 + 5x - 4}{5x^2 - x + 7};$$

   (d)
$$\lim_{r \to -\infty} \frac{(2r + 1)^2}{(r - 1)(r + 3)};$$

   (e)
$$\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}.$$

6. Express the function e^x as the sum of an even function and an odd function.
7. Sketch the graph, from $x = -5$ to $x = 5$ of the odd function, $f(x)$, defined in the interval $0 < x < 5$ by the formula
$$f(x) \equiv \cos \frac{\pi x}{10}.$$
8. Classify the function
$$f(x) \equiv \tan^2 x + \text{cosec}^3 x. \cos x$$
as even, odd or neither even nor odd.

10.1.8 ANSWERS TO EXERCISES

1. 4, 3, $\frac{7}{2}$.
2. $2x^2 \not\equiv (2x)^2$.

3. Both are identically equal to $x^2 + 2$.

4. (a)

$$f(x) \equiv 3x^3 \quad \text{and} \quad g(x) \equiv x^2 + 2;$$

(b)

$$f(x) \equiv x^{-\frac{1}{2}} \quad \text{and} \quad g(x) \equiv x^2 + 1;$$

(c)

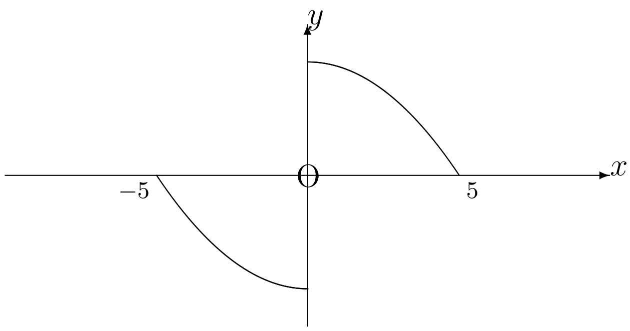
$$f(x) \equiv x^2 \quad \text{and} \quad g(x) \equiv \cos x.$$

5. (a) $-\frac{2}{3}$;
   (b) $\frac{1}{3}$;
   (c) $\frac{3}{5}$;
   (d) 4;
   (e) 3.

6.

$$e^x \equiv \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}.$$

7. The graph is as follows:



8. The function is neither even nor odd.

"JUST THE MATHS"

UNIT NUMBER

10.2

DIFFERENTIATION 2
(Rates of change)

by

A.J.Hobson

UNIT 10.2 - DIFFERENTIATION 2

RATES OF CHANGE

10.2.1 INTRODUCTION

The functional relationship

$$y = f(x)$$

can be represented diagramatically by drawing the graph of y against x to obtain, in general, some kind of curve.

Between one point of the curve and another, the values of both x and y will change, in general; and the purpose of this section is to introduce the concept of **the rate of increase of y with respect to x**.

A convenient practical illustration which will provide an aid to understanding is to think of y as the distance travelled by a moving object at time x; because, in this case, the rate of increase of y with respect to x becomes the familiar quantity which we know as **speed**.

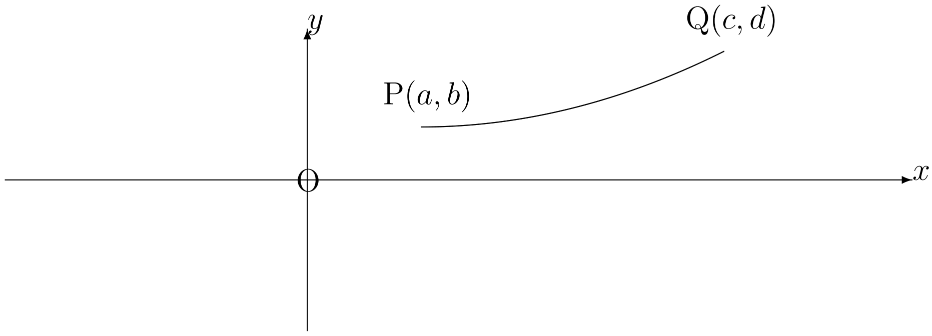
10.2.2 AVERAGE RATES OF CHANGE

Suppose that a vehicle travelled a distance of 280 miles in 7 hours, a journey which is likely to have included short stops, traffic jams, traffic lights and also some fairly high speed motoring. The ratio

$$\frac{280}{7} = 40$$

represents the **"average speed"** of 40 miles per hour over the whole journey. It is a convenient representation of the speed during the journey even though the vehicle might not have been travelling at that speed very often.

Consider now a graph representing the relationship, $y = f(x)$, between two arbitrary variables, x and y, not necessarily time and distance variables.



Between the two points $\mathrm{P}(a, b)$ and $\mathrm{Q}(c, d)$ an increase of $c - a$ in x gives rise to an increase of $d - b$ in y. Therefore, the average rate of increase of y with respect to x from P to Q is

$$\frac{d-b}{c-a}.$$

If it should happen that y **decreases** as x increases (between P and Q), this quantity will automatically turn out negative; hence,

all rates of increase which are POSITIVE correspond to an INCREASING function,

and

all rates of increase which are NEGATIVE correspond to a DECREASING function.

Note:
For the purposes of later work, the two points P and Q will need to be considered as very close together on the graph, and another way of expressing a rate of increase is to consider notations such as $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ for the pair of points.

Here, we are using the symbols δx and δy to represent **"a small fraction of** x**"** and **"a small fraction of** y**"**, respectively. We **do not** mean δ times x and δ times y. We normally consider that δx is positive, but δy may turn out to be negative.

The average rate of increase in this alternative notation is given by

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

In other words,

The average rate of increase is equal to

$$\frac{(\text{new value of } y) \quad \text{minus} \quad (\text{old value of } y)}{(\text{new value of } x) \quad \text{minus} \quad (\text{old value of } x)}$$

EXAMPLE
Determine the average rate of increase of the function

$$y = x^2$$

between the following pairs of points on its graph:

(a) $(3, 9)$ and $(3.3, 10.89)$;

(b) $(3, 9)$ and $(3.2, 10.24)$;

(c) $(3, 9)$ and $(3.1, 9.61)$.

Solution

The results are

(a) $\frac{\delta y}{\delta x} = \frac{1.89}{0.3} = 6.3$;

(b) $\frac{\delta y}{\delta x} = \frac{1.24}{0.2} = 6.2$;

(c) $\frac{\delta y}{\delta x} = \frac{0.61}{0.1} = 6.1$

10.2.3 INSTANTANEOUS RATES OF CHANGE

The results of the example at the end of the previous section seem to suggest that, by letting the second point become increasingly close to the first point along the curve, we could determine the **actual** rate of increase of y with respect to x at the first point only, rather than the **average** rate of increase between the two points.

In the above example, the indications are that the rate of increase of $y = x^2$ with respect to x at the point $(3, 9)$ is equal to 6; and this is called the **"instantaneous rate of increase of y with respect to x"** at the chosen point.

The instantaneous rate of increase in this example has been obtained by guesswork on the strength of just three points approaching $(3, 9)$. In general, we need to consider a limiting process in which an **infinite** number of points approach the chosen one along the curve.

This process is represented by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

and it forms the basis of our main discussion on differential calculus which now follows.

10.2.4 DERIVATIVES

(a) The Definition of a Derivative

In the functional relationship

$$y = f(x)$$

the **"derivative of y with respect to x"** at any point (x, y) on the graph of the function is defined to be the instantaneous rate of increase of y with respect to x at that point.

Assuming that a small increase of δx in x gives rise to a corresponding increase (positive or negative) of δy in y, the derivative will be given by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

This limiting value is usually denoted by one of the three symbols

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \quad f'(x) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}[f(x)].$$

Notes:

(i) In the third of these notations, the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ is called a **"differential operator"**; it cannot exist on its own, but needs to be operating on some function of x. In fact, the first alternative notation is really this differential operator operating on y, which we certainly know to be a function of x.

(ii) The second and third alternative notations are normally used when the derivative of a function of x is being considered without reference to a second variable, y.

(iii) The derivative of a constant function must be zero since the **rate of change** of something which **never changes** is obviously zero.

(iv) Geometrically, the derivative represents the **gradient of the tangent at the point** (x, y) to the curve whose equation is

$$y = f(x).$$

(b) Differentiation from First Principles

Ultimately, the derivatives of **simple** functions may be quoted from a table of standard results; but the establishing of such results requires the use of the definition of a derivative. We illustrate with two examples the process involved:

EXAMPLES

1. Differentiate the function x^4 from first principles.

   Solution

   Here we have a situation where the variable y is not mentioned; so, we could say "let $y = x^4$", and determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ from first principles in order to answer the question.

   However, we shall choose the alternative notation which does not require the use of y at all.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^4\right] = \lim_{\delta x \to 0} \frac{(x+\delta x)^4 - x^4}{\delta x}.$$

   Then, from Pascal's Triangle (Unit 2.2),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^4\right] = \lim_{\delta x \to 0} \frac{x^4 + 4x^3\delta x + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4 - x^4}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[4x^3 + 6x^2\delta x + 4x(\delta x)^2 + (\delta x)^3\right]$$

$$= 4x^3.$$

   Note:
   This result illustrates a general result which will not be proved here that

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$$

   for any constant value n, not necessarily an integer.

2. Differentiate the function $\sin x$ from first principles.

   Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x},$$

which, from Trigonometric Identities (Unit 3.5), becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}.$$

Finally, using the standard limit (Unit 10.1),

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x.$$

Note:
The derivative of $\cos x$ may be obtained in the same way (see EXERCISES 10.2.5, question 2) but it will also be possible to obtain this later (Unit 10.3) by regarding $\cos x$ as $\sin\left(\frac{\pi}{2} - x\right)$.

3. Differentiate from first principles the function

$$\log_b x$$

where b is any base of logarithms.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \lim_{\delta x \to 0} \frac{\log_b(x + \delta x) - \log_b x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log_b\left(1 + \frac{\delta x}{x}\right)}{\delta x}.$$

But writing

$$\frac{\delta x}{x} = r \quad \text{that is} \quad \delta x = rx,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \frac{1}{x} \lim_{r \to 0} \frac{\log_b(1 + r)}{r}$$

$$= \frac{1}{x} \lim_{r \to 0} \log_b (1+r)^{\frac{1}{r}}.$$

For convenience, we may choose a base of logarithms which causes the limiting value above to equal 1; and this will occur when

$$b = \lim_{r \to 0} (1+r)^{\frac{1}{r}}.$$

The appropriate value of b turns out to be approximately 2.71828 and this is the standard base of natural logarithms denoted by e.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x} [\log_e x] = \frac{1}{x}.$$

Note:
In scientific work, the natural logarithm of x is usually denoted by $\ln x$ and this notation will be used in future.

10.2.5 EXERCISES

1. Differentiate from first principles the function $x^3 + 2$.
2. Differentiate from first principles the function $\cos x$.
3. Differentiate from first principles the function $\sqrt{x}$.
   Hint:
   $$(\sqrt{x+\delta x} - \sqrt{x})(\sqrt{x+\delta x} + \sqrt{x}) = \delta x.$$

10.2.6 ANSWERS TO EXERCISES

1. $3x^2$.
2. $-\sin x$.
3. $\frac{1}{2\sqrt{x}}$.

"JUST THE MATHS"

UNIT NUMBER

10.3

DIFFERENTIATION 3
(Elementary techniques of differentiation)

by

A.J.Hobson

UNIT 10.3 - DIFFERENTIATION 3

ELEMENTARY TECHNIQUES OF DIFFERENTIATION

10.3.1 STANDARD DERIVATIVES

In Unit 10.2, reference was made to the use of a table of standard derivatives and such a table can be found in the appendix at the end of this Unit.

However, for the time being, a very short list of standard derivatives is all that is necessary since other derivatives may be developed from them using techniques to be discussed later in this and subsequent Units.

$f(x)$	$f'(x)$
a const.	0
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$

Note:
In the work which now follows, standard derivatives may be used which have not, here, been obtained from first principles; but the student is expected to be able to quote results from a table of derivatives including those for which no proof has been given.

10.3.2 RULES OF DIFFERENTIATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[Af(x) + Bg(x)\right] = A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

Proof:

The left-hand side is equivalent to

$$\lim_{\delta x\to 0}\frac{[Af(x+\delta x) + Bg(x+\delta x)] - [Af(x) + Bg(x)]}{\delta x}$$

$$= A\left[\lim_{\delta x\to 0}\frac{f(x+\delta x) - f(x)}{\delta x}\right] + B\left[\lim_{\delta x\to 0}\frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= A\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + B\frac{\mathrm{d}}{\mathrm{d}x}[g(x)].$$

The result, so far, deals with a **"linear combination"** of **two** functions of x but is easily extended to linear combinations of **three or more** functions of x.

EXAMPLES

1. Write down the expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = 6x^2 + 2x^6 + 13x - 7.$$

Solution

Using the linearity property, the standard derivative of x^n, and the derivative of a constant, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\frac{\mathrm{d}}{\mathrm{d}x}[x^2] + 2\frac{\mathrm{d}}{\mathrm{d}x}[x^6] + 13\frac{\mathrm{d}}{\mathrm{d}x}[x^1] - \frac{\mathrm{d}}{\mathrm{d}x}[7]$$

$$= 12x + 12x^5 + 13.$$

2. Write down the derivative with respect to x of the function

$$\frac{5}{x^2} - 4\sin x + 2\ln x.$$

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{5}{x^2} - 4\sin x + 2\ln x\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[5x^{-2} - 4\sin x + 2\ln x\right]$$

$$= -10x^{-3} - 4\cos x + \frac{2}{x}$$

$$= \frac{-10}{x^3} - 4\cos x + \frac{2}{x}.$$

(b) Composite Functions (or Functions of a Function)

(i) Functions of a Linear Function

Expressions such as $(5x+2)^{16}$, $\sin(2x+3)$ and $\ln(7-4x)$ may be called **"functions of a linear function"** and have the general form

$$f(ax+b),$$

where a and b are constants. The function $f(x)$ would, of course, be the one obtained on replacing $ax+b$ by a single x; hence, in the above illustrations, $f(x)$ would be x^{16}, $\sin x$ and $\ln x$, respectively.

Functions of a linear function may be differentiated as easily as $f(x)$ itself on the strength of the following argument:

Suppose we write

$$y = f(u) \quad \text{where} \quad u = ax+b.$$

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δy in y and δu in u. Then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u}\frac{\delta u}{\delta x}.$$

Assuming that δy and δu tend to zero as δx tends to zero, we can say that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}.$$

This rule is called the **"Function of a Function Rule"** or **"Composite Function Rule"** or **"Chain Rule"** and has applications to a much wider class of composite functions than has so far been discussed. But, for the moment we restrict the discussion to functions of a linear function.

EXAMPLES

1. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = (5x + 2)^{16}$.

   Solution

   First, we write $y = u^{16}$ where $u = 5x + 2$.

   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = 16u^{15}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 5$.

   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = 16u^{15}.5 = 80(5x + 2)^{15}$.

2. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sin(2x + 3)$.

   Solution

   First, we write $y = \sin u$ where $u = 2x + 3$.

   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$.

   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos u.2 = 2\cos(2x + 3)$.

3. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \ln(7 - 4x)$.

   Solution

   First, we write $y = \ln u$ where $u = 7 - 4x$.

   Then, $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = -4$.

   Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{u}.(-4) = \frac{-4}{7-4x}$.

Note:
It is hoped that the student will quickly appreciate how the fastest way to obtain the derivative of a function of a linear function is to treat the expression $ax + b$ initially as if it were a single x; then, multiply the final result by the constant value, a.

(ii) Functions of a Function in general

The formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}$$

is in no way dependent on the fact that the examples so far used to illustrate it have involved functions of a linear function. Exactly the same formula may be used for the composite function

$$f[g(x)],$$

whatever the functions $f(x)$ and $g(x)$ happen to be. All we need to do is to write

$$y = f(u) \quad \text{where} \quad u = g(x),$$

then apply the formula.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (x^2 + 7x - 3)^4.$$

Solution

Letting $y = u^4$ where $u = x^2 + 7x - 3$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3.(2x + 7)$$
$$= 4(x^2 + 7x - 3)^3(2x + 7).$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \ln(x^2 - 3x + 1).$$

Solution

Letting $y = \ln u$ where $u = x^2 - 3x + 1$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}.(2x - 3) = \frac{2x - 3}{x^2 - 3x + 1}.$$

3. Determine the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 1$ in the case when

$$y = 2\sin(5x^2 - 1) + 19x.$$

Solution

Consider, first, the function $2\sin(5x^2 - 1)$ which we shall call z.
Its derivative is $\frac{\mathrm{d}z}{\mathrm{d}x}$, where $z = 2\sin(5x^2 - 1)$.

Let $z = 2\sin u$ where $u = 5x^2 - 1$; then,

$$\frac{dz}{dx} = \frac{dz}{du}.\frac{du}{dx} = 2\cos u.10x = 20x\cos(5x^2 - 1).$$

Hence, the complete derivative is given by

$$\frac{dy}{dx} = 20x\cos(5x^2 - 1) + 19.$$

Finally, when $x = 1$, this derivative has the value $20\cos 4 + 19$, which is approximately equal to 5.927, remembering that the calculator must be in **radian mode**.

Note:
Again, it is hoped that the student will appreciate how there is a faster way of differentiating composite functions in general. We simply treat $g(x)$ initially as if it were a single x, then multiply by $g'(x)$ afterwards.

For example,

$$\frac{d}{dx}\left[\sin^3 x\right] = \frac{d}{dx}\left[(\sin x)^3\right] = 3(\sin x)^2.\cos x = 3\sin^2 x.\cos x.$$

10.3.3 EXERCISES

1. Determine an expression for $\frac{dy}{dx}$ in the following cases:

   (a)
   $$y = 3x^3 - 8x^2 + 11x + 9;$$

   (b)
   $$y = 10\cos x + 5\sin x - 14x^7;$$

   (c)
   $$y = (2x - 7)^5;$$

   (d)
   $$y = (2 - 5x)^{-\frac{5}{2}};$$

(e)

$$y = \sin\left(\frac{\pi}{2} - x\right); \text{ that is, } \cos x;$$

(f)

$$y = \cos(4x + 1);$$

(g)

$$y = \ln(4 - 2x);$$

(h)

$$y = \ln\left[\frac{3x - 8}{6x + 2}\right].$$

2. Determine an expression for $\frac{dy}{dx}$ in the cases when

(a)

$$y = (4 - 7x^3)^8;$$

(b)

$$y = (x^2 + 1)^{\frac{3}{2}};$$

(c)

$$y = \cos^5 x;$$

(d)

$$y = \ln(\ln x).$$

3. If $y = \sin(\cos x)$, evaluate $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$.

4. If $y = \cos(7x^5 - 3)$, evaluate $\frac{dy}{dx}$ at $x = 1$.

10.3.4 ANSWERS TO EXERCISES

1. (a)

$$9x^2 - 16x + 11;$$

(b)

$$-10\sin x + 5\cos x - 98x^6;$$

(c)

$$10(2x-7)^4;$$

(d)

$$\frac{25}{2}(2-5x)^{-\frac{7}{2}};$$

(e)

$$-\cos\left(\frac{\pi}{2}-x\right);\text{ that is, }-\sin x;$$

(f)

$$-4\sin(4x+1);$$

(g)

$$\frac{-2}{4-2x}\text{ or }\frac{2}{2x-4};$$

(h)

$$\frac{3}{3x-8}-\frac{6}{6x+2}=\frac{54}{(3x-8)(6x+2)}.$$

2. (a)

$$-168x^2(4-7x^3)^7;$$

(b)

$$3x(x^2+1)^{\frac{1}{2}};$$

(c)

$$-5\cos^4 x.\sin x;$$

(d)

$$\frac{1}{x\ln x}.$$

3.

$$-1$$

4.

$$-35\sin 4\cong 26.488$$

APPENDIX - A Table of Standard Derivatives

$f(x)$	$f'(x)$
a (const.)	0
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\text{sec}^2 ax$
$\cot ax$	$-a\text{cosec}^2\ ax$
$\text{sec}\ ax$	$a\text{sec}\ ax.\tan ax$
$\text{cosec}\ ax$	$-a\text{cosec}\ ax.\cot ax$
$\ln x$	$1/x$
e^{ax}	ae^{ax}
a^x	$a^x.\ln a$
$\sinh ax$	$a\cosh ax$
$\cosh ax$	$a\sinh ax$
$\tanh ax$	$a\text{sech}^2\ ax$
$\text{sech}\ ax$	$-a\text{sech}\ ax.\tanh ax$
$\text{cosech}\ ax$	$-a\text{cosech}\ ax.\coth x$
$\ln(\sin x)$	$\cot x$
$\ln(\cos x)$	$-\tan x$
$\ln(\sinh x)$	$\coth x$
$\ln(\cosh x)$	$\tanh x$
$\sin^{-1}(x/a)$	$1/\sqrt{(a^2-x^2)}$
$\cos^{-1}(x/a)$	$-1/\sqrt{(a^2-x^2)}$
$\tan^{-1}(x/a)$	$a/(a^2+x^2)$
$\sinh^{-1}(x/a)$	$1/\sqrt{(x^2+a^2)}$
$\cosh^{-1}(x/a)$	$1/\sqrt{(x^2-a^2)}$
$\tanh^{-1}(x/a)$	$a/(a^2-x^2)$

"JUST THE MATHS"

UNIT NUMBER

10.4

DIFFERENTIATION 4
(Products and quotients)
&
(Logarithmic differentiation)

by

A.J.Hobson

UNIT 10.4 - DIFFERENTIATION 4

PRODUCTS, QUOTIENTS AND LOGARITHMIC DIFFERENTIATION

10.4.1 PRODUCTS

Suppose
$$y = u(x)v(x),$$
where $u(x)$ and $v(x)$ are two functions of x.

Suppose, also, that a small increase of δx in x gives rise to increases (positive or negative) of δu in u, δv in v and δy in y.

Then,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{(u + \delta u)(v + \delta v) - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{uv + u\delta v + v\delta u + \delta u\delta v - uv}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} \right].$$

Hence,
$$\frac{\mathrm{d}}{\mathrm{d}x}[u.v] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Hint: Think of this as
(FIRST x DERIVATIVE OF SECOND) + (SECOND x DERIVATIVE OF FIRST)

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when
$$y = x^7 \cos 3x.$$

   Solution
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^7. - 3\sin 3x + \cos 3x.7x^6 = x^6[7\cos 3x - 3x\sin 3x].$$

2. Evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = -1$ in the case when

$$y = (x^2 - 8)\ln(2x + 3).$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 - 8).\frac{1}{2x+3}.2 + \ln(2x+3).2x = 2\left[\frac{x^2-8}{2x+3} + x\ln(2x+3)\right].$$

When $x = -1$, this has value -14 since $\ln 1 = 0$.

10.4.2 QUOTIENTS

Suppose, this time, that

$$y = \frac{u(x)}{v(x)}.$$

Then, we may write

$$y = u(x).[v(x)]^{-1}$$

in order to use the rule already known for products.

We obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u.(-1)[v]^{-2}.\frac{\mathrm{d}v}{\mathrm{d}x} + v^{-1}.\frac{\mathrm{d}u}{\mathrm{d}x},$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

EXAMPLES

1. Using the formula for the derivative of a quotient, show that the derivative with respect to x of the function $\tan x$ is the function $\sec^2 x$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x).(\cos x) - (\sin x).(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{2x+1}{(5x-3)^3}.$$

Solution

Using $u(x) \equiv 2x+1$ and $v(x) \equiv (5x-3)^3$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3)^3.2 - (2x+1).3(5x-3)^2.5}{(5x-3)^6}.$$

The expression $(5x-3)^2$ may be cancelled as a common factor of both numerator and denominator, leaving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 15(2x+1)}{(5x-3)^4} = -\frac{20x+21}{(5x-3)^4}.$$

Note:

The step in the second example above, where a common factor could be cancelled, may be avoided if we use a modified version of the rule for quotients when the function can be considered in the form

$$\frac{u}{v^n}.$$

It can be shown that, if

$$y = \frac{u}{v^n},$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - nu\frac{\mathrm{d}v}{\mathrm{d}x}}{v^{n+1}}.$$

For instance, in Example 2 above, we could write

$$u \equiv 2x+1 \quad v \equiv 5x-3 \quad \text{and} \quad n = 3$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(5x-3).2 - 3(2x+1).5}{(5x-3)^4},$$

as before.

10.4.3 LOGARITHMIC DIFFERENTIATION

The algebraic properties of natural logarithms (see Unit 1.4), together with the standard derivative of $\ln x$ and the rules of differentiation, enable us to differentiate two specific kinds of function as described below:

(a) Functions containing a variable index

The most familiar function with which to introduce this technique is the **"exponential function"**, e^x.

Suppose we let

$$y = e^x;$$

then, by properties of natural logarithms, we can write

$$\ln y = x;$$

and, if we differentiate both sides **with respect to** x, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y = e^x.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x\right] = e^x.$$

Notes:

(i) After taking logarithms, we could have differentiated the statement $x = \ln y$ with respect to y, obtaining

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{y}.$$

But it can be shown that, for most functions,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

so that the same result is obtained as before.

(ii) The derivative of e^x may easily be used to establish the standard derivatives of the hyperbolic functions, $\sinh x$, $\cosh x$ and $\tanh x$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = \cosh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cosh x] = \sinh x, \quad \frac{\mathrm{d}}{\mathrm{d}x}[\tanh x] = \mathrm{sech}^2\, x.$$

The first two of these follow from the definitions

$$\sinh x \equiv \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x \equiv \frac{e^x + e^{-x}}{2},$$

while the third may be obtained using the definition

$$\tanh x \equiv \frac{\sinh x}{\cosh x},$$

together with the Quotient Rule.

FURTHER EXAMPLES

1. Write down the derivative with respect to x of the function

$$e^{\sin x}$$

Solution

All that is required in this example is the standard derivative of e^x together with the Function of a Function Rule. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin x}\right] = e^{\sin x}.\cos x.$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = (3x+2)^x.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x \ln(3x+2).$$

Differentiating both sides with respect to x and using the Product Rule gives

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x.\frac{3}{3x+2} + \ln(3x+2).1.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x+2)^x\left[\frac{3x}{3x+2} + \ln(3x+2)\right].$$

(b) Products or Quotients with more than two elements

We have already discussed the rules for differentiating products and quotients; but, in certain cases, it is easier to make use of logarithmic differentiation. Essentially, we use this alternative method when a product or a quotient involves more than the two functions $u(x)$ and $v(x)$ mentioned earlier.

We illustrate with examples:

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = e^{x^2}.\cos x.(x+1)^5.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x^2 + \ln(\cos x) + 5\ln(x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{\sin x}{\cos x} + \frac{5}{x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2}.\cos x.(x+1)^5\left[2x - \tan x + \frac{5}{x+1}\right].$$

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$y = \frac{e^x.\sin x}{(7x+1)^4}.$$

Solution

Taking natural logarithms of both sides,

$$\ln y = x + \ln(\sin x) - 4\ln(7x+1).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{\cos x}{\sin x} - 4.\frac{7}{7x+1}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x.\sin x}{(7x+1)^4}\left[1 + \cot x - \frac{28}{7x+1}\right].$$

Note:
In all examples on logarithmic differentiation, the original function will appear as a factor at the beginning of its derivative.

10.4.4 EXERCISES

1. Differentiate the following functions with respect to x:

   (a)
   $$\sin x.\cos x;$$

   (b)
   $$(x^2+3).\sin 2x;$$

   (c)
   $$x.(x^2+1)^{\frac{1}{2}};$$

   (d)
   $$x^2\ln(1-2x).$$

2. Differentiate the following functions with respect to x:

   (a)
   $$\frac{\cos x}{\sin x} \quad \text{(that is,} \quad \cot x);$$

(b)

$$\frac{x^2-2}{(x+1)^2};$$

(c)

$$\frac{\cos x+\sin x}{\cos x-\sin x};$$

(d)

$$\frac{x}{(2x-x^2)^{\frac{1}{2}}}.$$

3. Differentiate the following functions with respect to x:

(a)

$$e^{x^2+1};$$

(b)

$$e^{1-x-x^2};$$

(c)

$$(2x+1)e^{4-x^3};$$

(d)

$$\frac{e^{1-7x}}{3x+2};$$

(e)

$$x.\sinh(x^2+1);$$

(f)

$$\operatorname{sech} x.$$

4. Use logarithms to differentiate the following functions with respect to x:

(a)

$$a^x \quad (a \text{ constant});$$

(b)

$$(x^2+1)^{3x};$$

(c)

$$(\sin x)^x;$$

(d)

$$\frac{x(x-2)}{(x+1)(x+3)};$$

(e)

$$\frac{e^{2x}.\ln x}{(x-1)^3}.$$

10.4.5 ANSWERS TO EXERCISES

1. (a)

$$\cos^2 x - \sin^2 x \quad (\text{or} \quad \cos 2x);$$

(b)

$$2(x^2+3)\cos 2x + 2x\sin 2x;$$

(c)

$$\frac{2x^2+1}{(x^2+1)^{\frac{1}{2}}};$$

(d)

$$2x\ln(1-2x) - \frac{2x^2}{1-2x}.$$

2. (a)

$$-\text{cosec}^2 x;$$

(b)

$$\frac{4+2x}{(x+1)^3};$$

(c)

$$\frac{2}{(\cos x - \sin x)^2};$$

(d)

$$\frac{x}{(2x - x^2)^{\frac{3}{2}}}.$$

3. (a)

$$2xe^{x^2+1};$$

(b)

$$-(1+2x)e^{1-x-x^2};$$

(c)

$$2.e^{4-x^3} - 3x^2(2x+1)e^{4-x^3};$$

(d)

$$-\frac{e^{1-7x}.(21x+17)}{(3x+2)^2};$$

(e)

$$\sinh(x^2+1) + 2x^2\cosh(x^2+1);$$

(f)

$$-\mathrm{cosech}^2\, x.$$

4. (a)

$$a^x.\ln a;$$

(b)

$$(x^2+1)^{3x}\left[3\ln(x^2+1) + \frac{6x^2}{x^2+1}\right];$$

(c)

$$(\sin x)^x\left[\ln\sin x + x\cot x\right];$$

(d)

$$\frac{x(x-2)}{(x+1)(x+3)}\left[\frac{1}{x} + \frac{1}{x-2} - \frac{1}{x+1} - \frac{1}{x+3}\right];$$

(e)

$$\frac{e^{2x}.\ln x}{(x-1)^3}\left[2 + \frac{1}{x\ln x} - \frac{3}{x-1}\right].$$

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UNIT NUMBER

10.5

DIFFERENTIATION 5
(Implicit and parametric functions)

by

A.J.Hobson

UNIT 10.5 - DIFFERENTIATION 5
IMPLICIT AND PARAMETRIC FUNCTIONS

10.5.1 IMPLICIT FUNCTIONS

Some relationships between two variables x and y do not give y explicitly in terms of x (or x explicitly in terms of y); but, nevertheless, it is **implied** that one of the two variables is a function of the other. In the work which follows, we shall normally assume that y is a function of x.

Consider, for instance, the relationship

$$x^2 + y^2 = 16,$$

which is not explicit for either x or y but could, if desired, be written in one of the two forms

$$y = \pm\sqrt{16 - x^2} \quad \text{or} \quad x = \pm\sqrt{16 - y^2}.$$

By contrast, consider the relationship

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

In this case, there is no apparent way of stating either variable explicitly in terms of the other; yet we may still wish to calculate $\frac{dy}{dx}$ or even $\frac{dx}{dy}$.

Such relationships between x and y are said to be **"implicit relationships"** and, in the technique of **"implicit differentiation"**, we simply differentiate each term in the relationship with respect to the same variable without attempting to rearrange the formula.

EXAMPLES

1. Determine an expression for $\frac{dy}{dx}$ in the case when

$$x^2 + y^2 = 16.$$

Solution

Treating y^2 as a function of a function, we have

$$2x + 2y\frac{dy}{dx} = 0.$$

Hence,

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

It is perfectly acceptable that the result is expressed in terms of both x and y; this will normally happen.

2. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2 + 2xy^3 + y^5 = 4.$$

Solution

Treating y^3 and y^5 as functions of a function and using the Product Rule in the second term on the left hand side,

$$2x + 2\left[x.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.1\right] + 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

On rearrangement,

$$\left[6xy^2 + 5y^4\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x + 2y^3).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + 2y^3}{6xy^2 + 5y^4}.$$

3. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when

$$x^2y^3 + 9\sin(5x - 7y) = 10.$$

Solution

Differentiating throughout with respect to x and using both the Product Rule and the Function of a Function Rule, we obtain

$$x^2.3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^3.2x + 9\cos(5x - 7y).\left[5 - 7\frac{\mathrm{d}y}{\mathrm{d}x}\right] = 0.$$

On rearrangment,

$$\left[3x^2y^2 - 63\cos(5x - 7y)\right]\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[2xy^3 + 45\cos(5x - 7y)\right].$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^3 + 45\cos(5x - 7y)}{3x^2y^2 - 63\cos(5x - 7y)}.$$

10.5.2 PARAMETRIC FUNCTIONS

In the geometry of straight lines, circles etc, we encounter **"parametric equations"** in which the variables x and y, related to each other by a formula, may each be expressed individually in terms of a third variable, usually t or θ, called a **"parameter"**.

In general, we write

$$x = x(t) \quad \text{and} \quad y = y(t);$$

but, in theory, we can imagine that t could be expressed explicitly in terms of x; so, essentially, y is a function of t, where t is a function of x. Hence, from the Function of a Function Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x}.$$

However, we are **not** given t explicitly in terms of x and it may not be practical to obtain it in this form. Therefore, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t}.$$

This is the standard formula for differentiating y with respect to x from a pair of parametric equations.

EXAMPLES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t in the case when

$$x = 3t^2 \quad \text{and} \quad y = 6t.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6 \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 6t.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of θ in the case when

$$x = \sin^3\theta \quad \text{and} \quad y = \cos^3\theta.$$

Solution

$$\frac{dx}{d\theta} = 3\sin^2\theta.\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = -3\cos^2\theta.\sin\theta.$$

Hence,

$$\frac{dy}{dx} = \frac{-3\cos^2\theta.\sin\theta}{3\sin^2\theta.\cos\theta}.$$

That is,

$$\frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$

10.5.3 EXERCISES

1. Determine an expression for $\frac{dy}{dx}$ in the following cases:

   (a)

   $$x^2 + y^2 = 10x;$$

   (b)

   $$x^3 + y^3 - 3xy^2 = 8;$$

   (c)

   $$x^4 + 2x^2y^2 + y^4 = x;$$

   (d)

   $$xe^y = \cos y.$$

2. Determine an expression for $\frac{dy}{dx}$ in terms of the appropriate parameter in the following cases:

   (a)

   $$x = 3\sin\theta \quad \text{and} \quad y = 4\cos\theta;$$

   (b)

   $$x = 4t \quad \text{and} \quad y = \frac{4}{t};$$

(c)

$$x = (1-t)^{\frac{1}{2}} \quad \text{and} \quad y = (1-t^2)^{\frac{1}{2}}.$$

10.5.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5-x}{y};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-x^2}{y^2-2xy};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-4x^3-4xy^2}{4\left(x^2y+y^3\right)};$$

(d)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^y}{xe^y+\sin y}.$$

2. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{3}\tan\theta;$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{t^2};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t}{(1+t)^{\frac{1}{2}}}.$$

"JUST THE MATHS"

UNIT NUMBER

10.6

DIFFERENTIATION 6
(Inverse trigonometric functions)

by

A.J.Hobson

UNIT 10.6 - DIFFERENTIATION 6

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

10.6.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric functions should be considered as standard results. They will be stated here first, before their proofs are discussed.

1. $$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}},$$

   where $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$.

2. $$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}},$$

   where $0 \leq \cos^{-1}x \leq \pi$.

3. $$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2},$$

   where $-\frac{\pi}{2} \leq \tan^{-1}x \leq \frac{\pi}{2}$.

10.6.2 THE DERIVATIVE OF AN INVERSE SINE

We shall consider the formula

$$y = \mathrm{Sin}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case S in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \sin y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \equiv \pm\sqrt{1-\sin^2 y} \equiv \pm\sqrt{1-x^2}.$$

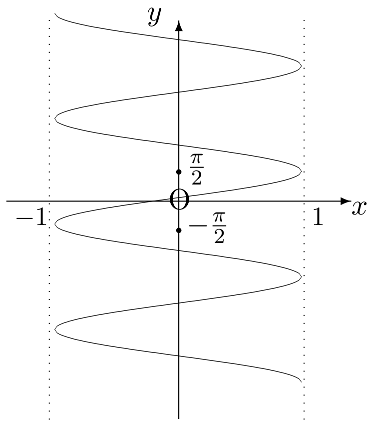
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sin}^{-1}x,$$

which may be obtained from the graph of $y = \sin x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i)The variable x must lie in the interval $-1 \leq x \leq 1$;

(ii) For each value of x in the interval $-1 \leq x \leq 1$, the variable y has infinitely many values which are spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each value of x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is positive and the other negative.

(iv) By restricting the discussion to the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y and one (positive) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $-1 \leq x \leq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse sine function and is denoted by $\sin^{-1}x$ using a lower-case s.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}.$$

10.6.3 THE DERIVATIVE OF AN INVERSE COSINE

We shall consider the formula

$$y = \mathrm{Cos}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \cos y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y \equiv \pm\sqrt{1-\cos^2 y} \equiv \pm\sqrt{1-x^2}.$$

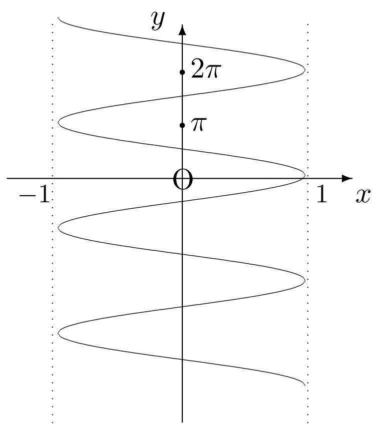
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cos}^{-1}x$$

which may be obtained from the graph of $y = \cos x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $-1 \leq x \leq 1$;

(ii) For each value of x in the interval $-1 \leq x \leq 1$, the variable y has infinitely many values which are spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each value of x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is positive and the other negative.

(iv) By restricting the discussion to the part of the graph from $y = 0$ to $y = \pi$, we may distinguish the results from those of the inverse sine function; and there will be only one value of y with one (negative) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $-1 \leq x \leq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse cosine function and is denoted by $\cos^{-1}x$ using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}}.$$

10.6.4 THE DERIVATIVE OF AN INVERSE TANGENT

We shall consider the formula

$$y = \mathrm{Tan}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case T in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \tan y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y \equiv 1 + \tan^2 y \equiv 1 + x^2.$$

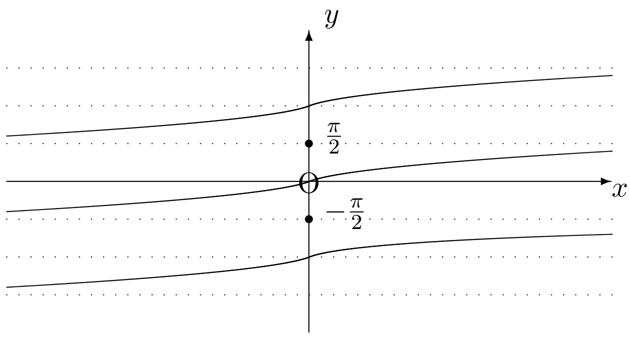
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tan}^{-1}x,$$

which may be obtained from the graph of $y = \tan x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x may lie anywhere in the interval $-\infty < x < \infty$;

(ii) For each value of x, the variable y has infinitely many values which are spaced at regular intervals of π.

(iii) For each value of x, there is only possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.

(iv) By restricting the discussion to the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y for each value of x.

The restricted part of the graph defines what is called the **"principal value"** of the inverse tangent function and is denoted by $\tan^{-1}x$ using a lower-case t.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2}.$$

ILLUSTRATIONS

1.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}2x] = \frac{2}{\sqrt{1-4x^2}}.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}(x+3)] = -\frac{1}{\sqrt{1-(x+3)^2}}.$$

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}(\sin x)] = \frac{\cos x}{1+\sin^2 x}.$$

4.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}\left(x^5\right)] = \frac{5x^4}{\sqrt{1-x^{10}}} \quad \text{(real only if } \; -1 < x < 1).$$

10.6.5 EXERCISES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases, assuming any necessary restrictions on the values of x:

   (a)

   $$y = \cos^{-1} 7x;$$

   (b)

   $$y = \tan^{-1}(\cos x);$$

   (c)

   $$y = \sin^{-1}(3-2x).$$

2. Show that

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)\right] = 1.$$

3. If

   $$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}},$$

   show that

   (a)

   $$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 1;$$

   (b)

   $$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

10.6.6 ANSWERS TO EXERCISES

1. (a)

$$-\frac{7}{\sqrt{1-49x^2}} \quad \left(\text{real only if} \quad -\frac{1}{7} < x < \frac{1}{7}\right);$$

(b)

$$-\frac{\sin x}{1+\cos^2 x};$$

(c)

$$-\frac{2}{\sqrt{1-(3-2x)^2}} \quad (\text{real only if} \quad 1 < x < 2).$$

"JUST THE MATHS"

UNIT NUMBER

10.7

DIFFERENTIATION 7
(Inverse hyperbolic functions)

by

A.J.Hobson

UNIT 10.7 - DIFFERENTIATION

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

10.7.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric and inverse hyperbolic functions should be considered as standard results. They will be stated here, first, before their proofs are discussed.

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}},$$
where $-\infty < \sinh^{-1}x < \infty$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2-1}},$$
where $\cosh^{-1}x \geq 0$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tanh^{-1}x] = \frac{1}{1-x^2},$$
where $-\infty < \tanh^{-1}x < \infty$.

10.7.2 THE DERIVATIVE OF AN INVERSE HYPERBOLIC SINE

We shall consider the formula
$$y = \mathrm{Sinh}^{-1}x$$
and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
The use of the upper-case S in the formula is temporary; and the reason will be explained shortly.

The formula is equivalent to
$$x = \sinh y,$$
so we may say that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cosh y \equiv \sqrt{1+\sin^2 y} \equiv \sqrt{1+x^2},$$

noting that $\cosh y$ is never negative.

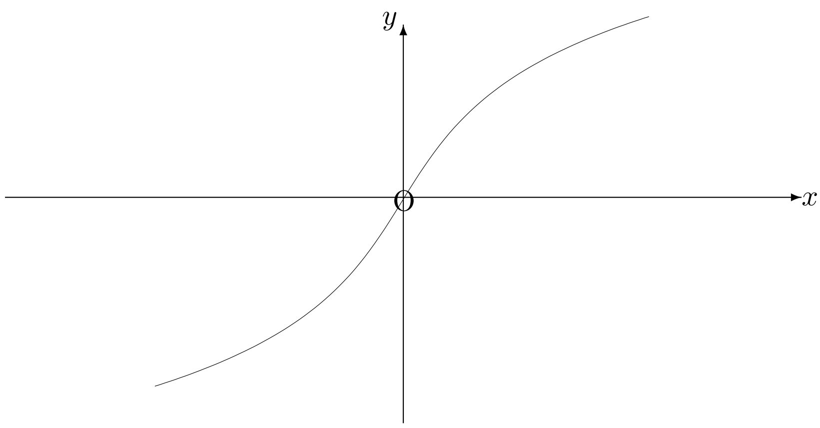
Thus,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

Consider now the graph of the formula

$$y = \text{Sinh}^{-1}x$$

which may be obtained from the graph of $y = \sinh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x may lie anywhere in the interval $-\infty < x < \infty$.

(ii) For each value of x, the variable y has only one value.

(iii) For each value of x, there is only one possible value of $\frac{dy}{dx}$, which is positive.

(iv) In this case (unlike the case of an inverse sine, in Unit 10.6) there is no need to distinguish between a general value and a principal value of the inverse hyperbolic sine function. This is because there is only one value of both the function and its derivative.

However, it is customary to denote the inverse function by $\sinh^{-1}x$, using a lower-case s rather than an upper-case S.

Hence,

$$\frac{d}{dx}[\sinh^{-1}x] = \frac{1}{\sqrt{1+x^2}}.$$

10.7.3 THE DERIVATIVE OF AN INVERSE HYPERBOLIC COSINE

We shall consider the formula

$$y = \mathrm{Cosh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula; the reason will be explained shortly.

The formula is equivalent to

$$x = \cosh y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y \equiv \pm\sqrt{\cosh^2 y - 1} \equiv \pm\sqrt{x^2 - 1}.$$

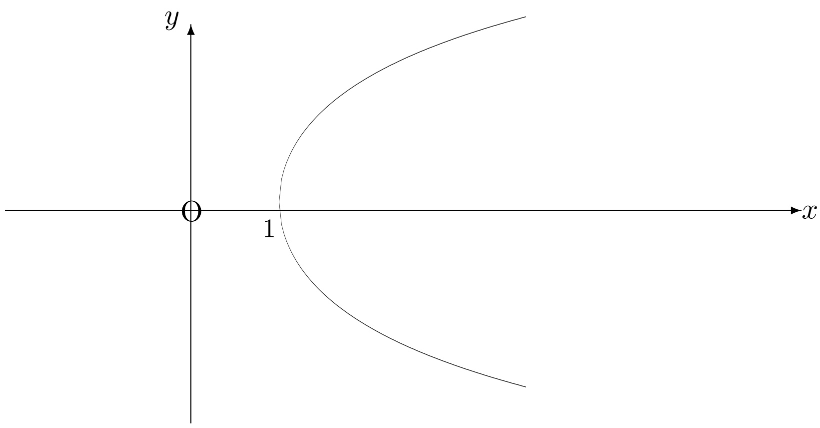
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}.$$

Consider now the graph of the formula

$$y = \mathrm{Cosh}^{-1}x,$$

which may be obtained from the graph of $y = \cosh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $x \geq 1$.

(ii) For each value of x in the interval $x > 1$, the variable y has two values one of which is positive and the other negative.

(iii) For each value of x in the interval $x > 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is postive and the other negative.

(iv) By restricting the discussion to the part of the graph for which $y \geq 0$, there will be only one value of y with one (positive) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $x \geq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse cosine function and is denoted by $\cosh^{-1}x$, using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}x] = \frac{1}{\sqrt{x^2 - 1}}.$$

10.7.4 THE DERIVATIVE OF AN INVERSE HYPERBOLIC TANGENT

We shall consider the formula

$$y = \mathrm{Tanh}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
The use of the upper-case T in the formula is temporary; and the reason will be explained later.

The formula is equivalent to

$$x = \tanh y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{sech}^2 y \equiv 1 - \tanh^2 y \equiv 1 - x^2.$$

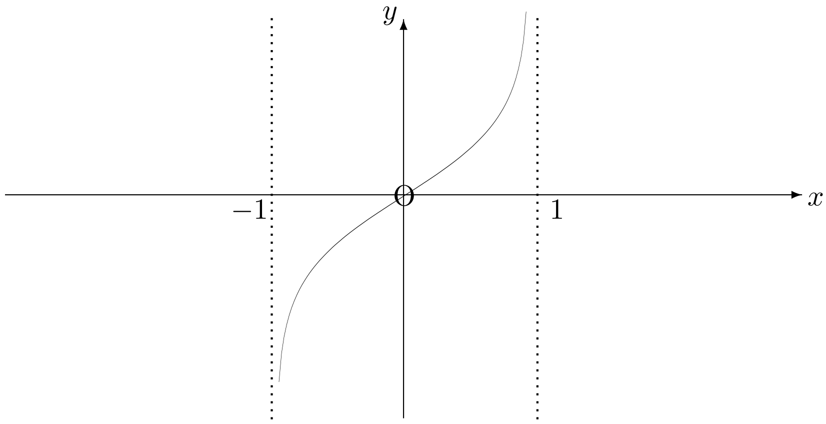
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1 - x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tanh}^{-1}x,$$

which may be obtained from the graph of $y = \tanh x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $-1 < x < 1$.

(ii) For each value of x in the interval $-1 < x < 1$, the variable y has just one value.

(iii) For each value of x in the interval $-1 < x < 1$, there is only possible value of $\frac{dy}{dx}$, which is positive.

(iv) In this case (unlike the case of an inverse tangent in Unit 10.6) there is no need to distinguish between a general value and a principal value of the inverse hyperbolic tangent function. This is because there is only one value of both the function and its derivative.

However, it is customary to denote the inverse hyperbolic tangent by $\tan^{-1}x$ using a lower-case t rather than an upper-case T.

Hence,

$$\frac{d}{dx}[\tanh^{-1}x] = \frac{1}{1-x^2}.$$

ILLUSTRATIONS

1.

$$\frac{d}{dx}[\sin^{-1}(\tanh x)] = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \text{sech} x.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cosh^{-1}(5x-4)] = \frac{5}{\sqrt{(5x-4)^2-1}},$$

assuming that $5x - 4 \geq 1$; that is, $x \geq 1$.

10.7.5 EXERCISES

Obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

1.

$$y = \cosh^{-1}(4+3x),$$

assuming that $4 + 3x \geq 1$; that is, $x \geq -1$.

2.

$$y = \cos^{-1}(\sinh x),$$

assuming that $-1 \leq \sinh x \leq 1$.

3.

$$y = \tanh^{-1}(\cos x).$$

4.

$$y = \cosh^{-1}(x^3),$$

assuming that $x \geq 1$.

5.

$$y = \tanh^{-1}\frac{2x}{1+x^2}.$$

10.7.6 ANSWERS TO EXERCISES

1.

$$\frac{3}{\sqrt{(4+3x)^2-1}}.$$

2.

$$-\frac{\cosh x}{\sqrt{1-\sinh^2 x}}.$$

3.

$$-\text{cosec}x.$$

4.

$$\frac{3x^2}{\sqrt{x^6-1}}.$$

5.

$$\frac{2}{1-x^2}.$$

"JUST THE MATHS"

UNIT NUMBER

10.8

DIFFERENTIATION 8
(Higher derivatives)

by

A.J.Hobson

UNIT 10.8 - DIFFERENTIATION 8

HIGHER DERIVATIVES

10.8.1 THE THEORY

In most of the examples (seen in earlier Units) on differentiating a function of x with respect to x, the result obtained has been **another** function of x. In general, this **will** be the case and the possibilty arises of differentiating again with respect to x.

This would occur, for example, in the case when the formula

$$y = f(x)$$

represents the distance, y, travelled by a moving object at time, x.

The **speed** of the moving object is the rate of increase of distance with respect to time; that is, $\frac{\mathrm{d}y}{\mathrm{d}x}$. But a second quantity called **acceleration** is defined as the rate of increase of speed with respect to time. It is therefore represented by the symbol

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right];$$

but this is usually written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

and is pronounced "d two y by dx squared".

We could, if necessary, differentiate over and over again to obtain the derivatives of order three, four, etc., namely

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} \text{ and } \frac{\mathrm{d}^4y}{\mathrm{d}x^4}, \quad \text{etc.}$$

EXAMPLES

1. If $y = \sin 2x$, show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 0.$$

Solution
Firstly,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x,$$

so that, on differentiating a second time, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\sin 2x = -4y.$$

Hence, the result follows.

2. If $y = x^4$, show that every derivative of y with respect to x after the fourth derivative is zero.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3;$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2;$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 24x;$$

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 24.$$

We now have a constant function, so that all future derivatives will be zero.

Note:
In general, every derivative of $y = x^n$ after the n-th derivative will be zero.

3. If $x = 3t^2$ and $y = 6t$, obtain an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}.$$

In order to differentiate again with respect to x, we observe that, in the formula for the first derivative with respect to x, we need to replace y with $\frac{\mathrm{d}y}{\mathrm{d}x}$.

That is,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{\mathrm{d}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\mathrm{d}x}.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

In the present example, therefore, we obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}\right]}{6t} = \frac{\frac{-1}{t^2}}{6t} = -\frac{1}{6t^3}.$$

Note:

For a function $f(x)$, an alternative notation for the derivatives of order two, three, four, etc. is

$$f''(x),\ f'''(x),\ f^{(iv)}(x),\ \text{etc.}$$

10.8.2 EXERCISES

1. Obtain expressions for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in the following cases:

   (a)

   $$y = 4x^3 - 7x^2 + 5x - 17;$$

   (b)

   $$y = (3x-2)^{10};$$

   (c)

   $$y = x^2e^{4x};$$

   (d)

   $$y = \frac{x-1}{x+1}.$$

2. If $y = \sin 3x$, evaluate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ when $x = \frac{\pi}{4}$.

3. If $x^2 + y^2 - 2x + 2y = 23$ determine the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point where $x = -2$ and $y = 3$.

4. If $x = 3(1 - \cos\theta)$ and $y = 3(\theta - \sin\theta)$, show that

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\frac{\theta}{2};$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{12\sin\frac{\theta}{2}\cos^3\frac{\theta}{2}}.$$

5. If $y = 3e^{2x}\cos(2x-3)$, verify that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 0.$$

10.8.3 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 14;$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 810(3x-2)^8;$$

(c)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = e^{4x}\left[16x^2 + 16x + 2\right];$$

(d)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{4}{(x+1^3}.$$

2.

$$-\frac{9}{\sqrt{2}}.$$

3.

$$\frac{3}{4} \text{ and } -\frac{25}{64}.$$

"JUST THE MATHS"

UNIT NUMBER

11.1

DIFFERENTIATION APPLICATIONS 1
(Tangents and normals)

by

A.J.Hobson

UNIT 11.1 - APPLICATIONS OF DIFFERENTIATION 1

TANGENTS AND NORMALS

11.1.1 TANGENTS

In the definition of a derivative (Unit 10.2), it is explained that the derivative of the function $f(x)$ can be interpreted as the gradient of the tangent to the curve $y = f(x)$ at the point (x, y).

We may now use this information, together with the geometry of the straight line, in order to determine the equation of the tangent to a given curve at a particular point on it.

We illustrate with examples which will then be used also in the subsequent paragraph dealing with normals.

EXAMPLES

1. Determine the equation of the tangent at the point $(-1, 2)$ to the curve whose equation is
$$y = 2x^3 + 5x^2 - 2x - 3.$$

   Solution
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 10x - 2,$$
   which takes the value -6 when $x = -1$.
   Hence the tangent is the straight line passing through the point $(-1, 2)$ having gradient -6. Its equation is therefore
$$y - 2 = -6(x + 1).$$
   That is,
$$6x + y + 4 = 0.$$

2. Determine the equation of the tangent at the point $(2, -2)$ to the curve to the curve whose equation is
$$x^2 + y^2 + 3xy + 4 = 0.$$

   Solution
$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + 3\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right] = 0.$$
   That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x+3y}{3x+2y},$$

which takes the value -2 at the point $(2, -2)$.
Hence, the equation of the tangent is

$$y + 2 = -2(x - 2).$$

That is,

$$2x + y - 2 = 0.$$

3. Determine the equation of the tangent at the point where $t = 2$ to the curve given parametrically by

$$x = \frac{3t}{1+t} \quad \text{and} \quad y = \frac{t^2}{1+t}.$$

Solution
We note first that the point at which $t = 2$ has co-ordinates $\left(2, \frac{4}{3}\right)$.
Furthermore,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{(1+t)^2} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2t+t^2}{(1+t)^2},$$

by the quotient rule.
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t+t^2}{3},$$

which takes the value $\frac{8}{3}$ when $t = 2$.
Hence, the equation of the tangent is

$$y - \frac{4}{3} = \frac{8}{3}(x - 2).$$

That is,

$$3y + 12 = 8x.$$

11.1.2 NORMALS

The normal to a curve at a point on it is defined to be a straight line passing through this point and perpendicular to the tangent there.

Using previous work on perpendicular lines (Unit 5.2), if the gradient of the tangent is m, then the gradient of the normal will be $-\frac{1}{m}$.

EXAMPLES

In the examples of section 11.1.1, therefore, the normals to each curve at the point given will have equations as follows:

1.
$$y - 2 = \frac{1}{6}(x + 1).$$

   That is,

$$6y = x + 13.$$

2.
$$y + 2 = \frac{1}{2}(x - 2).$$

   That is,

$$2y = x - 6.$$

3.
$$y - \frac{4}{3} = -\frac{3}{8}(x - 2).$$

   That is,

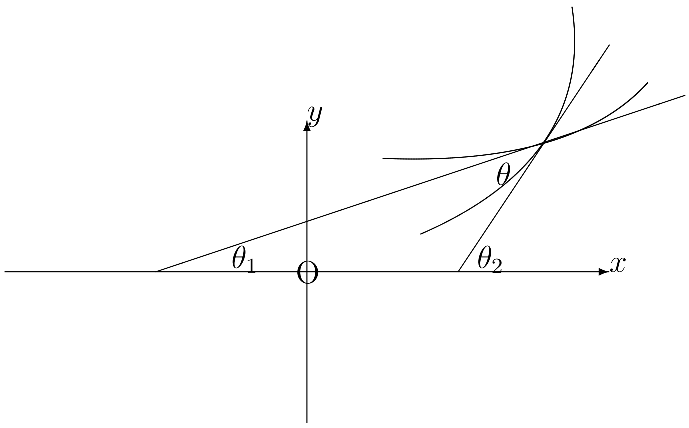
$$24y + 9x = 50.$$

Note:
It may occasionally be required to determine the angle, θ, between two curves at one of their points of intersection. This is defined to be the angle between the tangents at this point; and, if the gradients of the tangents are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, then the angle $\theta \equiv \theta_2 - \theta_1$ and is given by

$$\tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

That is,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$



11.1.3 EXERCISES

1. Determine the equations of the tangent and normal to the following curves at the point given:

   (a)

   $$8y = x^3 \quad \text{at} \quad (2, 1);$$

   (b)

   $$y = \frac{e^{2x}\cos x}{(1+x)^3} \quad \text{at} \quad (0, 1).$$

2. The parametric equations of a curve are

   $$x = 1 + \sin 2t, \quad y = 1 + \cos t + \cos 2t.$$

   Determine the equation of the tangent to the curve at the point for which $t = \frac{\pi}{2}$.

3. Determine the quation of the tangent at the point $(2, 3)$ to the curve whose equation is

$$3x^2 + 2y^2 = 30.$$

4. Determine the equation of the normal at the point $(-1, 2)$ to the curve whose equation is

$$2xy + 3xy^2 - x^2 + y^3 + 9 = 0.$$

11.1.4 ANSWERS TO EXERCISES

1. (a) The tangent is

$$2y = 3x - 4,$$

and the normal is

$$2x + 3y = 7;$$

(b) The tangent is

$$y = 1 - x,$$

and the normal is

$$y = x + 1.$$

2. The tangent is

$$2y = x - 1.$$

3. The tangent is

$$x + y = 5.$$

4. The normal is

$$x + 9y = 17.$$

"JUST THE MATHS"

UNIT NUMBER

11.2

DIFFERENTIATION APPLICATIONS 2
(Local maxima and local minima)
&
(Points of inflexion)

by

A.J.Hobson

UNIT 11.2 - APPLICATIONS OF DIFFERENTIATION 2

LOCAL MAXIMA, LOCAL MINIMA AND POINTS OF INFLEXION

11.2.1 INTRODUCTION

(a) Let us first suppose that the formula

$$s = f(t)$$

represents the distance s, travelled in time t, by a moving object from some previously chosen point on its journey.

The derivative , $\frac{\mathrm{d}s}{\mathrm{d}t}$, of s with respect to t gives the speed of the object at time t and can be represented by the slope of the tangent at the point (t, s) to the curve whose equation is $s = f(t)$.

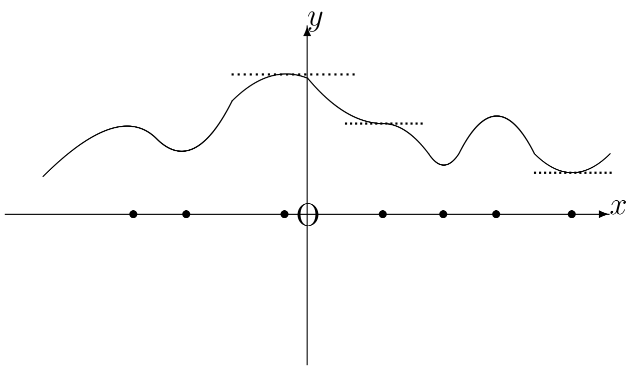
For any instant, t_0, of time, at which the object is stationary, the value of the derivative will be zero and hence, the slope of the tangent will be zero.

The corresponding point (t_0, s_0), on the graph may thus be called a **"stationary point"**.

(b) More generally, any relationship,

$$y = f(x),$$

between two variable quantities, x and y, can usually be represented by a graph of y against x and any point (x_0, y_0) on the graph at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ takes the value zero is called a "stationary point". The tangent to the curve at the point (x_0, y_0) will be parallel to x-axis.



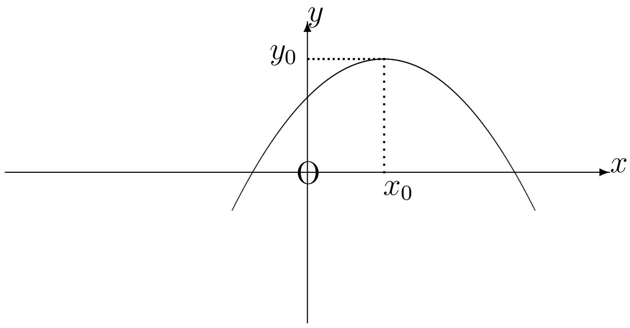
In the paragraphs which follow, we shall discuss the definitions and properties of particular kinds of stationary point.

11.2.2 LOCAL MAXIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local maximum"** if y_0 is greater than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



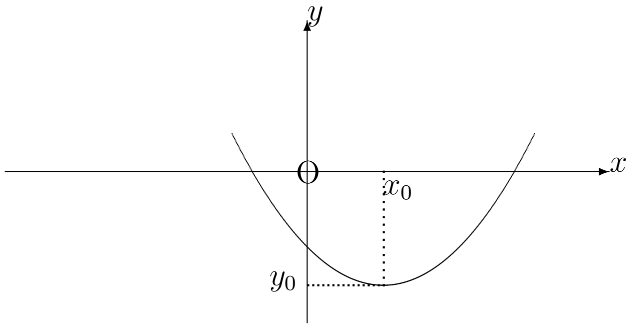
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are greater than y_0; hence, the definition of a local maximum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.3 LOCAL MINIMA

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"local minimum"** if y_0 is less than the y co-ordinates of all other points on the curve in the immediate neighbourhood of (x_0, y_0).



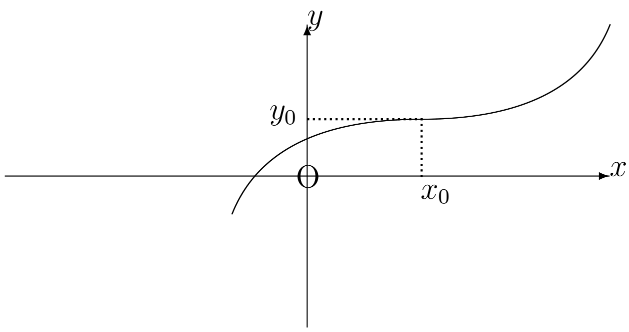
Note:
It may well happen that, for points on the curve which are some distance away from (x_0, y_0), their y co-ordinates are less than y_0; hence, the definition of a local minimum point must refer to the behaviour of y in the immediate neighbourhood of the point.

11.2.4 POINTS OF INFLEXION

A stationary point (x_0, y_0) on the graph whose equation is

$$y = f(x)$$

is said to be a **"point of inflexion"** if the curve exhibits a change in the direction bending there.



11.2.5 THE LOCATION OF STATIONARY POINTS AND THEIR NATURE

In order to determine the location of any stationary points on the curve whose equation is

$$y = f(x),$$

we simply obtain an expression for the derivate of y with respect to x, then equate it to zero. That is, we solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Having located a stationary point (x_0, y_0), we may then determine whether it is a local maximum, a local minimum, or a point of inflexion using two alternative methods. These methods will be illustrated by examples:

METHOD 1. - The "First Derivative" Method

Suppose ϵ denotes a number which is relatively small compared with x_0.

If we examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$, first at $x = x_0 - \epsilon$ and then at $x = x_0 + \epsilon$, the following conclusions may be drawn:

(a) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative, there is a local maximum at (x_0, y_0).

(b) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive, there is a local minimum at (x_0, y_0).

(c) If the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change, there is a point of inflexion at (x_0, y_0).

EXAMPLES

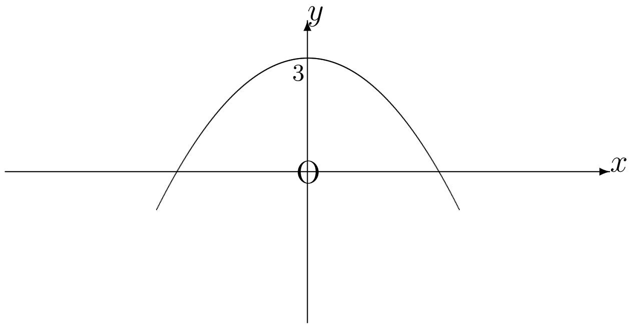
1. Determine the stationary point on the graph whose equation is

$$y = 3 - x^2.$$

Solution:

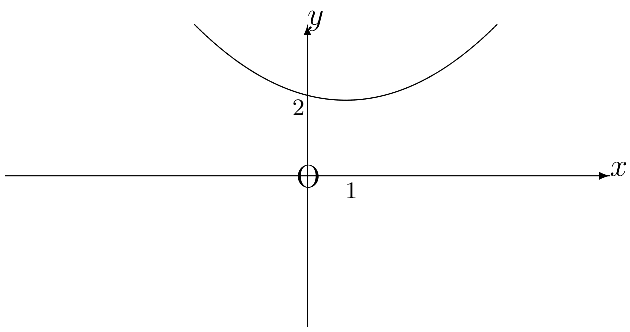
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x,$$

which is equal to zero at the point where $x = 0$ and hence, $y = 3$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.
Hence, there is a local maximum at the point $(0, 3)$.



2. Determine the stationary point on the graph whose equation is
$$y = x^2 - 2x + 3.$$

Solution:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2,$$
which is equal to zero at the point where $x = 1$ and hence, $y = 2$.
If $x = 1 - \epsilon$, (for example, $x = 1 - 0.01 = 0.99$), then $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ and
If $x = 1 + \epsilon$, (for example, $x = 1 + 0.01) = 1.01$, then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a local minimum at the point $(1, 2)$.

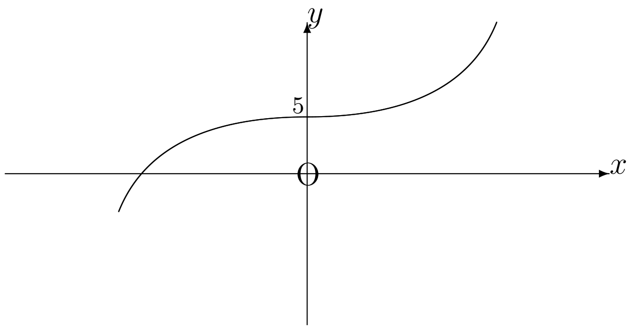


3. Determine the stationary point on the graph whose equation is
$$y = 5 + x^3.$$

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

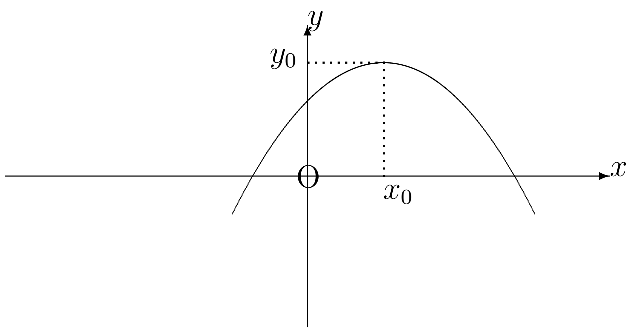
which is equal to zero at the point where $x = 0$ and hence, $y = 5$.
If $x = 0 - \epsilon$, (for example, $x = -0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and
If $x = 0 + \epsilon$, (for example, $x = 0.01$), then $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$.
Hence, there is a point of inflexion at $(0, 5)$.



METHOD 2. - The "Second Derivative" Method

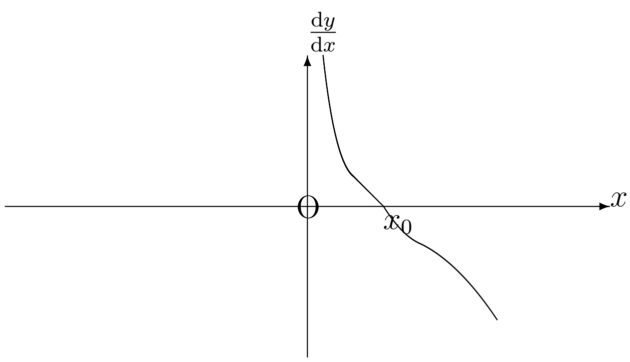
This method considers the general appearance of the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ against x, which is called the **"first derived curve"**. The properties of the first derived curve in the neighbourhood of a stationary point (x_0, y_0) may be used to predict the nature of this point.

(a) Local Maxima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ steadily decrease from large positive values to large negative values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going downwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **negative**. In other words,

$$\frac{d^2y}{dx^2} < 0 \text{ at } x = x_0.$$

This is the second derivative test for a local maximum.

EXAMPLE

For the curve whose equation is

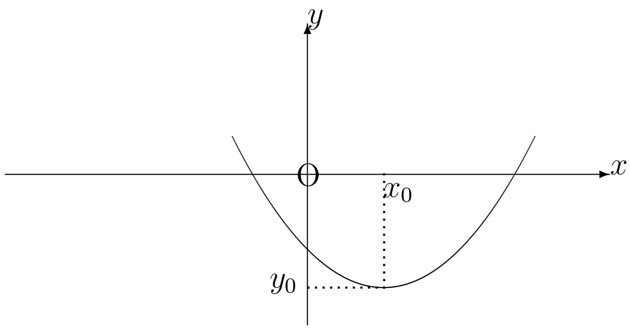
$$y = 3 - x^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2.$$

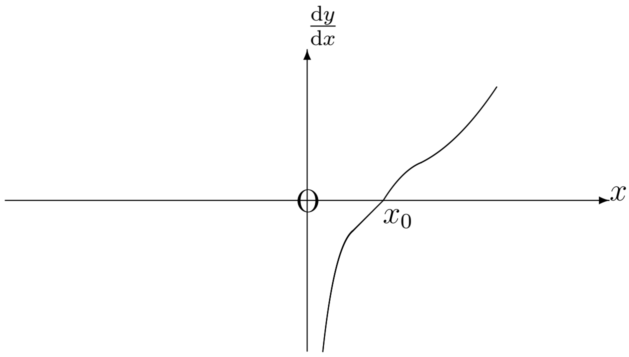
The second derivative is negative <u>everywhere</u>, so it is certainly negative at the stationary point $(0, 3)$ obtained in the previous method. Hence, $(0, 3)$ is a local maximum.

(b) Local Minima



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ steadily increase from large negative values to large positive values, passing through zero when $x = x_0$.

This suggests that the first derived curve exhibits a **"going upwards"** tendency at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is **positive**. In other words,

$$\frac{d^2y}{dx^2} > 0 \quad \text{at} \quad x = x_0.$$

This is the second derivative test for a local minimum.

EXAMPLE

For the curve whose equation is

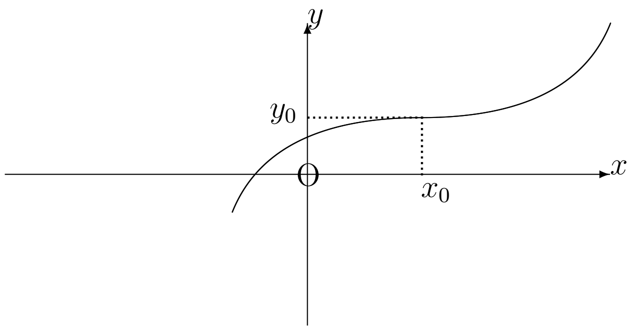
$$y = x^2 - 2x + 3,$$

we have

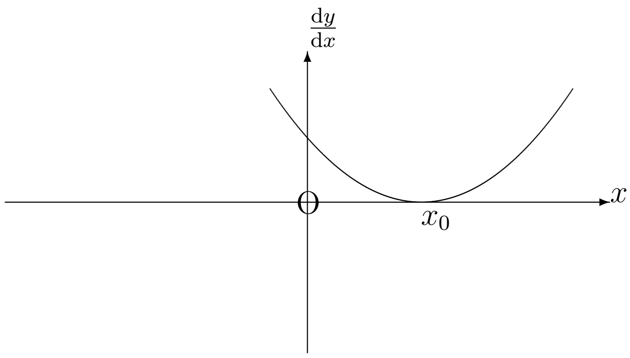
$$\frac{dy}{dx} = 2x - 2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 2.$$

The second derivative is positive everywhere, so it is certainly positive at the stationary point $(1, 2)$ obtained in the previous method. Hence, $(1, 2)$ is a local minimum.

(c) Points of inflexion



As x passes from values below x_0 to values above x_0, the corresponding values of $\frac{dy}{dx}$ appear to reach either a minimum or a maximum value at $x = x_0$.



It may be expected, therefore, that the slope at $x = x_0$ on the first derived curve is zero and changes sign as x passes through the value x_0.

$$\frac{d^2y}{dx^2} = 0 \ \text{ at } \ x = x_0 \ \text{ and } \ \text{changes } \ \text{sign.}$$

This is the second derivative test for a point of inflexion.

EXAMPLE

For the curve whose equation is

$$y = 5 + x^3,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

The second derivative is zero when $x = 0$ and changes sign as x passes through the value zero.

Hence, the stationary point $(0, 5)$ found previously is a point of inflexion.

Notes:
(i) For a stationary point of inflexion, it is not enough that

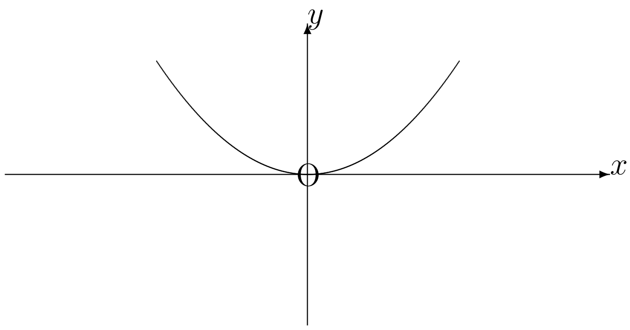
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

without also the change of sign.

For example, the curve whose equation is

$$y = x^4$$

is easily shown (by Method 1) to have a local minimum at the point $(0, 0)$; and yet, for this curve, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$.



(ii) Some curves contain what are called **"ordinary points of inflexion"**. They are not stationary points and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$; but the rest of the condition for a point of inflexion still holds. That is,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \text{and} \quad \text{changes} \quad \text{sign.}$$

EXAMPLE

For the curve whose equation is

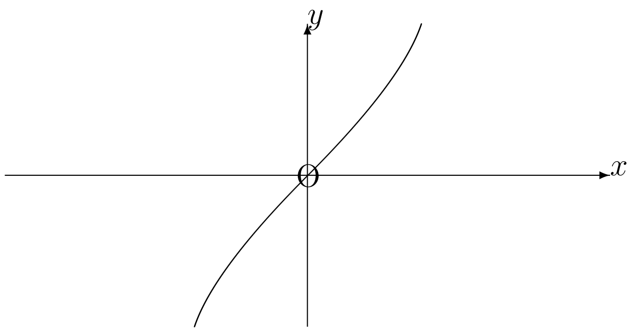
$$y = x^3 + x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

Hence, there are no stationary points at all; but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at $x = 0$ and changes sign as x passes through $x = 0$.

That is, there is an ordinary point of inflexion at $(0, 0)$.



Notes:
(i) In any interval of the x-axis, the greatest value of a function of x will be either the greatest maximum or possibly the value at one end of the interval. Similarly, the least value of the function will be either the smallest minimum or possibly the value at one end of the interval.

(ii) In sketching a curve whose maxima, minima and points of inflexion are known, it may also be necessary to determine, from the equation of the curve, its points of intersection with the axes of reference.

11.2.6 EXERCISES

1. Determine the local maxima, local minima and points of inflexion (including ordinary points of inflexion) on the curves whose equations are given in the following:

   (a)
   $$y = x^3 - 6x^2 + 9x + 6;$$

   (b)
   $$y = x + \frac{1}{x}.$$

   In each case, give also a sketch of the curve.

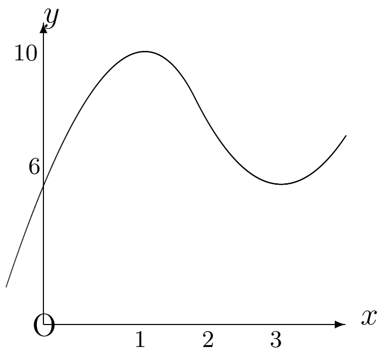
2. Show that the curve whose equation is
   $$y = \frac{1}{2x+1} + \ln(2x+1)$$
   has a local minimum at a point on the y-axis.

3. The horse-power, P, transmitted by a belt is given by
   $$P = k\left[Tv - \frac{wv^3}{g}\right],$$
   where k is a constant, v is the speed of the belt, T is the tension on the driving side and w is the weight per unit length of the belt. Determine the speed for which the horse-power is a maximum.

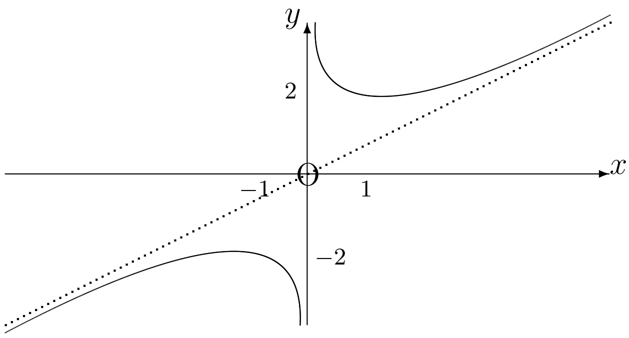
4. For x lying in the interval $-3 \leq x \leq 5$, determine the least and greatest values of the function
   $$x^3 - 12x + 20$$

11.2.7 ANSWERS TO EXERCISES

1. (a) Local maximum at $(1, 10)$, local minumum at $(3, 6)$, ordinary point of inflexion at $(2, 8)$;



(b) Local maximum at $(-1, -2)$, local minimum at $(1, 2)$.



2. Local minimum at the point $(0, 1)$.

3. The horse-power is maximum when

$$v = \sqrt{\frac{gT}{2w}}.$$

4. The greatest value is 85 at $(5, 85)$; the least value is 4 at $(2, 4)$.

"JUST THE MATHS"

UNIT NUMBER

11.3

DIFFERENTIATION APPLICATIONS 3
(Curvature)

by

A.J.Hobson

UNIT 11.3 - DIFFERENTIATION APPLICATIONS 3

CURVATURE

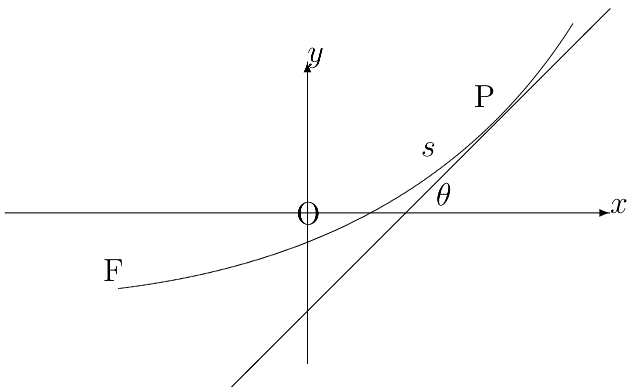
11.3.1 INTRODUCTION

In the discussion which follows, consideration will be given to a method of measuring the **"tightness of bends"** on a curve. This measure will be called **"curvature"** and its definition will imply that very tight bends have large curvature.

We shall also need to distinguish between curves which are **"concave upwards"** ($\cup$), having positive curvature, and curves which are **"concave downwards"** ($\cap$), having negative curvature.

DEFINITION

Suppose we are given a curve whose equation is $y = f(x)$; and suppose that θ is the angle made with the positive x-axis by the tangent to the curve at a point, P(x, y), on it. If s is the distance to P, measured along the curve from some fixed point, F, on it then the curvature, κ, at P, is defined as the rate of increase of θ with respect to s.

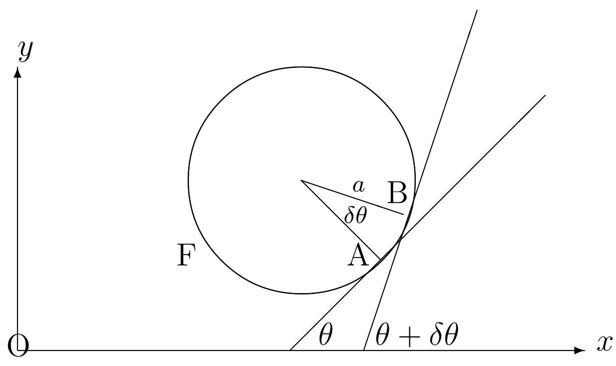


$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s}.$$

EXAMPLE

Determine the curvature at any point of a circle with radius a.

Solution



We shall let A be a point on the circle at which the tangent is inclined to the positive x-axis at an angle, θ, and let B be a point (close to A) at which the tangent is inclined to the positive x-axis at an angle, $\theta + \delta\theta$. The length of the arc, AB, will be called δs, where we shall assume that distances, s, are measured along the circle in a counter-clockwise sense from the fixed point, F.

The diagram shows that $\delta\theta$ is both the angle between the two tangents **and** the angle subtended at the centre of the circle by the arc, AB.

Thus, $\delta s = a\delta\theta$ which can be written

$$\frac{\delta\theta}{\delta s} = \frac{1}{a}.$$

Allowing $\delta\theta$, and hence δs, to approach zero, we conclude that

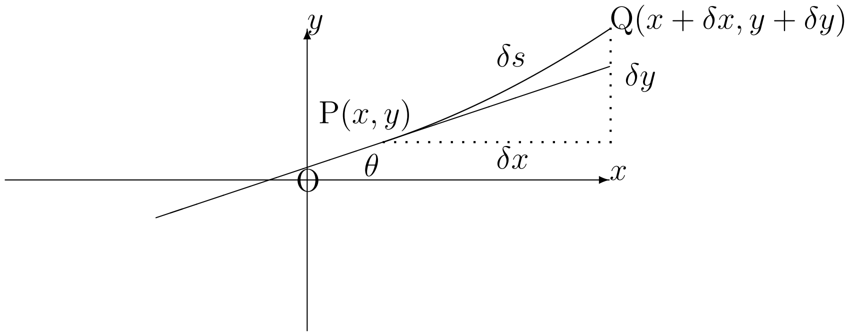
$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{a}.$$

We note, however, that, for the lower half of the circle, θ **increases** as s increases, while, in the upper half of the circle, θ **decreases** as s increases. The curvature will therefore be positive for the lower half (which is concave upwards) and negative for the upper half (which is concave downwards).

Summary
The curvature at any point of a circle is numerically equal to the reciprocal of the radius.

11.3.2 CURVATURE IN CARTESIAN CO-ORDINATES

Given a curve whose equation is $y = f(x)$, suppose P(x, y) and Q$(x + \delta x, y + \delta y)$ are two neighbouring points on it which are separated by a distance of δs along the curve.



In this diagram, we may observe that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan\theta$$

and also that

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \lim_{\delta s \to 0} \frac{\delta x}{\delta s} = \cos\theta.$$

The curvature may therefore be evaluated as follows:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}x}.\cos\theta.$$

But,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \frac{1}{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Finally,

$$\cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \pm\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}};$$

and so,

$$\kappa = \pm\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

Notes:

(i) For a curve which is concave upwards at a particular point, the gradient, $\frac{dy}{dx}$, will **increase** as x increases through the point. Hence, $\frac{d^2y}{dx^2}$ will be positive at the point.

(ii) For a curve which is concave downwards at a particular point, the gradient, $\frac{dy}{dx}$, will **decrease** as x increases through the point. Hence, $\frac{d^2y}{dx^2}$ will be negative at the point.

(ii) In future, therefore, we may allow the value of the curvature to take the same sign as $\frac{d^2y}{dx^2}$, giving the formula

$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left[1+\frac{dy}{dx}^2\right]^{\frac{3}{2}}}.$$

EXAMPLE

Use the cartesian formula to determine the curvature at any point on the circle, centre $(0,0)$ with radius a.

Solution

The equation of the circle is

$$x^2 + y^2 = a^2,$$

which means that, for the upper half,

$$y = \sqrt{a^2 - x^2}$$

and, for the lower half,

$$y = -\sqrt{a^2 - x^2}.$$

Considering, firstly, the upper half,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{a^2 - x^2}}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\sqrt{a^2 - x^2} + \frac{x^2}{\sqrt{a^2-x^2}}}{a^2 - x^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Therefore,

$$\kappa = \frac{-\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}}{\left(1 + \frac{x^2}{a^2-x^2}\right)^{\frac{3}{2}}} = -\frac{a^2}{a^3} = -\frac{1}{a}.$$

For the lower half of the circle,

$$\kappa = \frac{1}{a}.$$

11.3.3 EXERCISES

In the following questions, state your answer in decimals correct to three places of decimals:

1. Calculate the curvature at the point $(-1, 3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3.$$
2. Calculate the curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}.$$
3. Calculate the curvature at the point $(1, 1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0.$$
4. Calculate the curvature at the point for which $\theta = 30^\circ$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta.$$

11.3.4 ANSWERS TO EXERCISES

1. $\kappa = 0.023$
2. $\kappa = -0.707$
3. $\kappa = -5.650$
4. $\kappa = 0.179$

"JUST THE MATHS"

UNIT NUMBER

11.4

DIFFERENTIATION APPLICATIONS 4
(Circle, radius & centre of curvature)

by

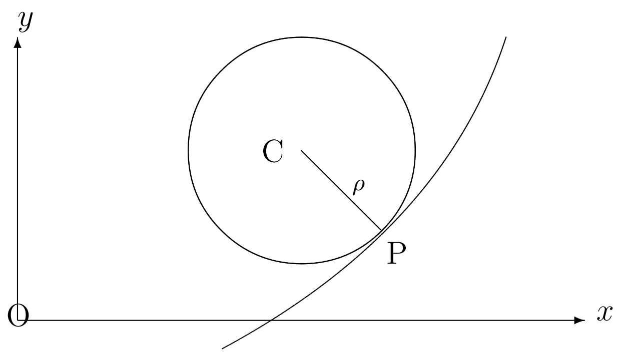
A.J.Hobson

UNIT 11.4 DIFFERENTIATION APPLICATIONS 4

CIRCLE, RADIUS AND CENTRE OF CURVATURE

11.4.1 INTRODUCTION

At a point, P, on a given curve, suppose we were to draw a circle which **just touches** the curve and has the same value of the curvature (including its sign). This circle is called the **"circle of curvature at P"**. Its radius, ρ, is called the **"radius of curvature at P"** and its centre is called the **"centre of curvature at P"**.



11.4.2 RADIUS OF CURVATURE

Using the earlier examples on the circle (Unit 11.3), we conclude that, if the curvature at P is κ, then $\rho = \frac{1}{\kappa}$ and, hence,

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\theta}.$$

Furthermore, in cartesian co-ordinates,

$$\rho = \frac{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}.$$

Note:
If we are interested in the radius of curvature simply as a length, then, for curves with

negative curvature, we would use only the **numerical** value obtained in the above formula. However, in a later discussion, it is necessary to use the appropriate sign for the radius of curvature.

EXAMPLE

Calculate the radius of curvature at the point $(0.5, -1)$ of the curve whose equation is

$$y^2 = 2x.$$

Solution

Differentiating implicitly,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2.$$

That is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}.$$

Also

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^2}.\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^3}.$$

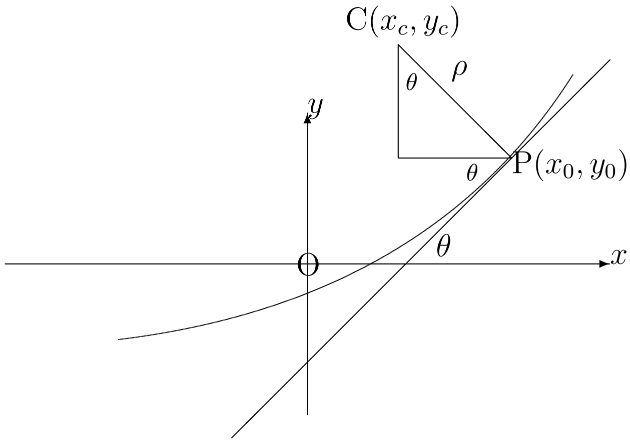
Hence, at the point $(0.5, -1)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$.

We conclude that

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{1} = 2\sqrt{2}.$$

11.4.3 CENTRE OF CURVATURE

We shall consider a point, (x_0, y_0), on an arc of a curve whose equation is $y = f(x)$ and for which the curvature is positive, the arc lying in the first quadrant. But it may be shown that the formulae obtained for the co-ordinates, (x_c, y_c), of the centre of curvature apply in any situation, provided that the curvature is associated with its appropriate sign.



From the diagram,

$$
\begin{aligned}
x_c &= x_0 - \rho \sin\theta, \\
y_c &= y_0 + \rho \cos\theta.
\end{aligned}
$$

Note:
Although the formulae apply in any situation, it is a good idea to sketch the curve in order estimate, roughly, where the centre of curvature is going to be. This is especially important where there is uncertainty about the precise value of the angle θ.

EXAMPLE

Determine the centre of curvature at the point $(0.5, -1)$ of the curve whose equation is

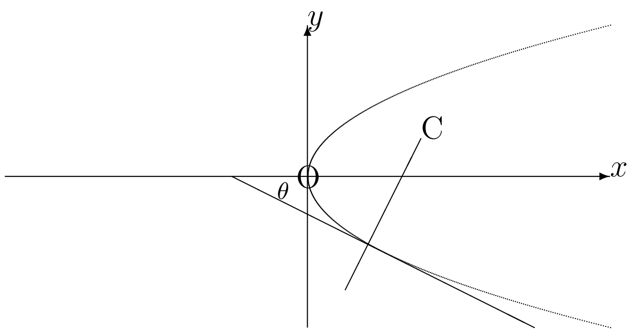
$$y^2 = 2x.$$

Solution

From the earlier example on calculating radius of curvature,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{y^3},$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$ and $\rho = 2\sqrt{2}$ at the point $(0.5, -1)$.



The diagram shows that the co-ordinates, (x_c, y_c), of the centre of curvature will be such that $x_c > 0.5$ and $y_c > -1$. This will be so provided that the angle, θ, is a negative acute angle; (that is, its cosine will be positive and its sine will be negative).

In fact,

$$\theta = \tan^{-1}(-1) = -45^\circ.$$

Hence,

$$\begin{aligned} x_c &= 0.5 - 2\sqrt{2}\sin(-45^\circ), \\ y_c &= -1 + 2\sqrt{2}\cos(-45^\circ). \end{aligned}$$

That is,

$$x_c = 2.5 \ \text{ and } \ y_c = 1.$$

11.4.4 EXERCISES

In the following questions, state your results in decimals correct to three places of decimals:

1. Calculate the radius of curvature at the point $(-1, 3)$ on the curve whose equation is
$$y = x + 3x^2 - x^3$$
and hence obtain the co-ordinates of the centre of curvature.
2. Calculate the radius of curvature at the origin on the curve whose equation is
$$y = \frac{x - x^2}{1 + x^2}$$
and hence obtain the co-ordinates of the centre of curvature.
3. Calculate the radius of curvature at the point $(1, 1)$ on the curve whose equation is
$$x^3 - 2xy + y^3 = 0$$
and hence obtain the co-ordinates of the centre of curvature.
4. Calculate the radius of curvature at the point for which $\theta = 30°$ on the curve whose parametric equations are
$$x = 1 + \sin\theta \quad \text{and} \quad y = \sin\theta - \frac{1}{2}\cos 2\theta$$
and hence obtain the co-ordinates of the centre of curvature.

11.4.5 ANSWERS TO EXERCISES

1. $\rho = 43.6705, \quad (x_c, y_c) = (42.333, 8.417).$
2. $\rho = -1.414 \quad (x_c, y_c) = (1, -1).$
3. $\rho = -0.177 \quad (x_c, y_c) = (0.875, 0.875).$
4. $\rho = 0.590 \quad (x_c, y_c = (-3.500, 2.750).$

"JUST THE MATHS"

UNIT NUMBER

11.5

DIFFERENTIATION APPLICATIONS 5
(Maclaurin's and Taylor's series)

by

A.J.Hobson

UNIT 11.5 - DIFFERENTIATION APPLICATIONS 5

MACLAURIN'S AND TAYLOR'S SERIES

11.5.1 MACLAURIN'S SERIES

One of the simplest kinds of function to deal with, in either algebra or calculus, is a polynomial (see Unit 1.8). Polynomials are easy to substitute numerical values into and they are easy to differentiate.

One useful application of the present section is to approximate, to a polynomial, functions which are not already in polynomial form.

THE GENERAL THEORY

Suppose $f(x)$ is a given function of x which is not in the form of a polynomial, and let us assume that it may be expressed in the form of an infinite series of ascending powers of x; that is, a **"power series"**, (see Unit 2.4).

More specifically, we assume that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ \ldots$$

This assumption cannot be justified unless there is a way of determining the **"coefficients"**, a_0, a_1, a_2, a_3, a_4, etc.; but this is possible as an application of differentiation as we now show:

(a) Firstly, if we substitute $x = 0$ into the assumed formula for $f(x)$, we obtain $f(0) = a_0$; in other words,

$$a_0 = f(0).$$

(b) Secondly, if we differentiate the assumed formula for $f(x)$ once with respect to x, we obtain

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$$

which, on substituting $x = 0$, gives $f'(0) = a_1$; in other words,

$$a_1 = f'(0).$$

(c) Differentiating a second time leads to the result that

$$f''(x) = 2a_2 + (3 \times 2)a_3x + (4 \times 3)a_4x^2 + \ldots$$

which, on substituting $x = 0$ gives $f''(0) = 2a_2$; in other words,

$$a_2 = \frac{1}{2}f''(0).$$

(d) Differentiating yet again leads to the result that

$$f'''(x) = (3 \times 2)a_3 + (4 \times 3 \times 2)a_4x + \ldots$$

which, on substituting $x = 0$ gives $f'''(0) = (3 \times 2)a_3$; in other words,

$$a_3 = \frac{1}{3!}f'''(0).$$

(e) Continuing this process with further differentiation will lead to the general formula

$$a_n = \frac{1}{n!}f^{(n)}(0),$$

where $f^{(n)}(0)$ means the value, at $x = 0$ of the n-th derivative of $f(x)$.

Summary

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

This is called the **"Maclaurin's series for $f(x)$"**.

Notes:

(i) We must assume, ofcourse, that all of the derivatives of $f(x)$ exist at $x = 0$ in the first place; otherwise the above result is invalid.
It is also necessary to examine, for convergence or divergence, the Maclaurin's series obtained

for a particular function. The result may not be used when the series diverges; (see Units 2.3 and 2.4).

(b) If x is small and it is possible to neglect powers of x after the n-th power, then Maclaurin's series approximates $f(x)$ to a polynomial of degree n.

11.5.2 STANDARD SERIES

Here, we determine the Maclaurin's series for some of the functions which occur frequently in the applications of mathematics to science and engineering. The ranges of values of x for which the results are valid will be stated without proof.

1. **The Exponential Series**

   (i) $f(x) \equiv e^x$; hence, $f(0) = e^0 = 1$.

   (ii) $f'(x) = e^x$; hence, $f'(0) = e^0 = 1$.

   (iii) $f''(x) = e^x$; hence, $f''(0) = e^0 = 1$.

   (iv) $f'''(x) = e^x$; hence, $f'''(0) = e^0 = 1$.

   (v) $f^{(iv)}(x) = e^x$; hence, $f^{(iv)}(0) = e^0 = 1$.

   Thus,

   $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

   and it may be shown that this series is valid for all values of x.

2. **The Sine Series**

   (i) $f(x) \equiv \sin x$; hence, $f(0) = \sin 0 = 0$.

   (ii) $f'(x) = \cos x$; hence, $f'(0) = \cos 0 = 1$.

   (iii) $f''(x) = -\sin x$; hence, $f''(0) = -\sin 0 = 0$.

   (iv) $f'''(x) = -\cos x$; hence, $f'''(0) = -\cos 0 = -1$.

   (v) $f^{(iv)}(x) = \sin x$; hence, $f^{(iv)}(0) = \sin 0 = 0$.

   (vi) $f^{(v)}(x) = \cos x$; hence, $f^{(v)}(0) = \cos 0 = 1$.

   Thus,

   $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots$$

   and it may be shown that this series is valid for all values of x.

3. The Cosine Series

(i) $f(x) \equiv \cos x$; hence, $f(0) = \cos 0 = 1$.

(ii) $f'(x) = -\sin x$; hence, $f'(0) = -\sin 0 = 0$.

(iii) $f''(x) = -\cos x$; hence, $f''(0) = -\cos 0 = -1$.

(iv) $f'''(x) = \sin x$; hence, $f'''(0) = \sin 0 = 0$.

(v) $f^{(iv)}(x) = \cos x$; hence, $f^{(iv)}(0) = \cos 0 = 1$.

Thus,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ \ldots$$

and it may be shown that this series is valid for all values of x.

4. The Logarithmic Series

It is not possible to find a Maclaurin's series for the function $\ln x$, since neither the function nor its derivatives exist at $x = 0$.

As an alternative, we may consider the function $\ln(1 + x)$ instead.

(i) $f(x) \equiv \ln(1 + x)$; hence, $f(0) = \ln 1 = 0$.

(ii) $f'(x) = \frac{1}{1+x}$; hence, $f'(0) = 1$.

(iii) $f''(x) = -\frac{1}{(1+x)^2}$; hence, $f''(0) = 1$.

(iv) $f'''(x) = \frac{2}{(1+x)^3}$; hence, $f'''(0) = 2$.

(v) $f^{(iv)}(x) = -\frac{2\times 3}{(1+x)^4}$; hence, $f^{(iv)}(0) = -(2 \times 3)$.

Thus,

$$\ln(1 + x) = x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - (2 \times 3)\frac{x^4}{4!} + \ \ldots$$

which simplifies to

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ \ldots$$

and it may be shown that this series is valid for $-1 < x \leq 1$.

5. The Binomial Series

The statement of the Binomial Formula has already appeared in Unit 2.2; and it was seen there that

(a) When n is a positive integer, the expansion of $(1 + x)^n$ in ascending powers of x is a **finite** series;

(b) When n is a negative integer or a fraction, the expansion of $(1+x)^n$ in ascending powers of x is an **infinite** series.

Here, we examine the proof of the Binomial Formula.

(i) $f(x) \equiv (1+x)^n$**;** hence, $f(0) = 1$**.**

(ii) $f'(x) = n(1+x)^{n-1}$**;** hence, $f'(0) = n$**.**

(iii) $f''(x) = n(n-1)(1+x)^{n-2}$**;** hence, $f''(0) = n(n-1)$**.**

(iv) $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$**;** hence, $f'''(0) = n(n-1)(n-2)$**.**

(v) $f^{(iv)}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$**;** hence, $f^{(iv)}(0) = n(n-1)(n-2)(n-3)$**.**

Thus,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ \ldots$$

If n is a positive integer, all of the derivatives of $(1+x)^n$ after the n-th derivative are identically equal to zero; so the series is a finite series ending with the term in x^n.

In all other cases, the series is an infinite series and it may be shown that it is valid whenever $-1 < x \leq 1$.

EXAMPLES

1. Use the Maclaurin's series for $\sin x$ to evaluate

$$\lim_{x \to 0} \frac{x + \sin x}{x(x+1)}.$$

Solution

Substituting the series for $\sin x$ gives

$$\lim_{x \to 0} \frac{x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ \ldots}{x^2 + x}$$

$$= \lim_{x \to 0} \frac{2x - \frac{x^3}{6} + \frac{x^5}{120} - \ \ldots}{x^2 + x}$$

$$= \lim_{x \to 0} \frac{2 - \frac{x^2}{6} + \frac{x^4}{120} - \ \ldots}{x + 1} = 2.$$

2. Use a Maclaurin's series to evaluate $\sqrt{1.01}$ correct to six places of decimals.

   Solution

   We shall consider the expansion of the function $(1+x)^{\frac{1}{2}}$ and then substitute $x = 0.01$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ \ldots$$

   That is,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ \ldots$$

   Substituting $x = 0.01$ gives

$$\sqrt{1.01} = 1 + \frac{1}{2} \times 0.01 - \frac{1}{8} \times 0.0001 + \frac{1}{16} \times 0.000001 - \ \ldots$$

$$= 1 + 0.005 - 0.0000125 + 0.0000000625 - \ \ldots$$

   The fourth term will not affect the sixth decimal place in the result given by the first three terms; and this is equal to 1.004988 correct to six places of decimals.

3. Assuming the Maclaurin's series for e^x and $\sin x$ and assuming that they may be multiplied together term-by-term, obtain the expansion of $e^x \sin x$ in ascending powers of x as far as the term in x^5.

   Solution

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{120} + \ldots\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24} + \ \ldots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \ \ldots$$

11.5.3 TAYLOR'S SERIES

A useful consequence of Maclaurin's series is known as Taylor's series and one form of it may be stated as follows:

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ . \ . \ .$$

Proof:

To obtain this result from Maclaurin's series, we simply let $f(x+h) \equiv F(x)$. Then,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0) + \ . \ .$$

But, $F(0) = f(h)$, $F'(0) = f'(h)$, $F''(0) = f''(h)$, $F'''(0) = f'''(h)$,. . . which proves the result.

Note: An alternative form of Taylor's series, often used for approximations, may be obtained by interchanging the symbols x and h to give

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ . \ . \ .$$

EXAMPLE

Given that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, use Taylor's series to evaluate $\sin(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{4}$ and $h = 0.01$.

Solution

Using the sequence of derivatives as in the Maclaurin's series for $\sin x$, we have

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + ...$$

Subsituting $x = \frac{\pi}{4}$ and $h = 0.01$, we obtain

$$\sin\left(\frac{\pi}{4} + 0.01\right) = \frac{1}{\sqrt{2}}\left(1 + 0.01 - \frac{(0.01)^2}{2!} - \frac{(0.01)^3}{3!} + \ . \ . \ .\right)$$

$$= \frac{1}{\sqrt{2}}(1 + 0.01 - 0.00005 - 0.000000017 + ...)$$

The fourth term does not affect the fifth decimal place in the sum of the first three terms; and so

$$\sin\left(\frac{\pi}{4}+0.01\right) \simeq \frac{1}{\sqrt{2}} \times 1.00995 \simeq 0.71414$$

11.5.4 EXERCISES

1. Determine the first three non-vanishing terms of the Maclaurin's series for the function $\sec x$.
2. Determine the Maclaurin's series for the function $\tan x$ as far as the term in x^5.
3. Determine the Maclaurin's series for the function $\ln(1+e^x)$ as far as the term in x^4.
4. Use the Maclaurin's series for the function e^x to deduce the expansion, in ascending powers of x of the function e^{-x} and then use these two series to obtain the expansion, in ascending powers of x, of the functions

   (a)
   $$\frac{e^x+e^{-x}}{2}(\equiv \cosh x);$$

   (b)
   $$\frac{e^x-e^{-x}}{2}(\equiv \sinh x).$$

5. Use the Maclaurin's series for the function $\cos x$ and the Binomial Series for the function $\frac{1}{1+x}$ to obtain the expansion of the function
   $$\frac{\cos x}{1+x}$$
   in ascending powers of x as far as the term in x^4.
6. From the Maclaurin's series for the function $\cos x$, deduce the expansions of the functions $\cos 2x$ and $\sin^2 x$ as far as the term in x^4.

7. Use appropriate Maclaurin's series to evaluate the following limits:

   (a)

$$\lim_{x \to 0} \left[\frac{e^x + e^{-x} - 2}{2\cos 2x - 2} \right];$$

   (b)

$$\lim_{x \to 0} \left[\frac{\sin^2 x - x^2 \cos x}{x^4} \right].$$

8. Use a Maclaurin's series to evaluate $\sqrt[3]{1.05}$ correct to four places of decimals.

9. Expand $\cos(x+h)$ as a series of ascending powers of h.

   Given that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, evaluate $\cos(x+h)$, correct to five places of decimals, in the case when $x = \frac{\pi}{6}$ and $h = -0.05$.

11.5.5 ANSWERS TO EXERCISES

1.

$$1 + \frac{x^2}{2} + \frac{x^4}{8} + \ ..$$

2.

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \ ...$$

3.

$$\ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ ...$$

4. (a)

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ ...;$$

   (b)

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ ..$$

5.

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13x^4}{24} - \ \ldots$$

6.

$$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} - \ \ldots$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \ \ldots$$

7. (a) $-\frac{1}{4}$, (b) $\frac{1}{6}$

8. 1.0164

9. 0.74156

"JUST THE MATHS"

UNIT NUMBER

11.6

DIFFERENTIATION APPLICATIONS 6
(Small increments and small errors)

by

A.J.Hobson

UNIT 11.6 - DIFFERENTIATION APPLICATIONS 6

SMALL INCREMENTS AND SMALL ERRORS

11.6.1 SMALL INCREMENTS

Given that a dependent variable, y, and an independent variable, x are related by means of the formula

$$y = f(x),$$

suppose that x is subject to a small **"increment"**, δx,

In the present context we use the term "increment" to mean that δx is positive when x is **increased**, but negative when x is **decreased**.

The exact value of the corresponding increment, δy, in y is given by

$$\delta y = f(x + \delta x) - f(x),$$

but this can often be a cumbersome expression to evaluate.

However, since δx is small, we may recall, from the definition of a derivative (Unit 10.2), that

$$\frac{f(x + \delta x) - f(x)}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x}.$$

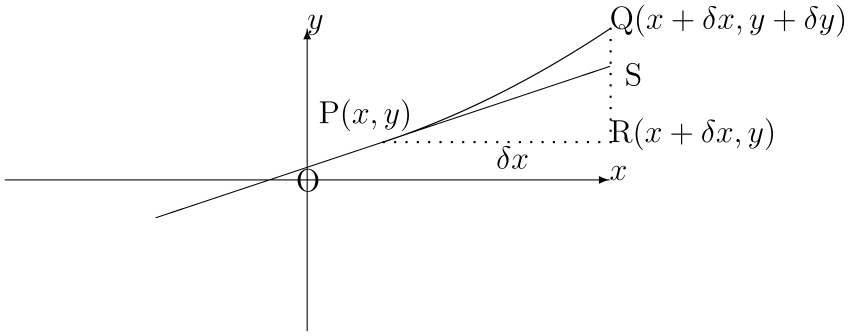
That is,

$$\frac{\delta y}{\delta x} \simeq \frac{\mathrm{d}y}{\mathrm{d}x};$$

and we may conclude that

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

For a diagramatic approach to this approximation for the increment in y, let us consider the graph of y against x in the neighbourhood of the two points P(x, y) and Q$(x + \delta x, y + \delta y)$ on the curve whose equation is $y = f(x)$.



In the diagram, PR $= \delta x$, QR $= \delta y$ and the gradient of the line PS is given by the value of $\frac{dy}{dx}$ at P.

Taking SR as an approximation to QR, we obtain

$$\frac{\text{SR}}{\text{PR}} = \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\text{P}}.$$

In other words,

$$\frac{\text{SR}}{\delta x} = \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\text{P}}.$$

Hence,

$$\delta y \simeq \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{\text{P}} \delta x,$$

which is the same result as before.

Notes:

(i) The quantity $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$ is known as the **"total differential of y"** (or simply the "differential of y"). It provides an approximation **(including the appropriate sign)** for the increment, δy, in y subject to an increment of δx in x.

(ii) It is important **not** to use the word "differential" when referring to a "derivative". Rather, the correct alternative to "derivative" is "differential coefficient".

(iii) A more rigorous approach to the calculation of δy is to use the result known as "Taylor's Theorem" (see Unit 11.5) which, in this context, would give the formula

$$f(x+\delta x) = f(x) + f'(x)\delta x + \frac{f''(x)}{2!}(\delta x)^2 + \frac{f'''(x)}{3!}(\delta x)^3 + \ldots .$$

Hence, if δx is small enough for powers of two and above to be neglected, then

$$f(x+\delta x) - f(x) \simeq f'(x)\delta x$$

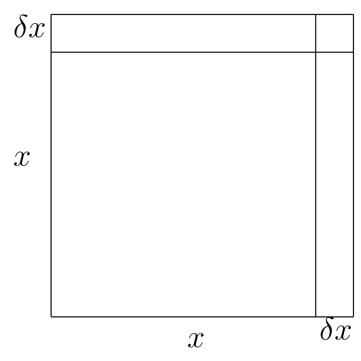
to the first order of approximation.

EXAMPLES

1. If a square has side $x$cms., determine both the exact and the approximate values of the increment in the area Acms^2. when x is increased by δx.

   Solution

   (a) Exact Method



The area is given by the formula

$$A = x^2.$$

If x increases by δx, then the increase, δA, in A may be obtained from the formula

$$A + \delta A = (x + \delta x)^2 = x^2 + 2x\delta x + (\delta x)^2.$$

That is,

$$\delta A = 2x\delta x + (\delta x)^2.$$

(b) Approximate Method

Here, we use

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

to give

$$\delta A \simeq 2x\delta x;$$

and we observe from the diagram that the two results differ only by the area of the small square, with side δx.

2. If

$$y = xe^{-x},$$

calculate, approximately, the change in y when x increases from 5 to 5.03.

Solution

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}(1 - x),$$

so that

$$\delta y \simeq e^{-x}(1 - x)\delta x,$$

where $x = 5$ and $\delta x = 0.3$.

Hence,

$$\delta y \simeq e^{-5}.(1 - 5).(0.3) \simeq -0.00809,$$

showing a **decrease** of 0.00809 in y.

We may compare this with the exact value which is given by

$$\delta y = 5.3e^{-5.3} - 5e^{-5} \simeq -0.00723$$

3. If

$$y = xe^{-x},$$

determine, in terms of x, the percentage change in y when x is increased by 2%.

Solution

Once again, we have

$$\delta y = e^{-x}(1-x)\delta x;$$

but, this time, $\delta x = 0.02x$, so that

$$\delta y = e^{-x}(1-x) \times 0.02x.$$

The **percentage** change in y is given by

$$\frac{\delta y}{y} \times 100 = \frac{e^{-x}(1-x) \times 0.02x}{xe^{-x}} \times 100 = 2(1-x).$$

That is, y increases by $2(1-x)$%, which will be positive when $x < 1$ and negative when $x > 1$.

Note:

It is usually more meaningful to discuss increments in the form of a percentage, since this gives a better idea of how much a variable has changed in porportion to its original value.

11.6.2 SMALL ERRORS

In the functional relationship

$$y = f(x),$$

let us suppose that x is known to be subject to an error in measurement; then we consider what error will be likely in the calculated value of y.

In particular, suppose x is known to be **too large** by a small amount, δx, in which case the correct value of x could be obtained if we **decreased** it by δx; or, what amounts to the same thing, if we **increased** it by $-\delta x$.

Correspondingly, the value of y will **increase** by approximately $-\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; that is, y will **decrease** by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$.

Summary

We conclude that, if x is too large by an amount δx, then y is too large by approximately $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; though, ofcourse, if $\frac{\mathrm{d}y}{\mathrm{d}x}$ itself is negative, y will be too small when x is too large and vice versa.

EXAMPLES

1. If

   $$y = x^2 \sin x,$$

   calculate, approximately, the error in y when x is measured as 3, but this measurement is subsequently discovered to be too large by 0.06.

   Solution

   We have

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \cos x + 2x \sin x$$

   and, hence,

   $$\delta y \simeq (x^2 \cos x + 2x \sin x)\delta x,$$

   where $x = 3$ and $\delta x = 0.06$.

   The error in y is therefore given approximately by

   $$\delta y \simeq (3^2 \cos 3 + 6 \sin 3) \times 0.06 \simeq -0.4838$$

   That is, y is too small by approximately 0.4838.

2. If

   $$y = \frac{x}{1+x},$$

   determine approximately, in terms of x, the percentage error in y when x is subject to an error of 5%.

   Solution

   We have

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2},$$

   so that

$$\delta y \simeq \frac{1}{(1+x)^2}\delta x,$$

where $\delta x = 0.05x$.

The **percentage** error in y is thus given by

$$\frac{\delta y}{y} \times 100 \simeq \frac{1}{(1+x)^2} \times 0.05x \times \frac{x+1}{x} \times 100 = \frac{5}{1+x}.$$

Hence, y is too large by approximately $\frac{5}{1+x}$% which will be positive when $x > -1$ and negative when $x < -1$.

11.6.3 EXERCISES

1. If
$$y = \frac{e^{2x}}{x},$$
calculate, approximately, the change in y when x is increased from 1 to 1.0025.
State your answer correct to three significant figures.

2. If
$$y = (2x+1)^5,$$
determine approximately, in terms of x, the percentage change in y when x increases by 0.1%.

3. If
$$y = x^3 \ln x,$$
calculate approximately, correct to the nearest integer, the error in y when x is measured as 4, but this measurement is subsequently discovered to be too small by 0.12.

4. If
$$y = \cos(3x^2+2),$$
determine approximately, in terms of x, the percentage error in y if x is too large by 2%.
You may assume that $3x^2+2$ lies between π and $\frac{3\pi}{2}$.

11.6.4 ANSWERS TO EXERCISES

1. y increases by approximately 0.0185.
2. y increases by approximately $\frac{x}{(2x+1)}\%$
3. y is too small by approximately 10.
4. y is too small by approximately $-12x^2\tan(3x^2+2)$.

"JUST THE MATHS"

UNIT NUMBER

12.1

INTEGRATION 1
(Elementary indefinite integrals)

by

A.J.Hobson

UNIT 12.1 - INTEGRATION 1 - ELEMENTARY INDEFINITE INTEGRALS

12.1.1 THE DEFINITION OF AN INTEGRAL

In Differential Calculus, we are given functions of x and asked to obtain their derivatives; but, in Integral Calculus, we are given functions of x and asked what they are the derivatives of. The process of answering this question is called **"integration"**.

In other words **integration is the reverse of differentiation**.

DEFINITION
Given a function $f(x)$, another function, z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x)$$

is called an integral of $f(x)$ with respect to x.

Notes:
(i) The above definition refers to **an** integral of $f(x)$ rather than **the** integral of $f(x)$. This is because, having found a possible function, z, such that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x),$$

$z + C$ is also an integral for any constant value, C.

(ii) We call $z + C$ the **"indefinite integral of $f(x)$ with respect to x"** and we write

$$\int f(x)\mathrm{d}x = z + C.$$

(iii) C is an **arbitrary constant** called the **"constant of integration"**.

(iv) The symbol $\mathrm{d}x$ does not denote a number; it is to be taken as a label indicating the variable with respect to which we are integrating. It may seem obvious that this will be x, but it could happen, for instance, that x is already dependent upon some other variable, t, in which case it would be vital to indicate the variable with respect to which we are integrating.

(v) In any integration problem, the function being integrated is called the **"integrand"**.

Result:
Two functions z_1 and z_2 are both integrals of the same function $f(x)$ if and only if they differ by a constant.

Proof:
(a) Suppose, firstly, that

$$z_1 - z_2 = C,$$

where C is a constant.

Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

That is,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} - \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\mathrm{d}z_2}{\mathrm{d}x}.$$

From our definition, this shows that both z_1 and z_2 are integrals of the same function.

(b) Secondly, suppose that z_1 and z_2 are integrals of the same function. Then

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} = \frac{\mathrm{d}z_2}{\mathrm{d}x}.$$

That is,

$$\frac{\mathrm{d}z_1}{\mathrm{d}x} - \frac{\mathrm{d}z_2}{\mathrm{d}x} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}[z_1 - z_2] = 0.$$

Hence,

$$z_1 - z_2 = C,$$

where C may be any constant.

ILLUSTRATIONS
Any result so far encountered in differentiation could be re-stated in reverse as a result on integration as shown in the following illustrations:

1.
$$\int 3x^2 \mathrm{d}x = x^3 + C.$$

2.
$$\int x^2 \mathrm{d}x = \frac{x^3}{3} + C.$$

3.

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad \text{Provided} \quad n \neq -1.$$

4.

$$\int \frac{1}{x} \mathrm{d}x \quad \text{i.e.} \quad \int x^{-1} \mathrm{d}x = \ln x + C.$$

5.

$$\int e^x \mathrm{d}x = e^x + C.$$

6.

$$\int \cos x \mathrm{d}x = \sin x + C.$$

7.

$$\int \sin x \mathrm{d}x = -\cos x + C.$$

Note:
Basic integrals of the above kinds may simply be quoted from a table of standard integrals in a suitable formula booklet. More advanced integrals are obtainable using the rules to be discussed below.

12.1.2 ELEMENTARY TECHNIQUES OF INTEGRATION

(a) Linearity

Suppose $f(x)$ and $g(x)$ are two functions of x while A and B are constants. Then,

$$\int [Af(x) + Bg(x)] \mathrm{d}x = A \int f(x) \mathrm{d}x + B \int g(x) \mathrm{d}x.$$

The proof follows from the fact that differentiation is already linear and hence the derivative of the right hand side is the integrand of the left hand side. The result itself is easily extended to linear combinations of three or more functions.

ILLUSTRATIONS

1.

$$\int (x^2 + 3x - 7) \mathrm{d}x = \frac{x^3}{3} + 3\frac{x^2}{2} - 7x + C.$$

2.
$$\int(3\cos x + 4\sec^2 x)\mathrm{d}x = 3\sin x + 4\tan x + C.$$

(b) Functions of a Linear Function

(i) Inspection Method
Provided the method of **differentiating** functions of a linear function has been fully understood, the fastest method of **integrating** such functions is to examine, by inspection, what needs to be differentiated in order to arrive at them.

EXAMPLES

1. Determine the indefinite integral
$$\int (2x+3)^{12}\mathrm{d}x.$$

Solution

In order to arrive at the function $(2x+3)^{12}$ by a differentiation process, we would have to begin with a function related to $(2x+3)^{13}$.

In fact,
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+3)^{13}\right] = 13(2x+3)^{12}.2 = 26(2x+3)^{12}.$$
Since this is 26 times the function we are trying to integrate, we may say that
$$\int (2x+3)^{12} = \frac{(2x+3)^{13}}{26} + C.$$

2. Determine the indefinite integral
$$\int \cos(3-5x)\mathrm{d}x.$$

Solution

In order to arrive at the function $\cos(3-5x)$ by a differentiation process, we would have to begin with a function related to $\sin(3-5x)$.

In fact,
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(3-5x)] = \cos(3-5x). - 5 = -5\cos(3-5x).$$
Since this is -5 times the function we are trying to integrate, we may say that
$$\int \cos(3-5x) = -\frac{\sin(3-5x)}{5} + C.$$

3. Determine the indefinite integral
$$\int e^{4x+1}\mathrm{d}x.$$

Solution

In order to arrive at the function e^{4x+1} by a differentiation process, we would have to begin with a function related to e^{4x+1} itself because the derivative of a power of e always contains the **same** power of e.

In fact
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{4x+1}\right] = e^{4x+1}.4.$$

Since this is 4 times the function we are trying to integrate, we may say that
$$\int e^{4x+1}\mathrm{d}x = \frac{e^{4x+1}}{4} + C.$$

4. Determine the indefinite integral
$$\int \frac{1}{7x+3}\mathrm{d}x.$$

Solution

In order to arrive at the function $\frac{1}{7x+3}$ by a differentiation process, we would have to begin with a function related to $\ln(7x+3)$.

In fact,
$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(7x+3)] = \frac{1}{7x+3}.7 = \frac{7}{7x+3}$$

Since this is 7 times the function we are trying to integrate, we may say that
$$\int \frac{1}{7x+3}\mathrm{d}x = \frac{\ln(7x+3)}{7} + C.$$

Note:

In each of these examples, we are essentially treating the linear function as if it were a single x, then dividing the result by the coefficient of x in that linear function.

(ii) Substitution Method

The method to be discussed here will eventually be applied to functions other than functions of a linear function; but the latter serve as a useful way of introducing the technique of **"Integration by Substitution"**.

In the integral of the form $\int f(ax+b)\mathrm{d}x$, we may substitute $u = ax+b$ proceeding as follows:

Suppose

$$z = \int f(ax+b)\mathrm{d}x.$$

Then,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(ax+b).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(u).$$

But,

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{\mathrm{d}z}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}u} = f(u).\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Hence,

$$z = \int f(u)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u.$$

Note:
The secret of this integration by substitution formula is that, apart from putting $u = ax+b$ into $f(ax+b)$, we replace the symbol $\mathrm{d}x$ with $\frac{\mathrm{d}x}{\mathrm{d}u}.\mathrm{d}u$; almost as if we had divided $\mathrm{d}x$ by $\mathrm{d}u$ then immediately multiplied by it again, though, strictly, this would not be allowed since $\mathrm{d}x$ and $\mathrm{d}u$ are not numbers.

EXAMPLES

1. Determine the indefinite integral

$$z = \int (2x+3)^{12}\mathrm{d}x.$$

Solution
Putting $u = 2x+3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$.
Thus,

$$z = \int u^{12}.\frac{1}{2}\mathrm{d}u = \frac{u^{13}}{13} \times \frac{1}{2} + C.$$

That is,

$$z = \frac{(2x+3)^{13}}{26} + C,$$

as before.

2. Determine the indefinite integral
$$z = \int \cos(3-5x)\mathrm{d}x.$$

Solution
Putting $u = 3-5x$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = -5$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{5}$.
Thus,
$$z = \int \cos u . -\frac{1}{5}\mathrm{d}u = -\frac{1}{5}\sin u + C.$$
That is,
$$z = -\frac{1}{5}\sin(3-5x) + C,$$
as before.

3. Determine the indefinite integral
$$z = \int e^{4x+1}\mathrm{d}x.$$

Solution
Putting $u = 4x+1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 4$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{4}$.
Thus,
$$z = \int e^u.\frac{1}{4}\mathrm{d}u = \frac{e^u}{4} + C.$$
That is,
$$z = \frac{e^{4x+1}}{4} + C,$$
as before.

4. Determine the indefinite integral
$$z = \int \frac{1}{7x+3}\mathrm{d}x.$$

Solution
Putting $u = 7x+3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 7$ and, hence, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{7}$
Thus,
$$z = \int \frac{1}{u}.\frac{1}{7}\mathrm{d}u = \frac{1}{7}\ln u + C.$$
That is,
$$z = \frac{1}{7}\ln(7x+3) + C,$$
as before.

12.1.3 EXERCISES

1. Integrate the following functions with respect to x:

   (a)
$$x^5;$$
   (b)
$$x^{\frac{3}{2}};$$
   (c)
$$\frac{1}{x^6};$$
   (d)
$$2x^2 - x + 5;$$
   (e)
$$x^3 - 7x^2 + x + 1.$$

2. Use a substitution of the form $u = ax + b$ in order to determine the following integrals:

   (a)
$$\int \sin(5x - 6)\mathrm{d}x;$$
   (b)
$$\int e^{2x+11}\mathrm{d}x;$$
   (c)
$$\int (3x + 2)^6\mathrm{d}x.$$

3. Write down, by inspection, the indefinite integrals with respect to x of the following functions:

   (a)
$$(1 + 2x)^{10};$$

(b)

$$e^{12x+4};$$

(c)

$$\frac{1}{3x-1};$$

(d)

$$\sin(3-5x);$$

(e)

$$\frac{9}{(4-x)^5};$$

(f)

$$\text{cosec}^2(7x+1).$$

12.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{x^6}{6}+C;$$

(b)

$$\frac{2}{5}x^{\frac{5}{2}}+C;$$

(c)

$$-\frac{1}{5x^5}+C;$$

(d)

$$\frac{2}{3}x^3-\frac{1}{2}x^2+5x+C;$$

(e)

$$\frac{1}{4}x^4-\frac{7}{3}x^3+\frac{1}{2}x^2+x+C.$$

2. (a)
$$-\frac{1}{5}\cos(5x-6)+C;$$
(b)
$$\frac{1}{2}e^{2x+11}+C;$$
(c)
$$\frac{1}{21}(3x+2)^7+C.$$
3. (a)
$$\frac{1}{22}(1+2x)^{11}+C;$$
(b)
$$\frac{1}{12}e^{12x+4}+C;$$
(c)
$$\frac{1}{3}\ln(3x-1)+C;$$
(d)
$$\frac{1}{5}\cos(3-5x)+C;$$
(e)
$$\frac{9}{4(4-x)^4}+C;$$
(f)
$$-\frac{1}{7}\cot(7x+1)+C.$$

APPENDIX - A Table of Standard Integrals

$f(x)$	$\int f(x)\ \mathrm{d}x$
a (const.)	ax
x^n	$x^{n+1}/(n+1)\ n \neq -1$
$1/x$	$\ln x$
$\sin ax$	$-(1/a)\cos ax$
$\cos ax$	$(1/a)\sin ax$
$\sec^2 ax$	$(1/a)\tan ax$
$\mathrm{cosec}^2\ ax$	$-(1/a)\cot ax$
$\sec\ ax.\tan ax$	$(1/a)\mathrm{sec}ax$
$\mathrm{cosec}\ ax.\cot ax$	$-(1/a)\mathrm{cosec}ax$
e^{ax}	$(1/a)e^{ax}$
a^x	$a^x/\ln a$
$\sinh ax$	$(1/a)\cosh ax$
$\cosh ax$	$(1/a)\sinh ax$
$\mathrm{sech}^2\ ax$	$(1/a)\tanh ax$
$\mathrm{sech}\ ax.\tanh ax$	$-(1/a)\mathrm{sech}ax$
$\mathrm{cosech}\ ax.\coth ax$	$-(1/a)\mathrm{cosech}ax$
$\cot ax$	$(1/a)\ln(\sin ax)$
$\tan ax$	$-(1/a)\ln(\cos ax)$
$\tanh ax$	$(1/a)\ln(\cosh ax)$
$\coth ax$	$(1/a)\ln(\sinh ax)$
$1/\surd(a^2-x^2)$	$\sin^{-1}(x/a)$
$1/(a^2+x^2)$	$(1/a)\tan^{-1}(x/a)$
$1/\surd(x^2+a^2)$	$\sinh^{-1}(x/a)$ or $\ln(x+\sqrt{x^2+a^2})$
$1/\surd(x^2-a^2)$	$\cosh^{-1}(x/a)$ or $\ln(x+\sqrt{x^2-a^2})$
$1/(a^2-x^2)$	$(1/a)\tanh^{-1}(x/a)$ or $\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ when $\lvert x\rvert < a$, $\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right)$ when $\lvert x\rvert > a$.

"JUST THE MATHS"

UNIT NUMBER

12.2

INTEGRATION 2
(Introduction to definite integrals)

by

A.J.Hobson

UNIT 12.2 - INTEGRATION 2
INTRODUCTION TO DEFINITE INTEGRALS

12.2.1 DEFINITION AND EXAMPLES

So far, all the integrals considered have been **"indefinite integrals"** since each result has contained an arbitrary constant which cannot be assigned a value without further information.

In practical applications of integration, however, a different kind of integral, called a **"definite integral"**, is encountered and is represented by a numerical value rather than a function plus an arbitrary constant.

Suppose that

$$\int f(x)\mathrm{d}x = g(x) + C.$$

Then the symbol

$$\int_a^b f(x)\mathrm{d}x$$

is used to mean

(Value of $g(x) + C$ at $x = b$) minus (Value of $g(x) + C$ at $x = a$).

In other words, since C will cancel out,

$$\int_a^b f(x)\mathrm{d}x = g(b) - g(a).$$

The right hand side of this statement can also be written

$$[g(x)]_a^b,$$

a notation which is used as the middle stage of a definite integral calculation.

The values a and b are known as the **"lower limit"** and **"upper limit"**, respectively, of the definite integral (even when a is larger than b).

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos x\mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1.$$

2. Evaluate the definite integral

$$\int_1^3 (2x+1)^2 \mathrm{d}x.$$

Solution

Using the method of integrating a function of a linear function, we obtain

$$\int_1^3 (2x+1)^2 \mathrm{d}x = \left[\frac{(2x+1)^3}{6}\right]_1^3 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

Notes:

(i) If we had decided to multiply out the integrand $(2x+1)^2$ before integrating, giving

$$4x^2 + 4x + 1,$$

the integration process would have yielded the expression

$$4\frac{x^3}{3} + 2x^2 + x,$$

which differs only from the previous result by the constant value $\frac{1}{6}$; students may like to check this. Hence the numerical result for the definite integral will be the same.

(ii) An alternative method of evaluating the definite integral would be to make the substitution

$$u = 2x + 1.$$

But, whenever substitution is used for definite integrals, it is not necessary to return to the original variable at the end as long as the limits of integration are changed to the appropriate values for u.

Replacing $\mathrm{d}x$ by $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$ (that is, $\frac{1}{2}\mathrm{d}u$) and the limits $x = 1$ and $x = 3$ by $u = 2 \times 1 + 1 = 3$ and $u = 2 \times 3 + 1 = 7$, respectively, we obtain

$$\int_3^7 u^2 \frac{1}{2} \mathrm{d}u = \left[\frac{u^3}{6}\right]_3^7 = \frac{7^3}{6} - \frac{3^3}{6} \simeq 52.67$$

12.2.2 EXERCISES

Evaluate the following definite integrals:

1.
$$\int_0^{\frac{\pi}{3}} \sin 4x \mathrm{d}x.$$

2.
$$\int_{-1}^{1} (x+1)^7 \mathrm{d}x.$$

3.
$$\int_0^{\frac{\pi}{4}} \sec^2(x+\pi)\ \mathrm{d}x.$$

4.
$$\int_1^2 \frac{1}{4+3x}\ \mathrm{d}x.$$

5.
$$\int_0^1 e^{1-7x}\ \mathrm{d}x.$$

12.2.3 ANSWERS TO EXERCISES

1.
$$0.375$$

2.
$$32.$$

3.
$$1.$$

4.
$$\frac{1}{3}\ln\frac{10}{7} \quad \text{or} \quad 0.119 \quad \text{approximately}$$

5.
$$0.388 \text{ approximately}$$

"JUST THE MATHS"

UNIT NUMBER

12.3

INTEGRATION 3
(The method of completing the square)

by

A.J.Hobson

UNIT 12.3 - INTEGRATION 3
THE METHOD OF COMPLETING THE SQUARE

12.3.1 INTRODUCTION AND EXAMPLES

A substitution such as $u = \alpha x + \beta$ may also be used with integrals of the form

$$\int \frac{1}{px^2+qx+r}\mathrm{d}x \quad \text{and} \quad \int \frac{1}{\sqrt{px^2+qx+r}}\mathrm{d}x,$$

where, in the first of these, we assume that the quadratic will not factorise into simple linear factors; otherwise the method of partial fractions would be used to integrate it (see Unit 12.6).

Note:
The two types of integral here are often written, for convenience, as

$$\int \frac{\mathrm{d}x}{px^2+qx+r} \quad \text{and} \quad \int \frac{\mathrm{d}x}{\sqrt{px^2+qx+r}}.$$

In order to deal with such functions, we shall need to quote standard results which may be deduced from previous ones developed in the differentiation of inverse trigonometric and hyperbolic functions.

They are as follows:

1. $$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

2. $$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{a} + C.$$

3. $$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \sinh^{-1}\frac{x}{a} + C \quad \text{or} \quad \ln(x+\sqrt{x^2+a^2}) + C.$$

4. $$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \cosh^{-1}\frac{x}{a} + C \quad \text{or} \quad \ln(x+\sqrt{x^2-a^2}) + C.$$

5.

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C;$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

EXAMPLES

1. Determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{\sqrt{x^2 + 2x - 3}}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 + 2x - 3 \equiv (x+1)^2 - 4 \equiv (x+1)^2 - 2^2.$$

Hence,

$$z = \int \frac{\mathrm{d}x}{\sqrt{(x+1)^2 - 2^2}}.$$

Putting $u = x + 1$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.

Thus,

$$z = \int \frac{\mathrm{d}u}{\sqrt{u^2 - 2^2}},$$

giving

$$z = \ln\left[u + \sqrt{u^2 - 2^2}\right] + C.$$

Returning to the variable, x, we have

$$z = \ln\left[x + 1 + \sqrt{x^2 + 2x - 3}\right] + C.$$

2. Evaluate the definite integral

$$z = \int_3^7 \frac{\mathrm{d}x}{x^2 - 6x + 25}.$$

Solution

Completing the square in the quadratic expression gives

$$x^2 - 6x + 25 \equiv (x-3)^2 + 16.$$

Hence,

$$z = \int_3^7 \frac{\mathrm{d}x}{(x-3)^2+16}.$$

Putting $u = x - 3$, we obtain $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$; and so $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$.

Thus,

$$z = \int_0^4 \frac{\mathrm{d}u}{u^2+16},$$

giving

$$z = \left[\frac{1}{4}\tan^{-1}\frac{u}{4}\right]_0^4 = \frac{\pi}{16}.$$

Alternatively, without changing the original limits of integration,

$$z = \left[\frac{1}{4}\tan^{-1}\frac{x-3}{4}\right]_3^7.$$

Note:

In cases like the two examples discussed above, when $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and therefore $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$, it seems pointless to go through the laborious process of actually **making** the substitution in detail. All we need to do is to treat the linear expression within the completed square as if it were a single x, then write the result straight down !

12.3.2 EXERCISES

1. Use a table of standard integrals to write down the indefinite integrals of the following functions:

   (a)

   $$\frac{1}{\sqrt{4-x^2}};$$

   (b)

   $$\frac{1}{9+x^2};$$

(c)

$$\frac{1}{\sqrt{x^2-7}}.$$

2. By completing the square, evaluate the following definite integrals:

(a)

$$\int_{-1}^{\sqrt{3}-1} \frac{\mathrm{d}x}{x^2+2x+2};$$

(b)

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}}.$$

12.3.3 ANSWERS TO EXERCISES

1. (a)

$$\sin^{-1}\frac{x}{2}+C;$$

(b)

$$\frac{1}{3}\tan^{-1}\frac{x}{3}+C;$$

(c)

$$\ln(x+\sqrt{x^2-7})+C.$$

2. (a)

$$\left[\tan^{-1}(x+1)\right]_{-1}^{\sqrt{3}-1}=\frac{\pi}{3};$$

(b)

$$\left[\sin^{-1}\frac{x+1}{2}\right]_0^1=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}.$$

"JUST THE MATHS"

UNIT NUMBER

12.4

INTEGRATION 4
(Integration by substitution in general)

by

A.J.Hobson

UNIT 12.4 - INTEGRATION 4
INTEGRATION BY SUBSTITUTION IN GENERAL

12.4.1 EXAMPLES USING THE STANDARD FORMULA

With any integral

$$\int f(x)\mathrm{d}x,$$

it may be convenient to make some kind of substitution relating the variable, x, to a new variable, u. In such cases, we may use the formula discussed in Unit 12.1, namely

$$\int f(x)\mathrm{d}x = \int f(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u,$$

where it is assumed that, on the right hand side, the integrand has been expressed wholly in terms of u.

For this Unit, substitutions other than linear ones will be given in the problems to be solved.

EXAMPLES

1. Use the substitution $x = a\sin u$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

Solution

To be precise, we shall assume for simplicity that u is the **acute** angle for which $x = a\sin u$. In effect, we shall be making the substitution $u = \sin^{-1}\frac{x}{a}$ using the principal value of the inverse function; we can certainly do this because the expression $\sqrt{a^2 - x^2}$ requires that $-a < x < a$.

If $x = a\sin u$, then $\frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$, so that the integral becomes

$$\int \frac{a\cos u}{\sqrt{a^2 - a^2\sin^2 u}}\mathrm{d}u.$$

But, from trigonometric identities,

$$\sqrt{a^2 - a^2\sin^2 u} \equiv a\cos u,$$

both sides being positive when u is an acute angle.

We are thus left with

$$\int 1\mathrm{d}u = u + C = \sin^{-1}\frac{x}{a} + C.$$

2. Use the substitution $u = \frac{1}{x}$ to determine the indefinite integral

$$z = \int \frac{\mathrm{d}x}{x\sqrt{1+x^2}}.$$

Solution

Converting the substitution to the form

$$x = \frac{1}{u},$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}.$$

Hence,

$$z = \int \frac{1}{\frac{1}{u}\sqrt{1+\frac{1}{u^2}}} . -\frac{1}{u^2}\mathrm{d}u$$

That is,

$$z = \int -\frac{1}{\sqrt{u^2+1}} = -\ln(u + \sqrt{u^2+1}) + C.$$

Returning to the original variable, x, we have

$$z = -\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right) + C.$$

Note:

This example is somewhat harder than would be expected under examination conditions.

12.4.2 INTEGRALS INVOLVING A FUNCTION AND ITS DERIVATIVE

The method of integration by substitution provides two useful results applicable to a wide range of problems. They are as follows:

(a)

$$\int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1} + C$$

provided $n \neq -1$.

(b)

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln f(x) + C.$$

These two results are readily established by means of the substitution

$$u = f(x).$$

In both cases $\frac{\mathrm{d}u}{\mathrm{d}x} = f'(x)$ and hence $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{f'(x)}$. This converts the integrals, respectively, into

(a)

$$\int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + C$$

and (b)

$$\int \frac{1}{u} \mathrm{d}u = \ln u + C.$$

EXAMPLES

1. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \sin^3 x.\cos x \ \mathrm{d}x.$$

   Solution
   In this example we can consider $\sin x$ to be $f(x)$ and $\cos x$ to be $f'(x)$.
   Thus, by quoting result (a), we obtain

$$\int_0^{\frac{\pi}{3}} \sin^3 x.\cos x \mathrm{d}x = \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{3}} = \frac{9}{64},$$

   using $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

2. Integrate the function

$$\frac{2x+1}{x^2+x-11}$$

   with respect to x.
   Solution
   Here, we can identify $x^2 + x - 11$ with $f(x)$ and $2x + 1$ with $f'(x)$.
   Thus, by quoting result (b), we obtain

$$\int \frac{2x+1}{x^2+x-11} \mathrm{d}x = \ln(x^2 + x - 11) + C.$$

12.4.3 EXERCISES

1. Use the substitution $u = x + 3$ in order to determine the indefinite integral

$$\int x\sqrt{3+x}\ \mathrm{d}x.$$

2. Use the substitution $u = x^2 - 1$ in order to evaluate the definite integral

$$\int_1^5 x\sqrt{x^2-1}\ \mathrm{d}x.$$

3. Integrate the following functions with respect to x:

(a)

$$\sin^7 x.\cos x;$$

(b)

$$\cos^5 x.\sin x;$$

(c)

$$\frac{4x-3}{2x^2-3x+13};$$

(d)

$$\cot x.$$

12.4.4 ANSWERS TO EXERCISES

1.

$$\frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.$$

2.

$$\left[\frac{1}{3}(x^2-1)^{\frac{3}{2}}\right]_1^5 = \frac{1}{3}24^{\frac{3}{2}} \simeq 39.192$$

3. (a)

$$\frac{\sin^8 x}{8} + C;$$

(b)

$$-\frac{\cos^6 x}{6} + C;$$

(c)

$$\ln(2x^2 - 3x + 13) + C;$$

(d)

$$\ln \sin x + C.$$

"JUST THE MATHS"

UNIT NUMBER

12.5

INTEGRATION 5
(Integration by parts)

by

A.J.Hobson

UNIT 12.5 - INTEGRATION 5

INTEGRATION BY PARTS

12.5.1 THE STANDARD FORMULA

The technique to be discussed here provides a convenient method for integrating the product of two functions. However, it is possible to develop a suitable formula by considering, instead, the **derivative** of the product of two functions.

We consider, first, the following comparison:

$\frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] = x\cos x + \sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}[uv] = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
$x\cos x = \frac{\mathrm{d}}{\mathrm{d}x}[x\sin x] - \sin x$	$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[uv] - v\frac{\mathrm{d}u}{\mathrm{d}x}$
$\int\ x\cos x\ \mathrm{d}x = x\sin x - \int\ \sin x\ \mathrm{d}x$	$\int\ u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int\ v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$
$= x\sin x + \cos x + C$	

We see that, by labelling the product of two given functions as $u\frac{\mathrm{d}v}{\mathrm{d}x}$, we may express the integral of this product in terms of another integral which, it is anticipated, will be simpler than the original.

To summarise, the formula for **"integration by parts"** is

$$\int\ u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = uv - \int\ v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x.$$

EXAMPLES

1. Determine

$$I = \int\ x^2e^{3x}\ \mathrm{d}x.$$

Solution

In theory, it does not matter which element of the product x^2e^{3x} is labelled as u and which is labelled as $\frac{\mathrm{d}v}{\mathrm{d}x}$; but the integral obtained on the right-hand-side of the integration by parts formula must be simpler than the original.

In this case we shall take

$$u = x^2 \ \text{ and } \ \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Hence,

$$I = x^2\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2x \ \mathrm{d}x.$$

That is,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x} \ \mathrm{d}x.$$

The integral on the right-hand-side still contains the product of two functions and so we must use integration by parts a second time, setting

$$u = x \ \ \text{and} \ \ \frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}.$$

Thus,

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.1 \ \mathrm{d}x\right].$$

The integration may now be completed to obtain

$$I = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C,$$

or

$$I = \frac{e^{3x}}{27}\left[9x^2 - 6x + 2\right] + C.$$

2. Determine

$$I = \int x \ln x \ \mathrm{d}x.$$

Solution

In this case, we cannot effectively choose $\frac{\mathrm{d}v}{\mathrm{d}x} = \ln x$ since we would need to know the integral of $\ln x$ in order to find v. Hence, we choose

$$u = \ln x \ \ \text{and} \ \ \frac{\mathrm{d}v}{\mathrm{d}x} = x,$$

obtaining

$$I = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x}\ \mathrm{d}x.$$

That is,

$$I = \frac{1}{2}x^2 \ln x - \int \frac{x}{2}\ \mathrm{d}x,$$

giving

$$I = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C.$$

3. Determine

$$I = \int \ln x\ \mathrm{d}x.$$

Solution
It is possible to regard this as an integration by parts problem if we set

$$u = \ln x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = x \ln x - \int x.\frac{1}{x}\ \mathrm{d}x,$$

giving

$$I = x \ln x - x + C.$$

4. Evaluate

$$I = \int_0^1 \sin^{-1}x\ \mathrm{d}x.$$

Solution
In a similar way to the previous example, it is possible to regard this as an integration by parts problem if we set

$$u = \sin^{-1}x \ \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = 1.$$

We obtain

$$I = [x\sin^{-1}x]_0^1 - \int_0^1 x.\frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x.$$

That is,

$$I = [x\sin^{-1}x + \sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1.$$

5. Determine

$$I = \int e^{2x}\cos x\ \mathrm{d}x.$$

Solution

In this example, it makes little difference whether we choose e^{2x} or $\cos x$ to be u; but we shall set

$$u = e^{2x} \ \text{ and } \ \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x.$$

Hence,

$$I = e^{2x}\sin x - \int (\sin x).2e^{2x}\ \mathrm{d}x.$$

That is,

$$I = e^{2x}\sin x - 2\int e^{2x}\sin x\ \mathrm{d}x.$$

Now we need to integrate by parts again, setting

$$u = e^{2x} \ \text{ and } \ \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x.$$

Therefore,

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x - \int (-\cos x).2e^{2x}\ \mathrm{d}x\right].$$

In other words, the original integral has appeared again on the right hand side to give

$$I = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2I\right].$$

On simplification,

$$5I = e^{2x}\sin x + 2e^{2x}\cos x,$$

so that

$$I = \frac{1}{5}e^{2x}[\sin x + 2\cos x] + C.$$

Note:

The above examples suggest a priority order for choosing u in a typical integration by parts problem. For example, if the product to be integrated contains a logarithm or an inverse function, then we must choose the logarithm or the inverse function as u; but if there are powers of x without logarithms or inverse functions, then we choose the power of x to be u.

The order of priorities is as follows:

1. LOGARITHMS or INVERSE FUNCTIONS;
2. POWERS OF x;
3. POWERS OF e.

12.5.2 EXERCISES

1. Use integration by parts to evaluate the definite integral

$$\int_0^1 x^3 e^{2x}\ \mathrm{d}x.$$

2. Use integration by parts to integrate the following functions with respect to x:

   (a)

$$x^2\cos 2x;$$

   (b)

$$x^5\ln x;$$

   (c)

$$\tan^{-1}x;$$

(d)

$$x\tan^{-1}x.$$

3. Use integration by parts to evaluate the definite integral

$$\int_0^{\pi} e^{-2x} \sin 3x \ \mathrm{d}x.$$

12.5.3 ANSWERS TO EXERCISES

1.

$$\left[\frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3)\right]_0^1 = \frac{1}{8}\left(e^2 + 3\right) \simeq 1.299$$

2. (a)

$$\frac{1}{4}[2x^2 \sin 2x + 2x \cos 2x - \sin 2x] + C;$$

(b)

$$\frac{x^6}{36}[6 \ln x - 1] + C;$$

(c)

$$x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C;$$

(d)

$$\frac{1}{2}[x^2\tan^{-1}x - x + \tan^{-1}x] + C.$$

3.

$$\left[\frac{e^{-2x}}{13}(3\cos 3x - 2\sin 3x)\right]_0^{\pi} = -\frac{3}{13}(e^{-2\pi} + 1) \simeq -0.231$$

"JUST THE MATHS"

UNIT NUMBER

12.6

INTEGRATION 6
(Integration by partial fractions)

by

A.J.Hobson

UNIT 12.6 - INTEGRATION 6
INTEGRATION BY PARTIAL FRACTIONS

12.6.1 INTRODUCTION AND ILLUSTRATIONS

If the ratio of two polynomials, whose denominator has been factorised, is expressed as a sum of partial fractions, each partial fraction will be of a type whose integral can be determined by the methods of preceeding sections of this chapter.

The following summary of results will cover most elementary problems involving partial fractions:

RESULTS

1.

$$\int \frac{1}{ax+b}\ \mathrm{d}x = \frac{1}{a}\ln(ax+b) + C.$$

2.

$$\int \frac{1}{(ax+b)^n}\ \mathrm{d}x = \frac{1}{a}.\frac{(ax+b)^{-n+1}}{-n+1} + C \quad \text{provided} \quad n \neq 1.$$

3.

$$\int \frac{1}{a^2+x^2}\ \mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

4.

$$\int \frac{1}{a^2-x^2}\ \mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C,$$

or

$$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C \quad \text{when} \quad |x| < a,$$

and

$$\frac{1}{2a}\ln\left(\frac{x+a}{x-a}\right) + C \quad \text{when} \quad |x| > a.$$

5.

$$\int \frac{2ax+b}{ax^2+bx+c}\ \mathrm{d}x = \ln(ax^2+bx+c) + C.$$

ILLUSTRATIONS

We use some of the results of examples on partial fractions in Unit 1.8

1.

$$\int \frac{7x+8}{(2x+3)(x-1)}\,\mathrm{d}x = \int \left[\frac{1}{2x+3} + \frac{3}{x-1}\right]\,\mathrm{d}x$$

$$= \frac{1}{2}\ln(2x+3) + 3\ln(x-1) + C.$$

2.

$$\int_6^8 \frac{3x^2+9}{(x-5)(x^2+2x+7)}\,\mathrm{d}x = \int_6^8 \left[\frac{2}{x-5} + \frac{x+1}{x^2+2x+7}\right]\,\mathrm{d}x$$

$$= \left[2\ln(x-5) + \frac{1}{2}\ln(x^2+2x+7)\right]_6^8 \simeq 2.427$$

3.

$$\int \frac{9}{(x+1)^2(x-2)} = \int \left[\frac{-1}{x+1} - \frac{3}{(x+1)^2} + \frac{1}{x-2}\right]\,\mathrm{d}x$$

$$= -\ln(x+1) + \frac{3}{x+1} + \ln(x-2) + C.$$

4.

$$\int \frac{4x^2+x+6}{(x-4)(x^2+4x+5)}\,\mathrm{d}x = \int \left[\frac{2}{x-4} + \frac{2x+1}{x^2+4x+5}\right]\mathrm{d}x$$

$$= 2\ln(x-4) + \ln(x^2+4x+5) - 3\tan^{-1}(x+2) + C.$$

Note:
In the last example above, the second partial fraction has a numerator of $2x+1$ which is not the derivative of x^2+4x+5. But we simply rearrange the numerator as $(2x+4)-3$ to give a third integral which requires the technique of completing the square (discussed in Unit 12.3).

12.6.2 EXERCISES

Integrate the following functions with respect to x:

1. (a)
$$\frac{3x+5}{(x+1)(x+2)};$$
(b)
$$\frac{17x+11}{(x+1)(x-2)(x+3)};$$
(c)
$$\frac{3x^2-8}{(x-1)(x^2+x-7)}.$$
(d)
$$\frac{2x+1}{(x+2)^2(x-3)};$$
(e)
$$\frac{9+11x-x^2}{(x+1)^2(x+2)};$$
(f)
$$\frac{x^5}{(x+2)(x-4)}.$$

2. Evaluate the following definite integrals

(a)
$$\int_2^5 \frac{7x^2+11x+47}{(x-1)(x^2+2x+10)}\,\mathrm{d}x;$$
(b)
$$\int_1^3 \frac{4x^2+1}{x(2x-1)^2}\,\mathrm{d}x.$$

12.6.3 ANSWERS TO EXERCISES

1. (a)
$$2\ln(x+1)+\ln(x+2)+C;$$
(b)
$$\ln(x+1)+3\ln(x-2)-4\ln(x+3)+C;$$
(c)
$$\ln(x-1)+\ln(x^2+x-7)+C;$$
(d)
$$-\frac{3}{5(x+2)}-\frac{7}{25}\ln(x+2)+\frac{7}{25}\ln(x-3)+C;$$
(e)
$$-\frac{3}{(x+1)^2}+\frac{16}{x+1}-\frac{17}{x+2}$$
$$\frac{3}{x+1}+16\ln(x+1)-17\ln(x+2)+C;$$
(f)
$$\frac{x^4}{4}+\frac{2x^3}{3}+6x^2+40x+\frac{16}{3}\ln(x+2)+\frac{512}{3}\ln(x-4)+C.$$

2. (a)
$$\left[5\ln(x-1)+\ln(x^2+2x+10)+\frac{1}{3}\tan^{-1}\frac{x+1}{3}\right]_2^5\simeq -2.726;$$
(b)
$$\left[\ln x-\frac{2}{2x-1}\right]_1^3\simeq 2.699$$

"JUST THE MATHS"

UNIT NUMBER

12.7

INTEGRATION 7
(Further trigonometric functions)

by

A.J.Hobson

UNIT 12.7 - INTEGRATION 7 - FURTHER TRIGONOMETRIC FUNCTIONS

12.7.1 PRODUCTS OF SINES AND COSINES

In order to integrate the product of a sine and a cosine, or two cosines, or two sines, we may use one of the following trigonometric identities:

$$\text{sinAcosB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B})+\sin(\text{A}-\text{B})\right];$$

$$\text{cosAsinB} \equiv \frac{1}{2}\left[\sin(\text{A}+\text{B})-\sin(\text{A}-\text{B})\right];$$

$$\text{cosAcosB} \equiv \frac{1}{2}\left[\cos(\text{A}+\text{B})+\cos(\text{A}-\text{B})\right];$$

$$\text{sinAsinB} \equiv \frac{1}{2}\left[\cos(\text{A}-\text{B})-\cos(\text{A}+\text{B})\right].$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin 2x \cos 5x \, \mathrm{d}x.$$

Solution

$$\int \sin 2x \cos 5x \, \mathrm{d}x = \frac{1}{2}\int [\sin 7x - \sin 3x] \, \mathrm{d}x$$

$$= -\frac{\cos 7x}{14} + \frac{\cos 3x}{6} + C.$$

2. Determine the indefinite integral

$$\int \sin 3x \sin x \, \mathrm{d}x.$$

Solution

$$\int \sin 3x \sin x \, \mathrm{d}x = \frac{1}{2}\int [\cos 2x - \cos 4x] \, \mathrm{d}x$$

$$= \frac{\sin 2x}{4} - \frac{\sin 4x}{8} + C.$$

12.7.2 POWERS OF SINES AND COSINES

In this section, we consider the two integrals,

$$\int \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int \cos^n \, \mathrm{d}x,$$

where n is a positive integer.

(a) The Complex Number Method

A <u>single</u> method which will cover <u>both</u> of the above integrals requires us to use the methods of Unit 6.5 in order to express $\cos^n x$ and $\sin^n x$ as a sum of whole multiples of sines or cosines of whole multiples of x.

EXAMPLE

Determine the indefinite integral

$$\int \sin^4 x \, \mathrm{d}x.$$

Solution

By the complex number method,

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

The Working:

$$j^4 2^4 \sin^4 x \equiv \left(z - \frac{1}{z}\right)^4,$$

where $z \equiv \cos x + j\sin x$.

That is,

$$16\sin^4 x \equiv z^4 - 4z^3.\frac{1}{z} + 6z^2.\left(\frac{1}{z}\right)^2 - 4z.\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4;$$

or, after cancelling common factors,

$$16\sin^4 x \equiv z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \equiv z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6,$$

which gives

$$16\sin^4 x \equiv 2\cos 4x - 8\cos 2x + 6,$$

or

$$\sin^4 x \equiv \frac{1}{8}[\cos 4x - 4\cos 2x + 3].$$

Hence,

$$\int \sin^4 x \, \mathrm{d}x = \frac{1}{8}\left[\frac{\sin 4x}{4} - 4\frac{\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{32}[\sin 4x - 8\sin 2x + 12x] + C.$$

(b) Odd Powers of Sines and Cosines

The following method uses the facts that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = \sin x.$$

We illustrate with examples in which use is made of the trigonometric identity

$$\cos^2 A + \sin^2 A \equiv 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^3 x \, \mathrm{d}x.$$

Solution

$$\int \sin^3 x \ dx = \int \sin^2 x . \sin x \ dx.$$

That is,

$$\int \sin^3 x dx = \int \left(1 - \cos^2 x\right) \sin x \ dx$$

$$= \int \left(\sin x - \cos^2 x . \sin x\right) \ dx$$

$$= -\cos x + \frac{\cos^3 x}{3} + C.$$

2. Determine the indefinite integral

$$\int \cos^7 x \ dx.$$

Solution

$$\int \cos^7 x \ dx = \int \cos^6 x . \cos x \ dx.$$

That is,

$$\int \cos^7 x \ dx = \int \left(1 - \sin^2 x\right)^3 . \cos x \ dx$$

$$= \int \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right) . \cos x \ dx$$

$$= \sin x - \sin^3 x + 3. \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

(c) Even Powers of Sines and Cosines

The method illustrated here becomes tedious if the even power is higher than 4. In such cases, it is best to use the complex number method in paragraph (a) above.

In the examples which follow, we shall need the trigonometric identity

$$\cos 2A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \sin^2 x \ \mathrm{d}x.$$

Solution

$$\int \sin^2 x \ \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x) \ \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \ \mathrm{d}x.$$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 x \ \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \ \mathrm{d}x$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

3. Determine the indefinite integral

$$\int \cos^4 x \ \mathrm{d}x.$$

Solution

$$\int \cos^4 x \ \mathrm{d}x = \int \left[\cos^2 x\right]^2 \ \mathrm{d}x = \int \left[\frac{1}{2}(1 + \cos 2x)\right]^2 \ \mathrm{d}x.$$

That is,

$$\int \cos^4 x \ \mathrm{d}x = \int \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right) \ \mathrm{d}x$$

$$= \int \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2}[1 + \cos 4x]\right) \ \mathrm{d}x$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C.$$

12.7.3 EXERCISES

1. Determine the indefinite integral

$$\int \cos x \cos 3x \ \mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{3}} \cos 4x \sin 2x \ \mathrm{d}x.$$

3. Determine the following indefinite integrals:

(a)

$$\int \sin^5 x \ \mathrm{d}x;$$

(b)

$$\int \cos^3 x \ \mathrm{d}x.$$

4. Evaluate the following definite integrals:

(a)

$$\int_0^{\frac{\pi}{8}} \sin^4 x \ \mathrm{d}x;$$

(b)

$$\int_0^{\frac{\pi}{2}} \cos^6 x \ \mathrm{d}x.$$

12.7.4 ANSWERS TO EXERCISES

1.

$$\frac{\sin 4x}{8} + \frac{\sin 2x}{4} + C.$$

2.

$$-\frac{\sqrt{3}}{4} \simeq -0.433$$

3. (a)

$$-\frac{\cos^5 x}{5} + 2\frac{\cos^3 x}{3} - \cos x + C,$$

or

$$\frac{1}{16}\left[-\frac{\cos 5x}{5} + \frac{5\cos 3x}{3} - 10\cos x\right] + C \quad \text{by complex numbers;}$$

(b)

$$\sin x - \frac{\sin^3 x}{3} + C,$$

or

$$\frac{1}{4}\left[\frac{\sin 3x}{3} - 3\sin x\right] + C \quad \text{by complex numbers.}$$

4. (a)

$$1.735 \times 10^{-3} \text{ approx;}$$

(b)

$$-1.$$

"JUST THE MATHS"

UNIT NUMBER

12.8

INTEGRATION 8
(The tangent substitutions)

by

A.J.Hobson

UNIT 12.8 - INTEGRATION 8

THE TANGENT SUBSTITUTIONS

There are two types of integral, involving sines and cosines, which require a special substitution using a tangent function. They are described as follows:

12.8.1 THE SUBSTITUTION $t = \tan x$

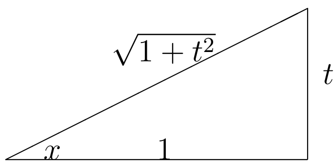
This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin^2 x + c\cos^2 x}\,\mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, at least one of these three constants will be zero.

A simple right-angled triangle will show that, if $t = \tan x$, then

$$\sin x \equiv \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos x \equiv \frac{1}{\sqrt{1+t^2}}.$$



Furthermore,

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \sec^2 x \equiv 1 + t^2 \quad \text{so that} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{1}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{4 - 3\sin^2 x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4 - 3\sin^2 x}\ \mathrm{d}x$$

$$= \int \frac{1}{4 - \frac{3t^2}{1+t^2}}.\frac{1}{1+t^2}\ \mathrm{d}t$$

$$= \int \frac{1}{4+t^2}\ \mathrm{d}t$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\ \mathrm{d}x.$$

Solution

$$\int \frac{1}{\sin^2 x + 9\cos^2 x}\ \mathrm{d}x$$

$$= \int \frac{1}{\frac{t^2}{1+t^2} + \frac{9}{1+t^2}}.\frac{1}{1+t^2}\ \mathrm{d}t$$

$$= \int \frac{1}{t^2+9}\ \mathrm{d}t$$

$$= \frac{1}{3}\tan^{-1}\frac{t}{3} + C = \frac{1}{3}\tan^{-1}\left[\frac{\tan x}{3}\right] + C.$$

12.8.2 THE SUBSTITUTION $t = \tan(x/2)$

This substitution is used for integrals of the form

$$\int \frac{1}{a + b\sin x + c\cos x}\ \mathrm{d}x,$$

where a, b and c are constants; though, in most exercises, one or more of these constants will be zero.

In order to make the substitution, we make the following observations:

(i)

$$\sin x \equiv 2\sin(x/2).\cos(x/2) \equiv 2\tan(x/2).\cos^2(x/2) \equiv \frac{2\tan(x/2)}{\sec^2(x/2)} \equiv \frac{2\tan(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\sin x \equiv \frac{2t}{1+t^2}.$$

(ii)

$$\cos x \equiv \cos^2(x/2) - \sin^2(x/2) \equiv \cos^2(x/2)\left[1-\tan^2(x/2)\right] \equiv \frac{1-\tan^2(x/2)}{\sec^2(x/2)} \equiv \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}.$$

Hence,

$$\cos x \equiv \frac{1-t^2}{1+t^2}.$$

(iii)

$$\frac{\mathrm{d}t}{\mathrm{d}x} \equiv \frac{1}{2}\sec^2(x/2) \equiv \frac{1}{2}\left[1+\tan^2(x/2)\right] \equiv \frac{1}{2}[1+t^2].$$

Hence,

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{2}{1+t^2}.$$

EXAMPLES

1. Determine the indefinite integral

$$\int \frac{1}{1+\sin x}\ \mathrm{d}x$$

Solution

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{1+t^2+2t}\,\mathrm{d}t$$

$$= \int \frac{2}{(1+t)^2}\,\mathrm{d}t$$

$$= -\frac{2}{1+t} + C = -\frac{2}{1+\tan(x/2)} + C.$$

2. Determine the indefinite integral

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x.$$

Solution

$$\int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x$$

$$= \int \frac{1}{4\frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

$$= \int \frac{2}{4-4t^2-6t}\,\mathrm{d}t = \int -\frac{1}{2t^2+3t-2}\,\mathrm{d}t$$

$$= \int -\frac{1}{(2t-1)(t+2)}\,\mathrm{d}t$$

$$= \int \frac{1}{5}\left[\frac{1}{t+2} - \frac{2}{2t-1}\right]\mathrm{d}t$$

$$= \frac{1}{5}[\ln(t+2) - \ln(2t-1)] + C = \frac{1}{5}\ln\left[\frac{\tan(x/2)+2}{2\tan(x/2)-1}\right] + C.$$

12.8.3 EXERCISES

1. Determine the indefinite integral

$$\int \frac{1}{4 + 12\cos^2 x} \, \mathrm{d}x.$$

2. Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x + 3\sin^2 x} \, \mathrm{d}x.$$

3. Determine the indefinite integral

$$\int \frac{1}{5 + 3\cos x} \, \mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_3^{3.1} \frac{1}{12\sin x + 5\cos x} \, \mathrm{d}x.$$

12.8.4 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\tan^{-1}\left[\frac{\tan x}{2}\right] + C.$$

2.

$$\left[\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)\right]_0^{\frac{\pi}{4}} \simeq 0.1702$$

3.

$$\frac{1}{2}\tan^{-1}\left[\frac{\tan(x/2)}{2}\right] + C.$$

4.

$$\left[\frac{1}{13}[5\ln(5\tan(x/2) + 1) - \ln(\tan(x/2) - 5)\right]_3^{3.1} \simeq 0.348$$

"JUST THE MATHS"

UNIT NUMBER

12.9

INTEGRATION 9
(Reduction formulae)

by

A.J.Hobson

UNIT 12.9 - INTEGRATION 9

REDUCTION FORMULAE

INTRODUCTION

For certain integrals, both definite and indefinite, the function being integrated (that is, the "integrand") consists of a product of two functions, one of which involves an unspecified integer, say n. Using the method of integration by parts, it is sometimes possible to express such an integral in terms of a similar integral where n has been replaced by $(n-1)$, or sometimes $(n-2)$. The relationship between the two integrals is called a **"reduction formula"** and, by repeated application of this formula, the original integral may be determined in terms of n.

12.9.1 INDEFINITE INTEGRALS

The method will be illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n e^x \, \mathrm{d}x$$

and, hence, determine I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^x$, we obtain

$$I_n = x^n e^x - \int e^x . n x^{n-1} \, \mathrm{d}x.$$

That is,

$$I_n = x^n e^x - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = x^3 e^x - 3I_2,$$

where

$$I_2 = x^2 e^x - 2I_1$$

and

$$I_1 = xe^x - I_0.$$

But

$$I_0 = \int e^x \, \mathrm{d}x = e^x + \text{constant},$$

which leads us to the conclusion that

$$I_3 = x^3 e^x - 3\left[x^2 e^x - 2\left(xe^x - e^x\right)\right] + \text{constant}.$$

In other words,

$$I_3 = e^x\left[x^3 - 3x^2 + 6x - 6\right] + C,$$

where C is an arbitrary constant.

2. Obtain a reduction formula for the indefinite integral

$$I_n = \int x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

Using integration by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = x^n \sin x - \int \sin x . nx^{n-1} \, \mathrm{d}x = x^n \sin x - n \int x^{n-1} \sin x \, \mathrm{d}x.$$

Using integration by parts in this last integral, with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$, we obtain

$$I_n = x^n \sin x - n\left\{-x^{n-1}\cos x + \int \cos x.(n-1)x^{n-2} \, \mathrm{d}x\right\}.$$

That is,

$$I_n = x^n \sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = x^2 \sin x + 2x\cos x - 2I_0,$$

where

$$I_0 = \int \cos x \, dx = \sin x + \text{constant}.$$

Hence,

$$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x + C,$$

where C is an arbitrary constant.
Also, substituting $n = 3$,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 3.2.I_1,$$

where

$$I_1 = \int x \cos x \, dx = x \sin x + \cos x + \text{constant}.$$

Therefore,

$$I_3 = x^3 \sin x - 3x^2 \cos x - 6x \sin x - 6 \cos x + D,$$

where D is an arbitrary constant.

12.9.2 DEFINITE INTEGRALS

Integrals of the type encountered in the previous section may also include upper and lower limits of integration. The process of finding a reduction formula is virtually the same, except that the limits of integration are inserted where appropriate. Again, the method is illustrated by examples.

EXAMPLES

1. Obtain a reduction formula for the definite integral

$$I_n = \int_0^1 x^n e^x \, dx$$

and, hence, determine I_3.

Solution

From the first example in section 12.9.1,

$$I_n = [x^n e^x]_0^1 - nI_{n-1} = e - nI_{n-1}.$$

Substituting $n = 3$,

$$I_3 = e - 3I_2,$$

where

$$I_2 = e - 2I_1$$

and

$$I_1 = e - I_0.$$

But

$$I_0 = \int_0^1 e^x \, \mathrm{d}x = e - 1,$$

which leads us to the conclusion that

$$I_3 = e - 3e + 6e - 6e + 6 = 6 - 2e.$$

2. Obtain a reduction formula for the definite integral

$$I_n = \int_0^\pi x^n \cos x \, \mathrm{d}x$$

and, hence, determine I_2 and I_3.

Solution

From the second example in section 12.9.1,

$$I_n = \left[x^n \sin x + nx^{n-1} \cos x\right]_0^\pi - n(n-1)I_{n-2} = -n\pi^{n-1} - n(n-1)I_{n-2}.$$

Substituting $n = 2$,

$$I_2 = -2\pi - 2I_0,$$

where

$$I_0 = \int_0^\pi \cos x \, \mathrm{d}x = [\sin x]_0^\pi = 0.$$

Hence,

$$I_2 = -2\pi.$$

Also, substituting $n = 3$,

$$I_3 = -3\pi^2 - 3.2.I_1,$$

where

$$I_1 = \int_0^\pi x \cos x \ \mathrm{d}x = [x \sin x + \cos x]_0^\pi = -2.$$

Therefore,

$$I_3 = -3\pi^2 + 12.$$

12.9.3 EXERCISES

1. Obtain a reduction formula for

$$I_n = \int \ x^n e^{2x} \ \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_3.

2. Obtain a reduction formula for

$$I_n = \int_0^1 \ x^n e^{2x} \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_4.

3. Obtain a reduction formula for

$$I_n = \int \ x^n \sin x \ \mathrm{d}x$$

when $n \geq 1$ and, hence, determine I_4.

4. Obtain a reduction formula for

$$I_n = \int_0^\pi \ x^n \sin x \ \mathrm{d}x$$

when $n \geq 1$ and, hence, evaluate I_3.

5. If

$$I_n = \int (\ln x)^n \, \mathrm{d}x,$$

where $n \geq 1$, show that

$$I_n = x(\ln x)^n - nI_{n-1}$$

and, hence, determine I_3.

6. If

$$I_n = \int \left(x^2 + a^2\right)^n \, \mathrm{d}x,$$

show that

$$I_n = \frac{1}{2n+1}\left[x\left(x^2 + a^2\right)^n + 2na^2 I_{n-1}\right].$$

Hint: Write $(x^2 + a^2)^n$ as $1.(x^2 + a^2)^n$.

12.9.4 ANSWERS TO EXERCISES

1.

$$I_n = \frac{1}{2}\left[x^n e^{2x} - nI_{n-1}\right],$$

giving

$$I_3 = \frac{e^{2x}}{8}\left[4x^3 - 6x^2 + 6x - 3\right] + C.$$

2.

$$I_n = \frac{1}{2}\left[e^2 - nI_{n-1}\right],$$

giving

$$I_4 = \frac{1}{4}\left[e^2 - 3\right].$$

3.

$$I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2},$$

giving

$$I_4 = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

4.

$$I_n = \pi^n - n(n-1)I_{n-2},$$

giving

$$I_3 = \pi^3 - 6\pi.$$

5.

$$I_3 = x \left[\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6\right] + C.$$

"JUST THE MATHS"

UNIT NUMBER

12.10

INTEGRATION 10
(Further reduction formulae)

by

A.J.Hobson

UNIT 12.10 - INTEGRATION 10

FURTHER REDUCTION FORMULAE

INTRODUCTION

As an extension to the idea of reduction formulae, there are two particular definite integrals which are worthy of special consideration. They are

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, dx.$$

But, first, we shall establish the reduction formulae for the equivalent indefinite integrals.

12.10.1 INTEGER POWERS OF A SINE

Suppose that

$$I_n = \int \sin^n x \, dx;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \sin^{n-1} x \sin x \, dx.$$

Using integration by parts, with $u = \sin^{n-1} x$ and $\frac{dv}{dx} = \sin x$, we obtain

$$I_n = \sin^{n-1} x(-\cos x) + \int (n-1)\sin^{n-2} x \cos^2 x \, dx.$$

But, since $\cos^2 x \equiv 1 - \sin^2 x$, this becomes

$$I_n = -\sin^{n-1} x \cos x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[-\sin^{n-1} x \cos x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \sin^6 x \, \mathrm{d}x.$$

Solution

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x + 5I_4\right],$$

where

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x + 3I_2\right], \quad I_2 = \frac{1}{2}\left[-\sin x \cos x + I_0\right]$$

and

$$I_0 = \int \mathrm{d}x = x + \text{constant}.$$

Hence,

$$I_2 = \frac{1}{2}\left[-\sin x \cos x + x + \text{constant}\right];$$

$$I_4 = \frac{1}{4}\left[-\sin^3 x \cos x - \frac{3}{2}\sin x \cos x + \frac{3}{2}x + \text{constant}\right];$$

$$I_6 = \frac{1}{6}\left[-\sin^5 x \cos x - \frac{5}{4}\sin^3 x \cos x - \frac{15}{8}\sin x \cos x + \frac{15}{8}x + \text{constant}\right].$$

$$\text{Thus,} \quad \int \sin^6 x \, \mathrm{d}x = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5x}{16} + C,$$

where C is an arbitrary constant.

12.10.2 INTEGER POWERS OF A COSINE

Suppose that

$$I_n = \int \cos^n x \, \mathrm{d}x;$$

then, by writing the integrand as the product of two functions, we have

$$I_n = \int \cos^{n-1}x \cos x \, \mathrm{d}x.$$

Using integration by parts, with $u = \cos^{n-1}x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$, we obtain

$$I_n = \cos^{n-1}x \sin x + \int (n-1)\cos^{n-2}x\sin^2 x \, \mathrm{d}x.$$

But, since $\sin^2 x \equiv 1 - \cos^2 x$, this becomes

$$I_n = \cos^{n-1}x \sin x + (n-1)[I_{n-2} - I_n].$$

Thus,

$$I_n = \frac{1}{n}\left[\cos^{n-1}x \sin x + (n-1)I_{n-2}\right].$$

EXAMPLE

Determine the indefinite integral

$$\int \cos^5 x \, \mathrm{d}x.$$

Solution

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + 4I_3\right],$$

where

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2I_1\right]$$

and

$$I_1 = \int \cos x \, \mathrm{d}x = \sin x + \text{constant}.$$

Hence,

$$I_3 = \frac{1}{3}\left[\cos^2 x \sin x + 2\sin x + \text{constant}\right];$$

$$I_5 = \frac{1}{5}\left[\cos^4 x \sin x + \frac{4}{3}\cos^2 x \sin x + \frac{8}{3}\sin x + \text{constant}\right];$$

We conclude that

$$\int \cos^5 x \, \mathrm{d}x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C,$$

where C is an arbitrary constant.

12.10.3 WALLIS'S FORMULAE

Here, we consider the definite integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x.$$

Denoting either of these integrals by I_n, the reduction formula reduces to

$$I_n = \frac{n-1}{n} I_{n-2}$$

in both cases, from the previous two sections.

Convenient results may be obtained from this formula according as n is an odd number or an even number, as follows:

(a) n is an odd number

Repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\ldots\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.I_1.$$

But

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \; \mathrm{d}x \;\; \text{or} \;\; I_1 = \int_0^{\frac{\pi}{2}} \cos x \; \mathrm{d}x,$$

both of which have a value of 1.

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5).\ldots 6.4.2}{n(n-2)(n-4).\ldots 7.5.3},$$

which is **the first of "Wallis's formulae"**.

(b) n is an even number

This time, repeated application of the reduction formula gives

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\ldots\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.I_0.$$

But

$$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}.$$

Therefore,

$$I_n = \frac{(n-1)(n-3)(n-5).\ldots 5.3.1}{n(n-2)(n-4).\ldots 6.4.2}\frac{\pi}{2},$$

which is **the second of "Wallis's formulae"**.

EXAMPLES

1. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x.$$

   Solution
$$\int_0^{\frac{\pi}{2}} \sin^5 x \, \mathrm{d}x = \frac{4.2}{5.3} = \frac{8}{15}.$$

2. Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x.$$

   Solution
$$\int_0^{\frac{\pi}{2}} \cos^4 x \, \mathrm{d}x = \frac{3.1}{4.2}\frac{\pi}{2} = \frac{3\pi}{16}.$$

12.10.4 COMBINATIONS OF SINES AND COSINES

Another type of problem to which Wallis's formulae may be applied is of the form
$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, \mathrm{d}x,$$

where either m or n (or both) is an even number. We simply use $\sin^2 x \equiv 1 - \cos^2 x$ or $\cos^2 x \equiv 1 - \sin^2 x$ in order to convert the problem to several integrals of the types already discussed.

EXAMPLE

Evaluate the definite integral
$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x.$$

Solution
$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin^2 x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^5 x \left(1 - \cos^2 x\right) \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\cos^5 x - \cos^7 x\right) \, \mathrm{d}x,$$

which may be interpreted as

$$I_5 - I_7 = \frac{4.2}{5.3} - \frac{5.4.3}{6.4.2} = \frac{8}{15} - \frac{16}{35} = \frac{8}{105}.$$

12.10.5 EXERCISES

1. Determine the indefinite integral

$$\int \cos^4 x \, \mathrm{d}x.$$

2. Determine the indefinite integral

$$\int \sin^7 x \, \mathrm{d}x.$$

3. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^6 x \, \mathrm{d}x.$$

4. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^9 x \, \mathrm{d}x.$$

5. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^2 x \sin^6 x \, \mathrm{d}x.$$

6. Evaluate the definite integral

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x \, \mathrm{d}x.$$

12.10.6 ANSWERS TO EXERCISES

1.

$$\frac{1}{4}\cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3x}{8} + C.$$

2.

$$-\frac{1}{7}\sin^6 x \cos x - \frac{6}{35}\sin^4 x \cos x - \frac{24}{105}\sin^2 x \cos x - \frac{16}{35} \cos x + C.$$

3.

$$\frac{5\pi}{32}.$$

4.

$$\frac{128}{315}.$$

5.

$$\frac{5\pi}{32}.$$

6.

$$-\frac{4}{105}.$$

"JUST THE MATHS"

UNIT NUMBER

13.1

INTEGRATION APPLICATIONS 1
(The area under a curve)

by

A.J.Hobson

UNIT 13.1 - INTEGRATION APPLICATIONS 1

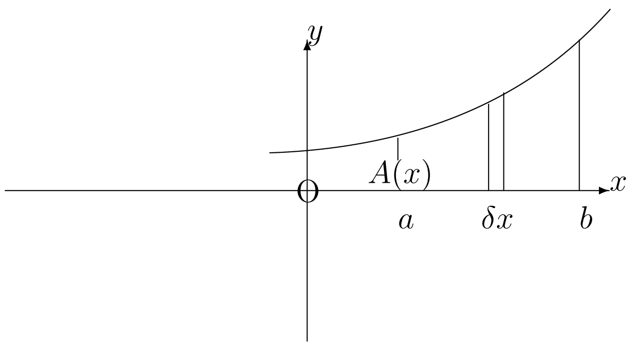
THE AREA UNDER A CURVE

13.1.1 THE ELEMENTARY FORMULA

We shall consider, here, a method of calculating the area contained between the x-axis of a cartesian co-ordinate system and the arc, from $x = a$ to $x = b$, of the curve whose equation is

$$y = f(x).$$

Suppose that $A(x)$ represents the area contained between the curve, the x-axis, the y-axis and the ordinate at some arbitrary value of x.



A small increase of δx in x will lead to a corresponding increase of δA in A approximating in area to that of a narrow rectangle whose width is δx and whose height is $f(x)$.

Thus,

$$\delta A \simeq f(x)\delta x,$$

which may be written

$$\frac{\delta A}{\delta x} \simeq f(x).$$

By allowing δx to tend to zero, the approximation disappears to give

$$\frac{\mathrm{d}A}{\mathrm{d}x} = f(x).$$

Hence, on integrating both sides with respect to x,

$$A(x) = \int f(x) \, \mathrm{d}x.$$

The constant of integration would need to be such that $A = 0$ when $x = 0$; but, in fact, we do not need to know the value of this constant because the required area, from $x = a$ to $x = b$, is given by

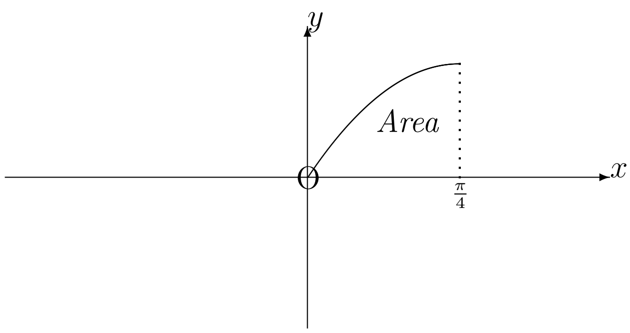
$$A(b) - A(a) = \int_a^b f(x) \, \mathrm{d}x.$$

EXAMPLES

1. Determine the area contained between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = 0$ to $x = \frac{\pi}{4}$.

   Solution

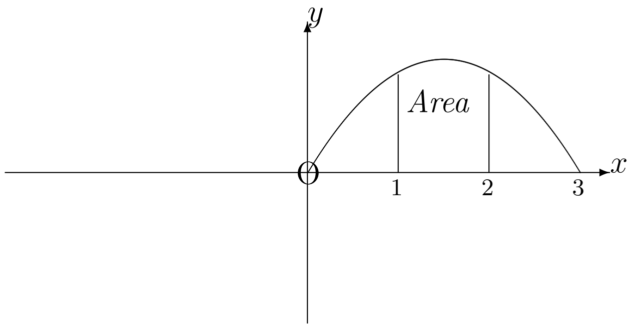
$$\int_0^{\frac{\pi}{4}} \sin 2x \, \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$



2. Determine the area contained between the x-axis and the curve whose equation is $y = 3x - x^2$, from $x = 1$ to $x = 2$.

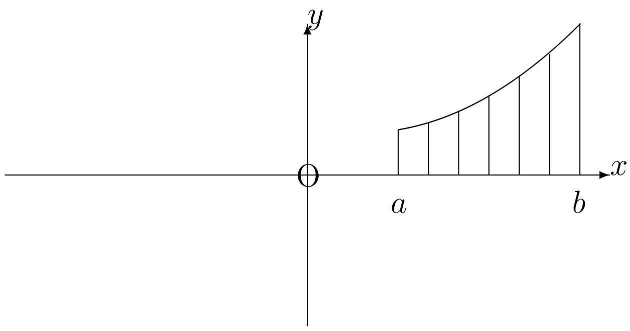
   Solution

$$\int_1^2 (3x - x^2)\,\mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_1^2 = \left(6 - \frac{8}{3}\right) - \left(\frac{3}{2} - \frac{1}{3}\right) = \frac{13}{6}.$$



13.1.2 DEFINITE INTEGRATION AS A SUMMATION

Consider, now, the same area as in the previous section, but regarded (approximately) as the sum of a large number of narrow rectangles with typical width δx and typical height $f(x)$. The narrower the strips, the better will be the approximation.



Hence, we may state an alternative expression for the area from $x = a$ to $x = b$ in the form

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x.$$

Since this new expression represents the same area as before, we may conclude that

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)\delta x.$$

Notes:

(i) The above result shows that an area which lies wholly **below** the x-axis will be **negative** and so care must be taken with curves which cross the x-axis between $x = a$ and $x = b$.

(ii) If c is any value of x between $x = a$ and $x = b$, the above result shows that

$$\int_a^b f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x + \int_c^b f(x)\ \mathrm{d}x.$$

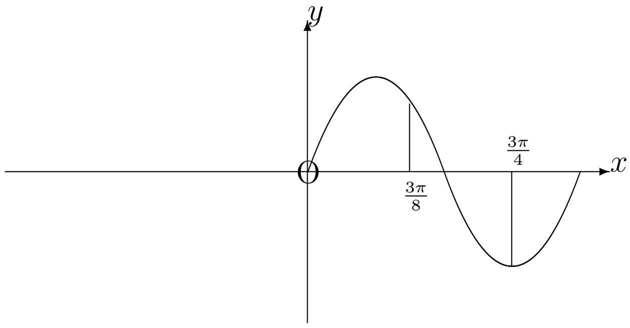
(iii) To calculate the TOTAL area contained between the x-axis and a curve which crosses the x-axis between $x = a$ and $x = b$, account must be taken of any parts of the area which are negative.

(iv) It is usually a good idea to sketch the area under consideration before evaluating the appropriate definite integrals.

(v) It will be seen shortly that the formula obtained for definite integration as a summation has a wider field of application than simply the calculation of areas.

EXAMPLES

1. Determine the total area between the x-axis and the curve whose equation is $y = \sin 2x$, from $x = \frac{3\pi}{8}$ and $x = \frac{3\pi}{4}$.
   Solution
   $$\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sin 2x\ \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin 2x\ \mathrm{d}x.$$
   That is,
   $$\left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} - \left[-\frac{\cos 2x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) - \left(0 - \frac{1}{2}\right) = 1 - \frac{1}{2\sqrt{2}}.$$



2. Evaluate the definite integral,

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \, \mathrm{d}x.$$

Solution

$$\int_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} \sin 2x \, \mathrm{d}x = \left[-\frac{\cos 2x}{2}\right]_{\frac{3\pi}{8}}^{\frac{3\pi}{4}} = -\frac{1}{2\sqrt{2}}.$$

13.1.3 EXERCISES

1. Determine the areas bounded by the following curves and the x-axis between the ordinates $x = 1$ and $x = 3$:
   (a)
   $$y = 2x^2 + x + 1;$$
   (b)
   $$y = (1 - x)^2;$$
   (c)
   $$y = 2\sqrt{x}.$$

2. Sketch the curve whose equation is
   $$y = (1 - x)(2 + x)$$
   and determine the area contained between the x-axis and the portion of the curve above the x-axis.

3. To the nearest whole number, determine the area bounded between $x = 1$ and $x = 2$ by the curves whose equations are
$$y = 3e^{2x} \text{ and } y = 3e^{-x}.$$

4. Determine the area bounded between $x = 0$ and $x = \frac{\pi}{3}$ by the curves whose equations are
$$y = \sin x \text{ and } y = \sin 2x.$$

5. Determine the total area, from $x = 0$ to $x = \frac{3\pi}{10}$, contained between the x-axis and the curve whose equation is
$$y = \cos 5x.$$

13.1.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{70}{3};$$
(b)
$$\frac{8}{3};$$
(c)
$$4\sqrt{3} - \frac{4}{3}.$$

2.
$$\frac{9}{2}.$$

3.
$$70.$$

4.
$$0.25$$

5.
$$\frac{2\sqrt{2}-1}{5\sqrt{2}} - \simeq 0.259$$

"JUST THE MATHS"

UNIT NUMBER

13.2

INTEGRATION APPLICATIONS 2
(Mean values)
&
(Root mean square values)

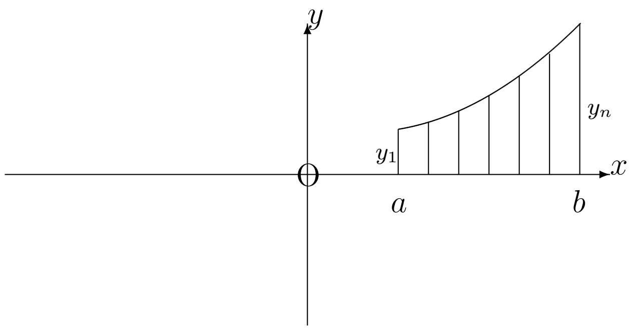
by

A.J.Hobson

UNIT 13.2 - INTEGRATION APPLICATIONS 2

MEAN AND ROOT MEAN SQUARE VALUES

13.2.1 MEAN VALUES



On the curve whose equation is

$$y = f(x),$$

suppose that $y_1, y_2, y_3, ..., y_n$ are the y-coordinates which correspond to n different x-coordinates, $a = x_1, x_2, x_3, ..., x_n = b$.

The average (that is, the arithmetic mean) of these n y-coordinates is

$$\frac{y_1 + y_2 + y_3 + ... + y_n}{n}.$$

But now suppose that we wished to determine the average (arithmetic mean) of **all** the y-coordinates, from $x = a$ to $x = b$ on the curve whose equation is $y = f(x)$.

We could make a reasonable approximation by taking a very **large** number, n, of y-coordinates separated in the x-direction by very **small** distances. If these distances are typically represented by δx, then the required mean value could be written

$$\frac{y_1\delta x + y_2\delta x + y_3\delta x + ... + y_n\delta x}{n\delta x},$$

in which the denominator is equivalent to $(b - a + \delta x)$, since there are only $n - 1$ spaces between the n y-coordinates.

Allowing the number of y-coordinates to increase indefinitely, δx will to tend to zero and we obtain the formula for the **"Mean Value"** in the form

$$\text{M.V.} = \frac{1}{b-a} \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x.$$

That is,

$$\text{M.V.} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x.$$

Note:
In cases where the definite integral in this formula represents the area between the curve and the x-axis, the Mean Value provides the height of a rectangle, with base $b - a$, having the same area as that represented by the definite integral.

EXAMPLE

Determine the Mean Value of the function

$$f(x) \equiv x^2 - 5x$$

from $x = 1$ to $x = 4$.

Solution

The Mean Value is given by

$$\text{M.V.} = \frac{1}{4-1} \int_1^4 (x^2 - 5x) \, \mathrm{d}x = \frac{1}{3}\left[\frac{x^3}{3} - \frac{5x^2}{2}\right]_1^4 =$$

$$\frac{1}{3}\left[\left(\frac{64}{3} - 40\right) - \left(\frac{1}{3} - \frac{5}{2}\right)\right] = -\frac{33}{2}.$$

13.2.2 ROOT MEAN SQUARE VALUES

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$.

The **"Root Mean Square Value"** provides a measure of "central tendency" for the **numerical** values of $f(x)$ and is defined to be the square root of the Mean Value of $f(x)$ from $x = a$ to $x = b$.

Hence,

$$\text{R.M.S.V.} = \sqrt{\frac{1}{b-a}\int_a^b \ [f(x)]^2 \ \mathrm{d}x}.$$

EXAMPLE

Determine the Root Mean Square Value of the function, $f(x) \equiv x^2 - 5$, from $x = 1$ to $x = 3$.

Solution

The Root Mean Square Value is given by

$$\text{R.M.S.V.} = \sqrt{\frac{1}{3-1}\int_1^3 \ (x^2-5)^2 \ \mathrm{d}x}.$$

Temporarily ignoring the square root, we obtain the **"Mean Square Value"**,

$$\text{M.S.V.} = \frac{1}{2}\int_1^3 \ (x^4 - 10x^2 + 25) \ \mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{10x^3}{3} + 25x\right]_1^3 = \frac{1}{2}\left[\left(\frac{243}{5} - \frac{270}{3} + 75\right) - \left(\frac{1}{5} - \frac{10}{3} + 25\right)\right] = \frac{176}{30}.$$

Thus,

$$\text{R.M.S.V.} = \sqrt{\frac{176}{30}} \simeq 2.422$$

13.2.3 EXERCISES

1. (a) Determine the Mean Value of the function, $(x-1)(x-2)$, from $x = 1$ to $x = 2$;

   (b) Determine, correct to three significant figures, the Mean Value of the function, $\frac{1}{2x+5}$, from $x = 3$ to $x = 5$;

   (c) Determine the Mean Value of the function, $\sin 2t$, from $t = 0$ to $t = \frac{\pi}{2}$;

   (d) Determine, correct to three places of decimals, the Mean Value of the function, e^{-x}, from $x = 1$ to $x = 5$;

   (e) Determine, correct to three significant figures, the mean value of the function, xe^{-2x}, from $x = 0$ to $x = 2$.

2. (a) Determine the Root Mean Square Value of the function, $3x + 1$, from $x = -2$ to $x = 2$;

   (b) Determine the Root Mean Square Value, of the function, e^x, from $x = 0$ to $x = 1$, correct to three decimal places;

   (c) Determine the Root Mean Square Value of the function, $\cos x$, from $x = \frac{\pi}{2}$ to $x = \pi$;

   (d) Determine the Root Mean Square Value of the function, $(4x-5)^{\frac{3}{2}}$, from $x = 1.25$ to $x = 1.5$.

13.2.4 ANSWERS TO EXERCISES

1. (a) $-\frac{1}{6}$;

   (b) 0.0775;

   (c) $\frac{2}{\pi}$;

   (d) -0.076;

   (e) 0.114

2. (a) $\sqrt{13} \simeq= 3.606$;

   (b) 1.787;

   (c) $\frac{1}{\sqrt{2}}$;

   (d) $\frac{1}{2}$.

"JUST THE MATHS"

UNIT NUMBER

13.3

INTEGRATION APPLICATIONS 3
(Volumes of revolution)

by

A.J.Hobson

UNIT 13.3 - INTEGRATION APLICATIONS 3

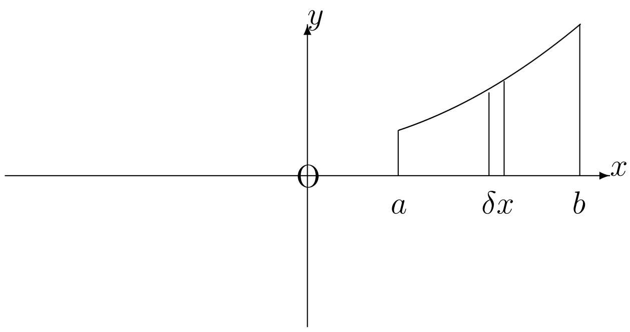
VOLUMES OF REVOLUTION

13.3.1 VOLUMES OF REVOLUTION ABOUT THE X-AXIS

Suppose that the area between a curve whose equation is

$$y = f(x)$$

and the x-axis, from $x = a$ to $x = b$, lies wholly above the x-axis; suppose, also, that this area is rotated through 2π radians about the x-axis. Then a solid figure is obtained whose volume may be determined as an application of definite integration.



When a narrow strip of width, δx, and height, y, is rotated through 2π radians about the x-axis, we obtain a disc whose volume, δV, is given approximately by

$$\delta V \simeq \pi y^2 \delta x.$$

Thus, the total volume, V, obtained is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

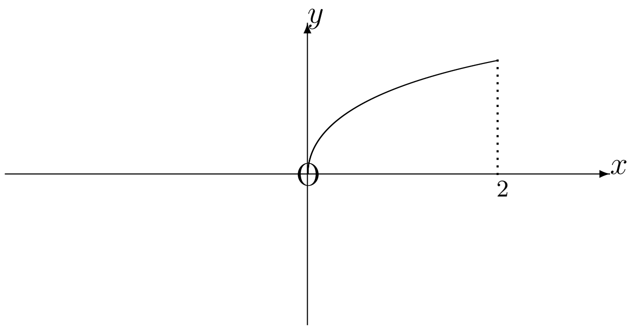
That is,

$$V = \int_a^b \pi y^2 \, \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line, $x = 2$, and the parabola, $y^2 = 8x$, is rotated through 2π radians about the x-axis.

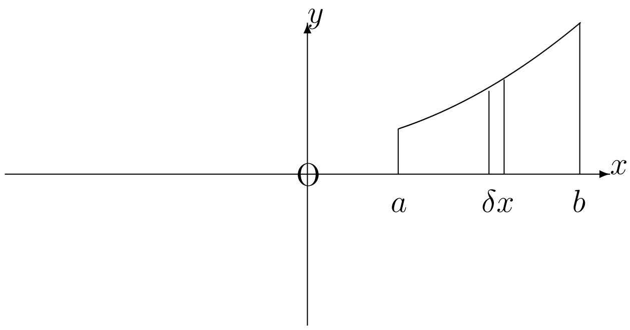
Solution



$$V = \int_0^2 \pi \times 8x \, \mathrm{d}x = [4\pi x^2]_0^2 = 16\pi.$$

13.3.2 VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

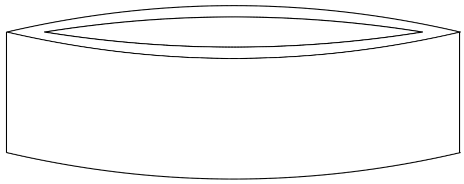
First we consider the same diagram as in the previous section:



This time, if the narrow strip of width, δx, is rotated through 2π radians about the y-axis, we obtain, approximately, a cylindrical shell of internal radius, x, external radius, $x + \delta x$ and height, y.

The volume, δV, of the shell is thus given by

$$\delta V \simeq 2\pi x y \delta x.$$



The total volume is given by

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x y \delta x.$$

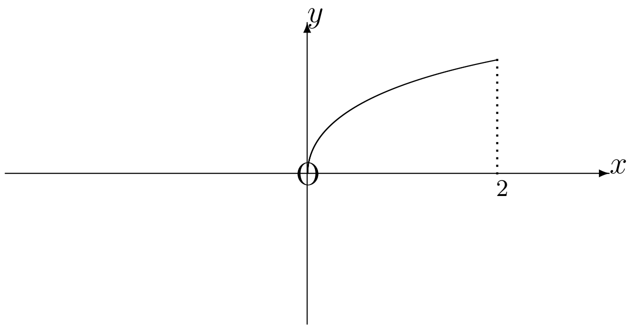
That is,

$$V = \int_a^b 2\pi x y \ \mathrm{d}x.$$

EXAMPLE

Determine the volume obtained when the area, bounded in the first quadrant by the x-axis, the y-axis, the straight line $x = 2$ and the parabola $y^2 = 8x$ is rotated through 2π radians about the y-axis.

Solution



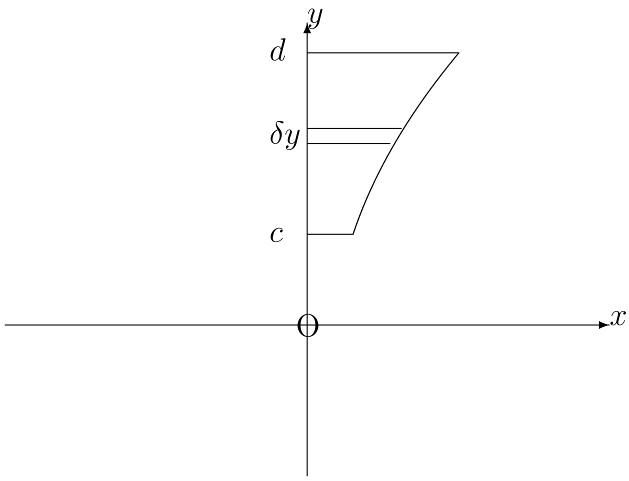
$$V = \int_0^2 \ 2\pi x \times \sqrt{8x} \ \mathrm{d}x.$$

In other words,

$$V = \pi 4\sqrt{2}\int_0^2 \ x^{\frac{3}{2}}\mathrm{d}x = \pi 4\sqrt{2}\left[\frac{2x^{\frac{5}{2}}}{5}\right]_0^2 = \frac{64\pi}{5}.$$

Note:

It may be required to find the volume of revolution about the y-axis of an area which is contained between a curve and the y-axis from $y = c$ to $y = d$.



But here we simply interchange the roles of x and y in the original formula for rotation about the x-axis; that is

$$V = \int_c^d \pi x^2 \ \mathrm{d}y.$$

Similarly, the volume of rotation of the above area about the x-axis is given by

$$V = \int_c^d 2\pi yx \ \mathrm{d}y.$$

13.3.3 EXERCISES

1. By using a straight line through the origin, obtain a formula for the volume, V, of a solid right-circular cone with height, h, and base radius, r.

2. Determine the volume obtained when the segment straight line

$$y = 5 - 4x,$$

lying between $x = 0$ and $x = 1$, is rotated through 2π radians about (a) the x-axis and (b) the y-axis.

3. Determine the volume obtained when the part of the curve

$$y = \cos 3x,$$

lying between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$, is rotated through 2π radians about the x-axis.

4. Determine the volume obtained when the part of the curve

$$y = \frac{1}{x\sqrt{2+x}},$$

lying between $x = 2$ and $x = 7$, is rotated through 2π radians about the x-axis.

5. Determine the volume obtained when the part of the curve

$$y = \frac{1}{(x-1)(x-5)},$$

lying between $x = 6$ and $x = 8$, is rotated through 2π radians about the y-axis.

6. Determine the volume obtained when the part of the curve

$$x = ye^{-y},$$

lying between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.

7. Determine the volume obtained when the part of the curve

$$y = \sin 2x,$$

lying between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, is rotated through 2π radians about the y-axis.

8. Determine the volume obtained when the part of the curve

$$y = x(1-x^3)^{\frac{1}{4}},$$

lying between $x = 0$ and $x = 1$, is rotated through 2π radians about the x-axis.

9. Determine the volume obtained when the part of the curve

$$x = (4-y^2)^2,$$

lying between $y = 1$ and $y = 2$, is rotated through 2π radians about the x-axis.

10. Determine the volume obtained when the part of the curve

$$y = x\sec(x^3),$$

lying between $x = 0$ and $x = 0.5$, is rotated through 2π radians about the x-axis.

11. Determine the volume obtained when the part of the curve

$$y = \frac{1}{x^2-1},$$

lying between $x = 2$ and $x = 3$ is rotated through 2π radians about the y-axis.

13.3.4 ANSWERS TO EXERCISES

1.

$$V = \frac{1}{3}\pi r^2 h.$$

2.

$$\text{(a)} \quad \frac{\pi}{3} \simeq 1.047 \quad \text{(b)} \quad \frac{7\pi}{3} \simeq 7.330$$

3.

$$\frac{\pi^2}{12} \simeq 0.822$$

4.

0.214 approximately.

5.

8.010 approximately.

6.

0.254 approximately.

7.

3.364 approximately.

8.

$$\frac{2\pi}{9} \simeq 0.698$$

9.

$$9\pi \simeq 28.274$$

10.

0.132 approximately.

11.

3.081 approximately.

"JUST THE MATHS"

UNIT NUMBER

13.4

INTEGRATION APPLICATIONS 4
(Lengths of curves)

by

A.J.Hobson

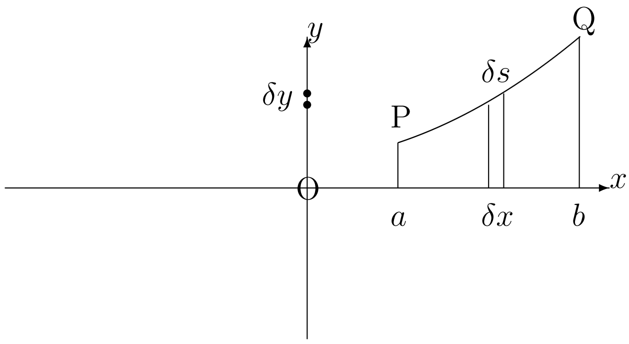
UNIT 13.4 - INTEGRATION APPLICATIONS 4 - LENGTHS OF CURVES

13.4.1 THE STANDARD FORMULAE

The problem, in this unit, is to calculate the length of the arc of the curve with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

The total length, s, of arc is thus given by

$$s = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x.$$

That is,

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Notes:
(i) If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$s = \pm \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(ii) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y, so that the length of the arc is given by

$$s = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

EXAMPLES

1. A curve has equation

$$9y^2 = 16x^3.$$

Determine the length of the arc of the curve between the point $\left(1, \frac{4}{3}\right)$ and the point $\left(4, \frac{32}{3}\right)$.

Solution

We may write the equation of the curve in the form

$$y = \frac{4x^{\frac{3}{2}}}{3};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^{\frac{1}{2}}.$$

Hence,

$$s = \int_1^4 \sqrt{1+4x}\ \mathrm{d}x = \left[\frac{(1+4x)^{\frac{3}{2}}}{6}\right]_1^4 = \frac{17^{\frac{3}{2}}}{6} - \frac{5^{\frac{3}{2}}}{6} \simeq 13.55$$

2. A curve is given parametrically by

$$x = t^2 - 1, \quad y = t^3 + 1.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 1$.

Solution

Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2,$$

we have

$$s = \int_0^1 \sqrt{4t^2 + 6t^4}\ \mathrm{d}t = \int_0^1 t\sqrt{4 + 6t^2}\ \mathrm{d}t = \left[\frac{1}{18}\left(4 + 6t^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{18}\left(10^{\frac{3}{2}} - 8\right) \simeq 1.31$$

13.4.2 EXERCISES

1. A straight line has equation

$$y = 3x + 2.$$

Use (a) elementary trigonometry and (b) definite integration to determine the length of the line segment joining the point where $x = 3$ and the point where $x = 7$.

2. A curve has equation

$$y = \frac{1}{2}x^2 - \frac{1}{4}\ln x.$$

Determine the length of the arc of the curve between $x = 1$ and $x = e$.

3. A curve has equation

$$x = 2(y + 3)^{\frac{3}{2}}.$$

Determine the length of the arc of the curve between $y = -2$ and $y = 1$, stating your answer in decimals correct to four significant figures.

4. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

Determine the length of the arc of the curve between the point where $t = 0$ and the point where $t = 2\pi$.

5. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

Determine the length of the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{4}$.

6. A curve is given parametrically by

$$x = e^u \sin u, \quad y = e^u \cos u.$$

Determine the length of the arc of the curve between the point where $u = 0$ and the point where $u = 1$.

13.4.3 ANSWERS TO EXERCISES

1.

$$4\sqrt{10} \simeq 12.65$$

2.

$$\frac{2e^2 - 1}{4} \simeq 3.44$$

3.

$$14.33$$

4.

$$8.$$

5.

$$\frac{\pi^2}{8}.$$

6.

$$\sqrt{2}(e - 1) \simeq 2.43$$

"JUST THE MATHS"

UNIT NUMBER

13.5

INTEGRATION APPLICATIONS 5
(Surfaces of revolution)

by

A.J.Hobson

UNIT 13.5 - INTEGRATION APPLICATIONS 5

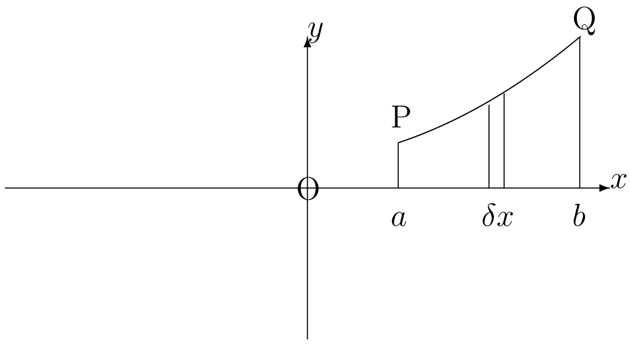
SURFACES OF REVOLUTION

13.5.1 SURFACES OF REVOLUTION ABOUT THE X-AXIS

The problem, in this unit, is to calculate the surface area obtained when the arc of the curve, with equation

$$y = f(x),$$

joining the two points, P and Q, on the curve, at which $x = a$ and $x = b$, is rotated through 2π radians about the x-axis or the y-axis.



For two neighbouring points along the arc, the part of the curve joining them may be considered, approximately, as a straight line segment.

Hence, if these neighbouring points are separated by distances of δx and δy, parallel to the x-axis and the y-axis, respectively, then the length, δs, of arc between them is given, approximately, by

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

using Pythagoras's Theorem.

When the arc, of length δs, is rotated through 2π radians about the x-axis, it generates a thin band whose area is, approximately,

$$2\pi y \delta s = 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

The total surface area, S, is thus given by

$$S = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x.$$

That is,

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that

$$S = \pm\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

EXAMPLES

1. A curve has equation

$$y^2 = 2x.$$

Determine the surface area obtained when the arc of the curve between the point $(2, 2))$ and the point $(8, 4)$ is rotated through 2π radians about the x-axis.

Solution

We may write the equation of the arc of the curve in the form

$$y = \sqrt{2x} = \sqrt{2}x^{\frac{1}{2}};$$

and so,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2x}}.$$

Hence,

$$S = \int_2^8 2\pi\sqrt{2x}\sqrt{1+\frac{1}{2x}}\ \mathrm{d}x = \int_2^8 \sqrt{2x+1}\ \mathrm{d}x = \left[\frac{(2x+1)^{\frac{3}{2}}}{3}\right]_2^8.$$

Thus,

$$S = \frac{17^{\frac{3}{2}}}{3} - \frac{5^{\frac{3}{2}}}{3} \simeq 19.64$$

2. A curve is given parametrically by

$$x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta.$$

Determine the surface area obtained when the arc of the curve between the point $(0, \sqrt{2})$ and the point $(1, 1)$ is rotated through 2π radians about the x-axis.

Solution

The parameters of the two points are $\frac{\pi}{2}$ and $\frac{\pi}{4}$, respectively; and, since

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sqrt{2}\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = \sqrt{2}\cos\theta,$$

we have

$$S = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 2\sqrt{2}\pi\sin\theta\sqrt{2\sin^2\theta + 2\cos^2\theta}\ \mathrm{d}\theta = -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} 4\pi\sin\theta\ \mathrm{d}\theta.$$

Thus,

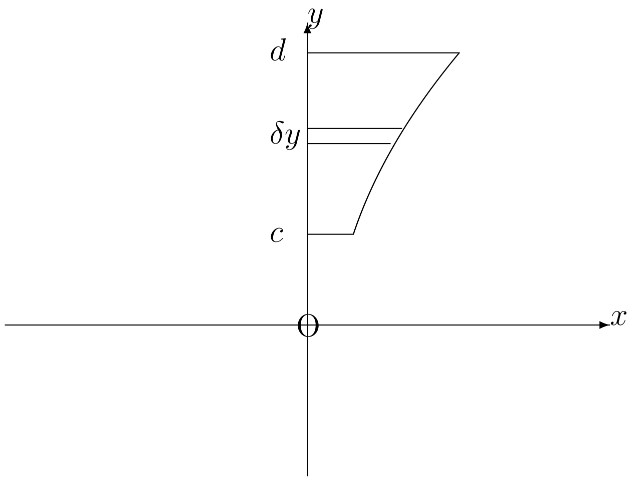
$$S = -\left[-4\pi\cos\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}} = \frac{4\pi}{\sqrt{2}} \simeq 8.89$$

13.5.2 SURFACES OF REVOLUTION ABOUT THE Y-AXIS

For a curve whose equation is of the form $x = g(y)$, the surface of revolution about the y-axis of an arc joining the two points at which $y = c$ and $y = d$ is given by

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$

We simply reverse the roles of x and y in the previous section.



Alternatively, if the curve is given parametrically,

$$S = \pm \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

acccording as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

EXAMPLE

If the arc of the parabola, with equation

$$x^2 = 2y,$$

joining the two points $(2, 2)$ and $(4, 8)$, is rotated through 2π radians about the y-axis, determine the surface area obtained.

Solution

Using the result from the previous section, the surface area obtained is given by

$$S = \int_2^8 2\pi\sqrt{2y}\sqrt{1 + \frac{1}{2y}}\ \mathrm{d}y \simeq 19.64$$

13.5.3 EXERCISES

1. Use a straight line through the origin to determine the surface area of a right-circular cone with height, h, and base radius, r.
2. Determine the surface area obtained when the arc of the curve $x = y^3$, between $y = 0$ and $y = 1$, is rotated through 2π radians about the y-axis.
3. A curve is given parametrically by

$$x = t - \sin t, \quad y = 1 - \cos t.$$

   Determine the surface area obtained when the arc of the curve between the point where $t = 0$ and the point where $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

   State your answer correct to three places of decimals.
4. A curve is given parametrically by

$$x = 4(\cos\theta + \theta\sin\theta), \quad y = 4(\sin\theta - \theta\cos\theta).$$

   Determine the surface area obtained when the arc of the curve between the point where $\theta = 0$ and the point where $\theta = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.
5. A curve is given parametrically by

$$x = e^u\cos u, \quad y = e^u\sin u.$$

   Determine the surface area obtained when the arc of the curve between the point where $u = 0$ and the point where $u = \frac{\pi}{4}$ is rotated through 2π radians about the y-axis.

   State your answer correct to three places of decimals.

13.5.4 ANSWERS TO EXERCISES

1.

$$\pi r\sqrt{r^2+h^2}.$$

2.

$$\frac{\pi(10\sqrt{10}-1)}{27} \simeq 3.56$$

3.

$$3.891$$

4.

$$32\pi\left(3-\left(\frac{\pi}{2}\right)^2\right) \simeq 53.54$$

5.

$$1.037$$

"JUST THE MATHS"

UNIT NUMBER

13.6

INTEGRATION APPLICATIONS 6
(First moments of an arc)

by

A.J.Hobson

UNIT 13.6 - INTEGRATION APPLICATIONS 6

FIRST MOMENTS OF AN ARC

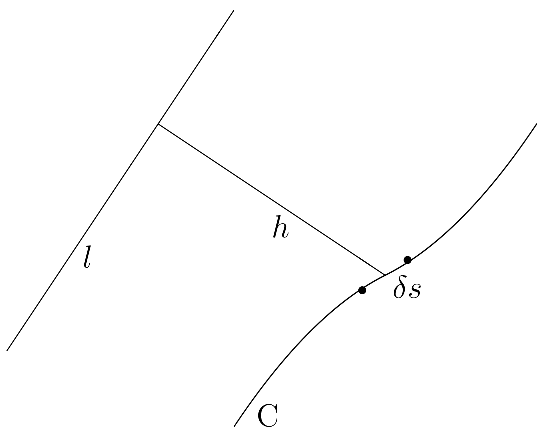
13.6.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates; and suppose that δs is the length of a small element of this arc.

Then the **"first moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h\delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

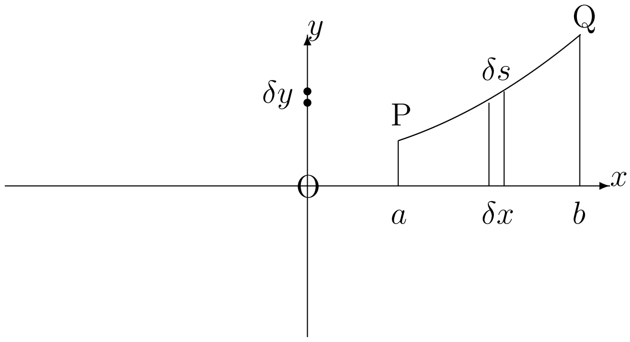


13.6.2 FIRST MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve, whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The first moment of each element about the y-axis is x times the length of the element; that is $x\delta s$, implying that the total first moment of the arc about the y-axis is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} x\delta s.$$

But, from Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x,$$

so that the first moment of the arc becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x = \int_a^b x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the y-axis is given by

$$\pm \int_{t_1}^{t_2} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.6.3 FIRST MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the first moment about the x-axis will be

$$\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the first moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

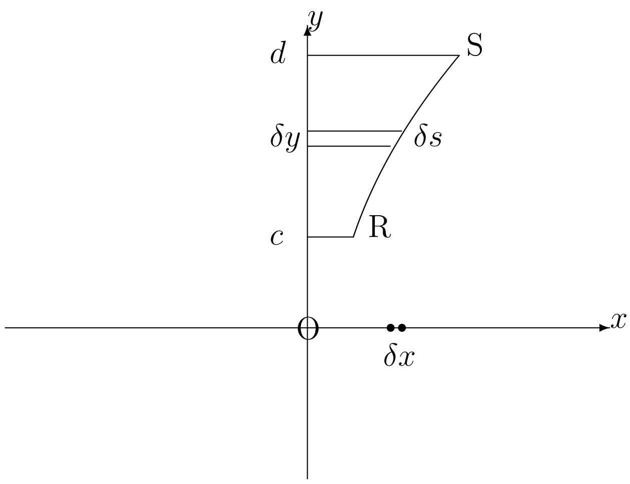
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.6.2 so that the first moment of the arc about the x-axis is given by

$$\int_c^d y\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the first moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

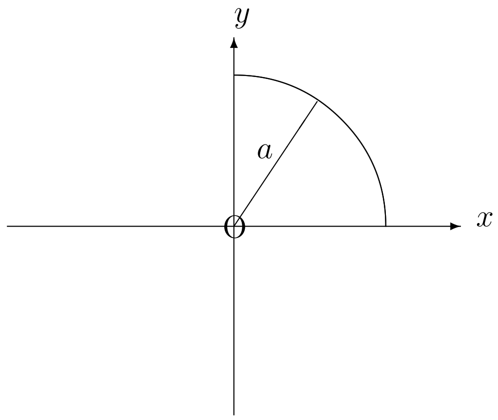
EXAMPLES

1. Determine the first moments about the x-axis and the y-axis of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The first moment of the arc about the y-axis is therefore given by

$$\int_0^a x\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and $y = \sqrt{a^2 - x^2}$.

Hence,

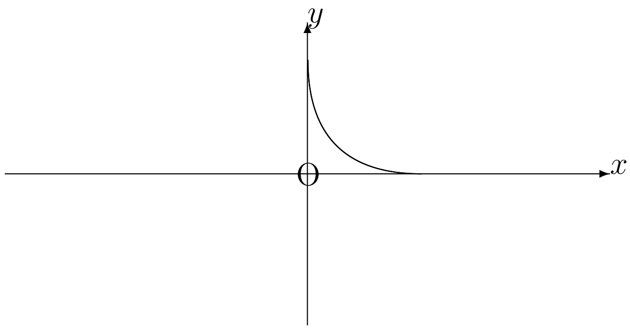
$$\text{first moment} = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\ \mathrm{d}x = \left[-a\sqrt{(a^2 - x^2)}\right]_0^a = a^2.$$

By symmetry, the first moment of the arc about the x-axis will also be a^2.

2. Determine the first moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} y\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\ \mathrm{d}\theta$$

$$= 3a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5}.$$

Similarly, the first moment of the arc about the y-axis is given by

$$\int_0^{\frac{\pi}{2}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a\cos^3\theta.(3a\cos\theta\sin\theta)\ \mathrm{d}\theta$$

$$= 3a^2\int_0^{\frac{\pi}{2}} \cos^4\theta\sin\theta\ \mathrm{d}\theta = 3a^2\left[-\frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3a^2}{5},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.6.4 THE CENTROID OF AN ARC

Having calculated the first moments of an arc about both the x-axis and the y-axis it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $s\overline{x}$, where s is the total length of the arc;

and

(b) The first moment about the x-axis is given by $s\overline{y}$, where s is the total length of the arc.

The point is called the **"centroid"** or the **"geometric centre"** of the arc and, for an arc of the curve with equation $y = f(x)$, between $x = a$ and $x = b$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} \text{ and } \overline{y} = \frac{\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an arc about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δs, to the y-axis, the first moment about the given axis will be

$$\sum_{\mathrm{C}}(x-\overline{x})\delta s = \sum_{\mathrm{C}} x\delta s - \overline{x}\sum_{\mathrm{C}}\delta s = s\overline{x} - s\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole arc at a single point for the purposes of considering first moments. In practice, it corresponds, for example, to the position of the centre of mass of a thin wire with uniform density.

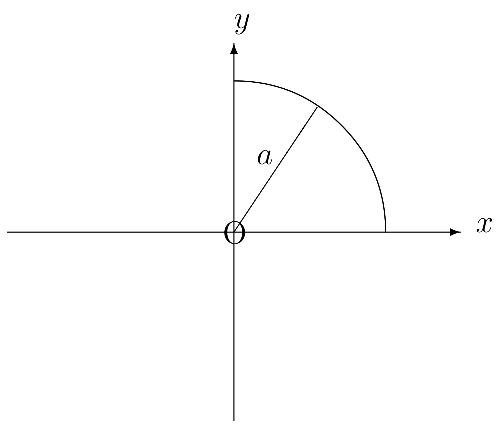
EXAMPLES

1. Determine the cartesian co-ordinates of the centroid of the arc of the circle, with equation

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an earlier example in this unit, we know that the first moments of the arc about the x-axis and the y-axis are both equal to a^2.

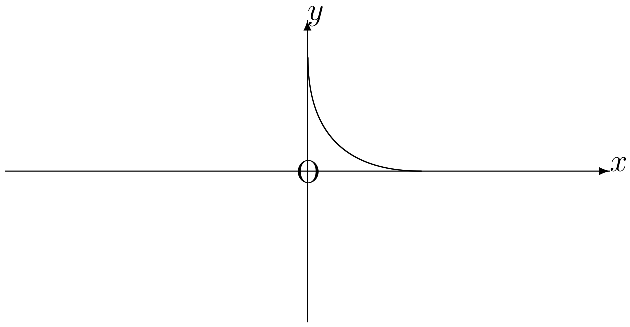
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that

$$\overline{x} = \frac{2a}{\pi} \text{ and } \overline{y} = \frac{2a}{\pi}.$$

2. Determine the cartesian co-ordinates of the centroid of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From an earlier example in this unit, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the first moments of the arc about the x-axis and the y-axis are both equal to $\frac{3a^2}{5}$.
Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus,

$$\overline{x} = \frac{2a}{5} \ \text{ and } \ \overline{y} = \frac{2a}{5}.$$

13.6.5 EXERCISES

1. Determine the first moment about the y-axis of the arc of the curve with equation
$$y = x^2,$$
lying between $x = 0$ and $x = 1$.

2. Determine the first moment about the x-axis of the arc of the curve with equation
$$x = 5y^2,$$
lying between $y = 0.1$ and $y = 0.5$.

3. Determine the first moment about the x-axis of the arc of the curve with equation
$$y = 2\sqrt{x},$$
lying between $x = 3$ and $x = 24$.

4. Verify, using integration, that the centroid of the straight line segment, defined by the equation
$$y = 3x + 2,$$
from $x = 0$ to $x = 1$, lies at its centre point.

5. Determine the cartesian co-ordinates of the centroid of the arc of the circle given parametrically by
$$x = 5\cos\theta, \quad y = 5\sin\theta,$$
from $\theta = -\frac{\pi}{6}$ to $\theta = \frac{\pi}{6}$.

6. For the curve whose equation is
$$9y^2 = x(3 - x)^2,$$
show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$
Hence show that the centroid of the first quadrant arch of this curve lies at the point $\left(\frac{7}{5}, \frac{\sqrt{3}}{4}\right)$.

13.6.6 ANSWERS TO EXERCISES

1.

$$\frac{5\sqrt{5}-1}{12} \simeq 0.85$$

2.

$$\frac{13\sqrt{26}-\sqrt{2}}{150} \simeq 0.43$$

3.

$$156.$$

4.

$$\overline{x} = \frac{1}{2} \quad \text{and} \quad \overline{y} = \frac{7}{2}.$$

5.

$$\overline{x} = \frac{15}{\pi} \simeq 4.77, \quad \overline{y} = 0.$$

"JUST THE MATHS"

UNIT NUMBER

13.7

INTEGRATION APPLICATIONS 7
(First moments of an area)

by

A.J.Hobson

UNIT 13.7 - INTEGRATION APPLICATIONS 7

FIRST MOMENTS OF AN AREA

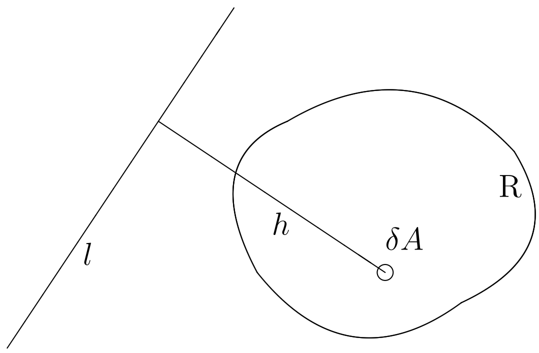
13.7.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane of cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"first moment"** of R about a fixed line, l, in the plane of R is given by

$$\lim_{\delta A \to 0} \sum_{\mathrm{R}} h\delta A,$$

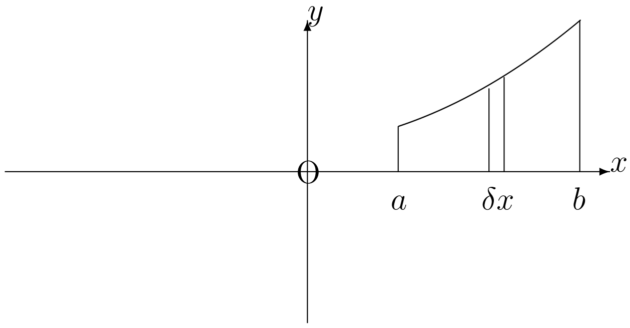
where h is the perpendicular distance, from l, of the element with area, δA.



13.7.2 FIRST MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may divided up into small elements by using a network, consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip' of width δx and height y (parallel to the y-axis) have the same perpendicular distance, x, from the y-axis.

Hence the first moment of this strip about the y-axis is x times the area of the strip; that is, $x(y\delta x)$, implying that the total first moment of the region about the y-axis is given by

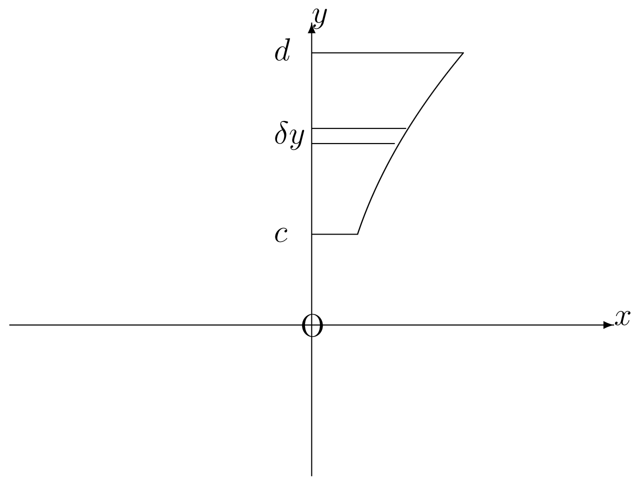
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} xy\delta x = \int_a^b xy \ \mathrm{d}x.$$

Note:
First moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment about the x-axis is given by

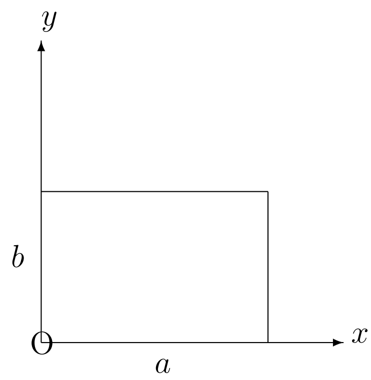
$$\int_c^d yx \ \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a rectangular region, with sides of lengths a and b about the side of length b.

   Solution



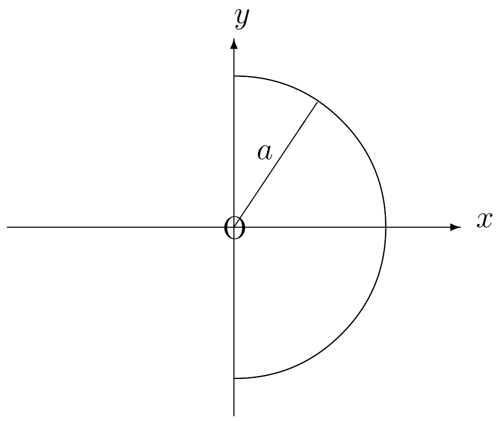
The first moment about the y-axis is given by

$$\int_0^a xb\ \mathrm{d}x = \left[\frac{x^2 b}{2}\right]_0^a = \frac{1}{2}a^2 b.$$

2. Determine the first moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the first moment about the y-axis is given by

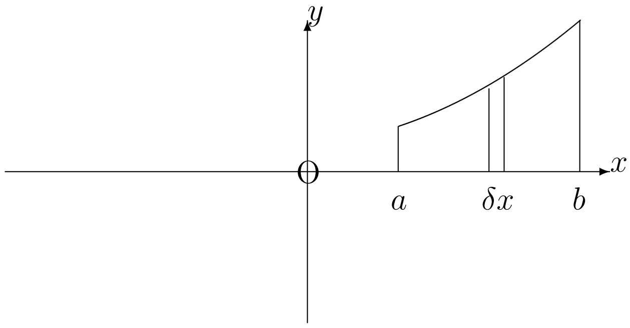
$$2\int_0^a x\sqrt{a^2 - x^2}\,\mathrm{d}x = \left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}a^3.$$

Note:
Although first moments about the x-axis will be discussed mainly in the next section of this Unit, we note that the symmetry of the above region shows that its first moment about the x-axis would be zero; this is because, for each $y(x\delta y)$, there will be a corresponding $-y(x\delta y)$ in calculating the first moments of the strips parallel to the x-axis.

13.7.3 FIRST MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the first moment of a rectangular region about one of its sides. This result may now be used to determine the first moment about the x-axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



If a narrow strip, of width δx and height y, is regarded as approximately a rectangle, its first moment about the x-axis is $\frac{1}{2}y^2\delta x$. Hence, the first moment of the whole region about the x-axis is given by

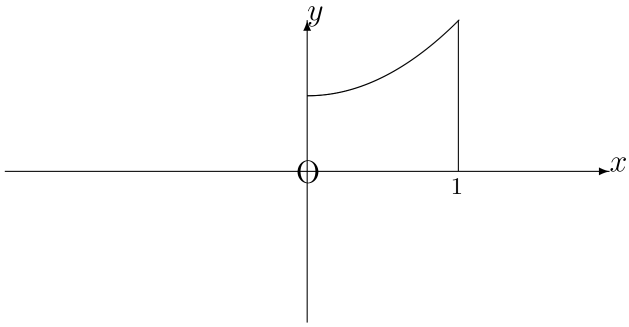
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{2}y^2\delta x = \int_a^b \frac{1}{2}y^2 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution

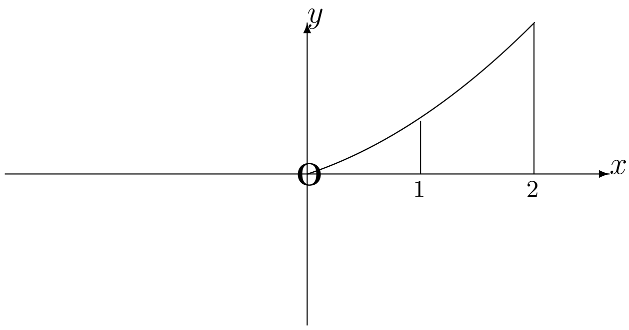


$$\text{First moment } = \int_0^1 \frac{1}{2}(x^2+1)^2 \, \mathrm{d}x = \frac{1}{2}\int_0^1 (x^4+2x^2+1) \, \mathrm{d}x = \frac{1}{2}\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_0^1 = \frac{28}{15}.$$

2. Determine the first moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the lines $x = 1$, $x = 2$ and the curve

$$y = xe^x.$$

Solution



$$\text{First moment } = \int_1^2 \frac{1}{2}x^2e^{2x} \, \mathrm{d}x$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \int_1^2 xe^{2x}\ \mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\left[x^2\frac{e^{2x}}{2}\right]_1^2 - \left[x\frac{e^{2x}}{2}\right]_1^2 + \int_1^2 \frac{e^{2x}}{2}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{2}\left[x^2\frac{e^{2x}}{2} - x\frac{e^{2x}}{2} + \frac{e^{2x}}{4}\right]_1^2 = \frac{5e^4 - e^2}{8} \simeq 33.20$$

13.7.4 THE CENTROID OF AN AREA

Having calculated the first moments of a two dimensional region about both the x-axis and the y-axis, it is possible to determine a point, $(\overline{x}, \overline{y})$, in the xy-plane with the property that

(a) The first moment about the y-axis is given by $A\overline{x}$, where A is the total area of the region;

and

(b) The first moment about the x-axis is given by $A\overline{y}$, where A is the toal area of the region.

The point is called the **"centroid"** or the **"geometric centre"** of the region and, in the case of a region bounded, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, its co-ordinates are given by

$$\overline{x} = \frac{\int_a^b\ xy\ \mathrm{d}x}{\int_a^b\ y\mathrm{d}x} \quad \text{and} \quad \overline{y} = \frac{\int_a^b\ \frac{1}{2}y^2\ \mathrm{d}x}{\int_a^b\ y\ \mathrm{d}x}.$$

Notes:

(i) The first moment of an area, about an axis through its centroid will, by definition, be zero. In particular, if we take the y-axis to be parallel to the given axis, with x as the perpendicular distance from an element, δA, to the y-axis, the first moment about the given axis will be

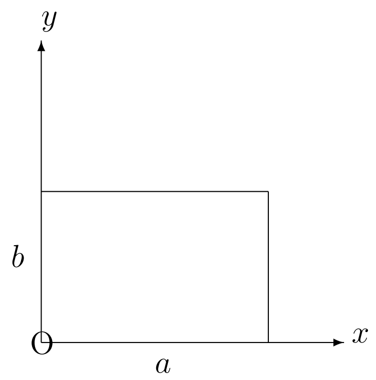
$$\sum_{\mathrm{R}}\ (x - \overline{x})\delta A = \sum_{\mathrm{R}}\ x\delta A - \overline{x}\sum_{\mathrm{R}}\ \delta A = A\overline{x} - A\overline{x} = 0.$$

(ii) The centroid effectively tries to concentrate the whole area at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass for a thin plate with uniform density, whose shape is that of the region which we have been considering.

EXAMPLES

1. Determine the position of the centroid of a rectangular region with sides of lengths, a and b.

   Solution

   The area of the rectangle is ab and, from Example 1 in section 13.7.2, the first moments about the y-axis and the x-axis are $\frac{1}{2}a^2b$ and $\frac{1}{2}b^2a$, respectively.

   Hence,

   $$\overline{x} = \frac{\frac{1}{2}a^2b}{ab} = \frac{1}{2}a$$

   and

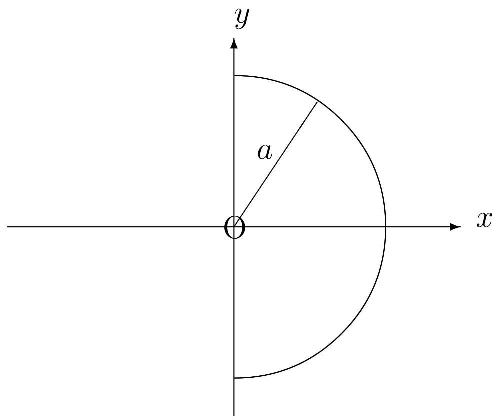
   $$\overline{y} = \frac{\frac{1}{2}b^2a}{ab} = \frac{1}{2}b,$$

   as we would expect for a rectangle.

2. Determine the position of the centroid of the semi-circular region bounded, in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



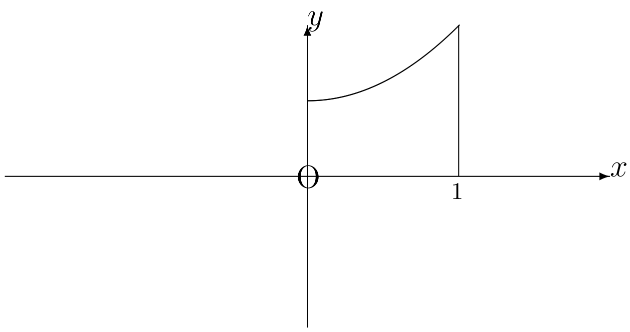
The area of the semi-circular region is $\frac{1}{2}\pi a^2$ and so, from Example 2, in section 13.7.2,

$$\overline{x} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \text{ and } \overline{y} = 0.$$

3. Determine the position of the centroid of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = x^2 + 1.$$

Solution



The first moment about the y-axis is given by

$$\int_0^1 x(x^2+1)\ \mathrm{d}x = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^1 = \frac{3}{4}.$$

The area is given by

$$\int_0^1 (x^2+1)\ \mathrm{d}x = \left[\frac{x^3}{3}+x\right]_0^1 = \frac{4}{3}.$$

Hence,

$$\overline{x} = \frac{3}{4} \div \frac{4}{3} = 1.$$

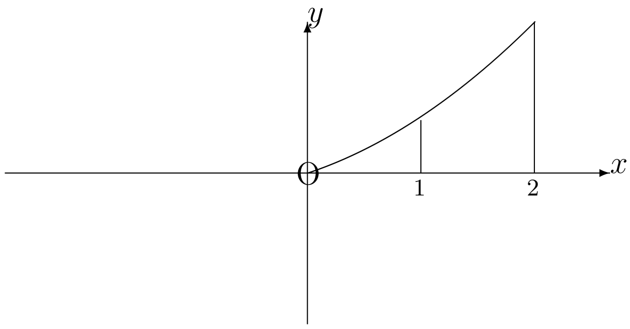
The first moment about the x-axis is $\frac{28}{15}$, from Example 1 in section 13.7.3; and, therefore,

$$\overline{y} = \frac{28}{15} \div \frac{4}{3} = \frac{7}{5}.$$

4. Determine the position of the centroid of the region bounded in the first quadrant by the x-axis, the lines $x=1$, $x=2$ and the curve whose equation is

$$y = xe^x.$$

Solution



The first moment about the y-axis is given by

$$\int_1^2 x^2e^x \, \mathrm{d}x = \left[x^2e^x - 2xe^x + 2e^x\right]_1^2 \simeq 12.06,$$

using integration by parts (twice).
The area is given by

$$\int_1^2 xe^x \, \mathrm{d}x = [xe^x - e^x]_1^2 \simeq 7.39$$

using integration by parts (once).
Hence,

$$\overline{x} \simeq 12.06 \div 7.39 \simeq 1.63$$

The first moment about the x-axis is approximately 33.20, from Example 2 in section 13.7.3; and so,

$$\overline{y} \simeq 33.20 \div 7.39 \simeq 4.47$$

13.7.5 EXERCISES

Determine the position of the centroid of each of the following regions of the xy-plane:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$x^2 + y^2 = a^2.$$

2. Bounded by the line $x = 1$ and the semi-circle whose equation is

$$(x-1)^2 + y^2 = 4, \ x > 1.$$

3. Bounded in the fourth quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 2x^2 - 1.$$

4. Bounded in the first quadrant by the x-axis and the curve whose equation is

$$y = \sin x.$$

5. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = xe^{-2x}.$$

13.7.6 ANSWERS TO EXERCISES

1.

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

2.

$$\left(\frac{11}{3\pi}, 0\right).$$

3.

$$\left(\frac{3\sqrt{2}}{16}, -\frac{13}{20}\right).$$

4.

$$\left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

5.

$$(0.28, 0.04).$$

"JUST THE MATHS"

UNIT NUMBER

13.8

INTEGRATION APPLICATIONS 8
(First moments of a volume)

by

A.J.Hobson

UNIT 13.8 - INTEGRATION APPLICATIONS 8

FIRST MOMENTS OF A VOLUME

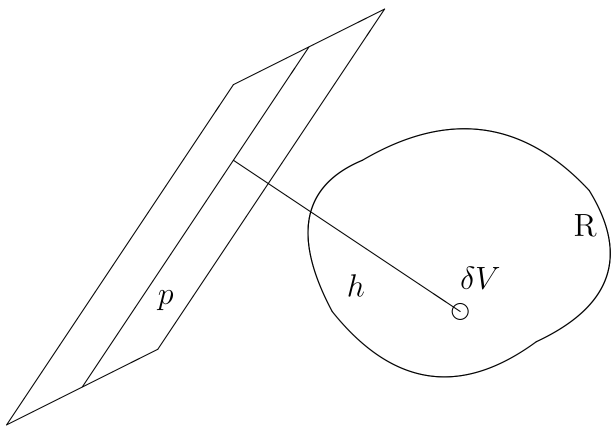
13.8.1 INTRODUCTION

Suppose that R denotes a region of space (with volume V) and suppose that δV is the volume of a small element of this region.

Then the **"first moment"** of R about a fixed plane, p, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h\delta V,$$

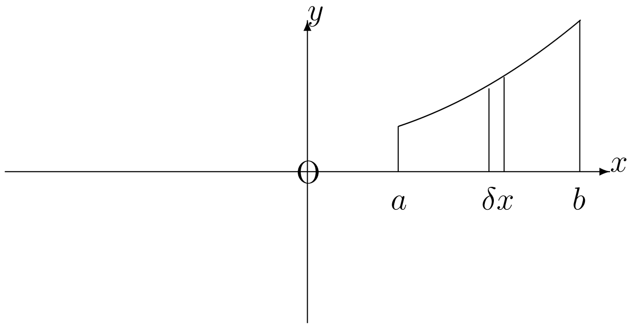
where h is the perpendicular distance, from p, of the element with volume, δV.



13.8.2 FIRST MOMENT OF A VOLUME OF REVOLUTION ABOUT A PLANE THROUGH THE ORIGIN, PERPENDICULAR TO THE X-AXIS.

Let us consider the volume of revolution about the x-axis of a region, in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



For a narrow 'strip' of width, δx, and height, y, parallel to the y-axis, the volume of revolution will be a thin disc with volume $\pi y^2 \delta x$ and all the elements of volume within it have the same perpendicular distance, x, from the plane about which moments are being taken.

Hence the first moment of this disc about the given plane is x times the volume of the disc; that is, $x(\pi y^2 \delta x)$, implying that the total first moment is given by

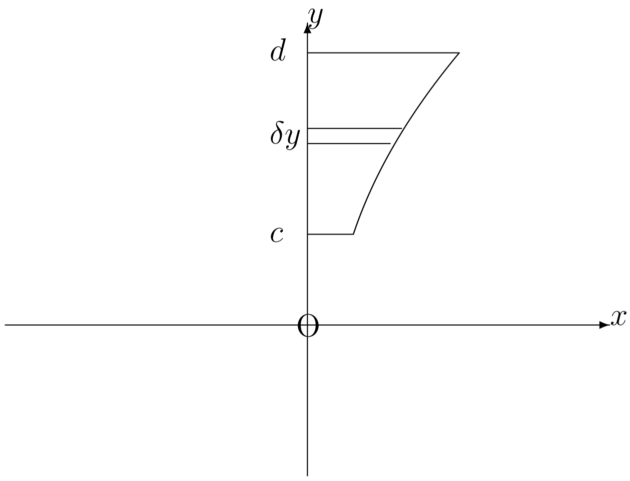
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi x y^2 \delta x = \int_a^b \pi x y^2 \, \mathrm{d}x.$$

Note:
For the volume of revolution about the y-axis of a region in the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the first moment of the volume about a plane through the origin, perpendicular to the y-axis, is given by

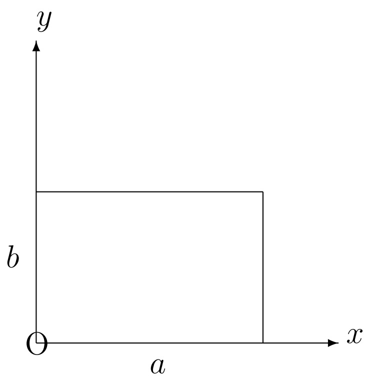
$$\int_c^d \pi y x^2 \, \mathrm{d}y.$$



EXAMPLES

1. Determine the first moment of a solid right-circular cylinder with height, a and radius b, about one end.

   Solution

   Let us consider the volume of revolution about the x-axis of the region, bounded in the first quadrant of the xy-plane, by the x-axis, the y-axis and the lines $x = a$, $y = b$.

   The first moment of the volume about a plane through the origin, perpendicular to the x-axis, is given by

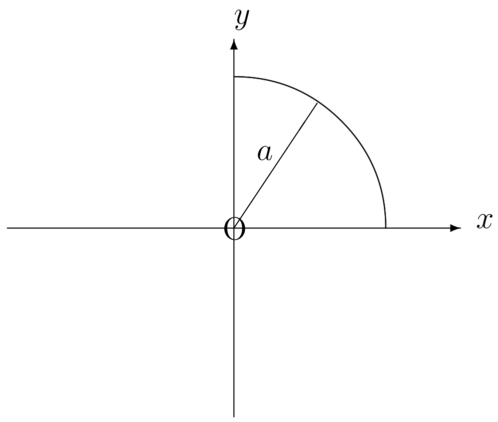
$$\int_0^a \pi x b^2 \, \mathrm{d}x = \left[\frac{\pi x^2 b^2}{2}\right]_0^a = \frac{\pi a^2 b^2}{2}.$$

2. Determine the first moment of volume, about its plane base, of a solid hemisphere with radius a.

   Solution

   Let us consider the volume of revolution about the x-axis of the region, bounded in the first quadrant, by the x-axis, y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$



The first moment of the volume about a plane through the origin, perpendicular to the x-axis is given by

$$\int_0^a \pi x(a^2 - x^2) \, \mathrm{d}x = \left[\pi(\left(\frac{a^2 x^2}{2} - \frac{x^4}{4}\right)\right]_0^a = \pi \left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{\pi a^4}{4}.$$

Note:

The symmetry of the solid figures in the above two examples shows that their first moments about a plane through the origin, perpendicular to the y-axis would be zero. This is because, for each $y\delta V$ in the calculation of the total first moment, there will be a corresponding $-y\delta V$.

In much the same way, the first moments of volume about the xy-plane (or indeed any plane of symmetry) would also be zero.

13.8.3 THE CENTROID OF A VOLUME

Suppose R denotes a volume of revolution about the x-axis of a region of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$

Having calculated the first moment of R about a plane through the origin, perpendicular to the x-axis (assuming that this is not a plane of symmetry), it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $V\overline{x}$, where V is the total volume of revolution about the x-axis.

The point is called the **"centroid"** or the **"geometric centre"** of the volume, and $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b \ \pi x y^2 \ \mathrm{d}x}{\int_a^b \ \pi y^2 \mathrm{d}x} = \frac{\int_a^b \ x y^2 \ \mathrm{d}x}{\int_a^b \ y^2 \mathrm{d}x}.$$

Notes:

(i) The centroid effectively tries to concentrate the whole volume at a single point for the purposes of considering first moments. It will always lie on the line of intersection of any two planes of symmetry.

(ii) In practice, the centroid corresponds to the position of the centre of mass for a solid with uniform density, whose shape is that of the volume of revolution which we have been considering.

(iii) For a volume of revolution about the y-axis, from $y = c$ to $y = d$, the centroid will lie on the y-axis, and its distance, $\overline{y}$, from the origin will be given by

$$\overline{y} = \frac{\int_c^d \ \pi y x^2 \ \mathrm{d}y}{\int_c^d \ \pi x^2 \mathrm{d}y} = \frac{\int_c^d \ y x^2 \ \mathrm{d}y}{\int_c^d \ x^2 \mathrm{d}y}.$$

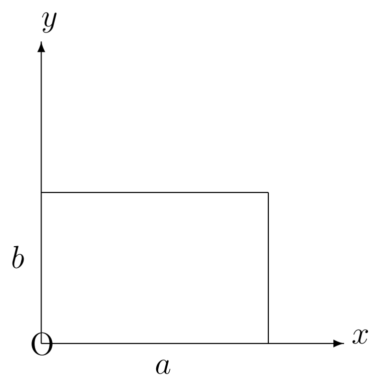
(iv) The first moment of a volume about a plane through its centroid will, by definition, be zero. In particular, if we take the plane through the y-axis, perpendicular to the x-axis to be parallel to the plane through the centroid, with x as the perpendicular distance from an element, δV, to the plane through the y-axis, the first moment about the plane through the centroid will be

$$\sum_{\mathrm{R}}(x-\overline{x})\delta V=\sum_{\mathrm{R}}x\delta V-\overline{x}\sum_{\mathrm{R}}\delta V=V\overline{x}-V\overline{x}=0.$$

EXAMPLES

1. Determine the position of the centroid of a solid right-circular cylinder with height, a, and radius, b.

   Solution



Using Example 1 in Section 13.8.2, the centroid will lie on the x-axis and the first moment about a plane through the origin, perpendicular to the x-axis is $\frac{\pi a^2b^2}{2}$.

Also, the volume is $\pi b^2 a$.

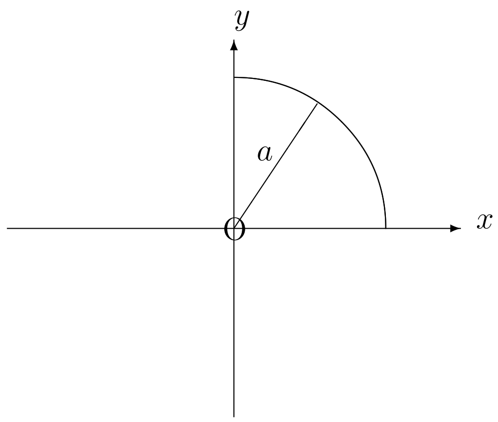
Hence,

$$\overline{x}=\frac{\frac{\pi a^2b^2}{2}}{\pi b^2 a}=\frac{a}{2},$$

as we would expect for a cylinder.

2. Determine the position of the centroid of a solid hemisphere with base-radius, a.

Solution



Let us consider the volume of revolution about the x-axis of the region bounded in the first quadrant by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2$$

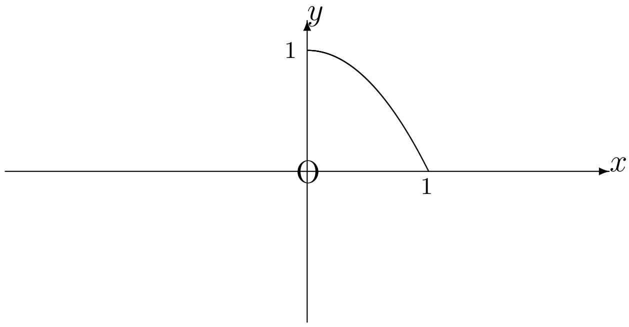
From Example 2 in Section 13.8.2, the centroid will lie on the x-axis and the first moment of volume about a plane through the origin, perpendicular to the x-axis is $\frac{\pi a^4}{4}$. Also, the volume of the hemisphere is $\frac{2}{3}\pi a^3$ and so,

$$\overline{x} = \frac{\frac{2}{3}\pi a^3}{\frac{\pi a^4}{4}} = \frac{3a}{8}.$$

3. Determine the position of the centroid of the volume of revolution about the y-axis of region bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - x^2.$$

Solution



Firstly, by symmetry, the centroid will lie on the y-axis.

Secondly, the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_0^1 \pi y(1-y)\ \mathrm{d}y = \pi\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

Thirdly, the volume is given by

$$\int_0^1 \pi(1-y)\ \mathrm{d}y = \left[y-\frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

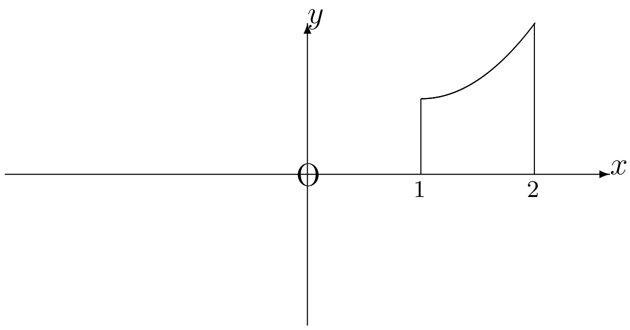
Hence,

$$\overline{y} = \frac{\pi}{6} \div \frac{\pi}{2} = \frac{1}{3}.$$

4. Determine the position of the centroid of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the lines $x = 1$, $x = 2$ and the curve whose equation is

$$y = e^x.$$

Solution



Firstly, by symmetry, the centroid will lie on the x axis.

Secondly, the First Moment about a plane through the origin, perpendicular to the x-axis is given by

$$\int_1^2 \pi x e^{2x} \, \mathrm{d}x = \pi \left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} \right]_1^2 \simeq 122.84,$$

using integration by parts.

The volume is given by

$$\int_1^2 \pi e^{2x} \, \mathrm{d}x = \pi \left[\frac{e^{2x}}{2} \right]_1^2 \simeq 74.15$$

Hence,

$$\overline{x} \simeq 122.84 \div 74.15 \simeq 1.66$$

13.8.4 EXERCISES

1. Determine the position of the centroid of the volume obtained when each of the following regions of the xy-plane is rotated through 2π radians about the x-axis:

   (a) Bounded in the first quadrant by the x-axis, the line $x = 1$ and the quarter-circle represented by

   $$(x-1)^2 + y^2 = 4, \; x > 1, y > 0;$$

(b) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - 2x^2;$$

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = \frac{\pi}{2}$ and the curve whose equation is

$$y = \sin x;$$

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = \sqrt{x}e^{-x}.$$

2. A solid right-circular cone, whose vertex is at the origin, has, for its central axis, the part of the y-axis between $y = 0$ and $y = h$. Determine the position of the centroid of the cone.

13.8.5 ANSWERS TO EXERCISES

1. (a)

$$\overline{x} = 1.75;$$

(b)

$$\overline{x} \simeq 0.22;$$

(c)

$$\overline{x} \simeq 1.10;$$

(d)

$$\overline{x} \simeq 0.36$$

2.

$$\overline{y} = \frac{3h}{4}.$$

"JUST THE MATHS"

UNIT NUMBER

13.9

INTEGRATION APPLICATIONS 9
(First moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.9 - INTEGRATION APPLICATIONS 9

FIRST MOMENTS OF A SURFACE OF REVOLUTION

13.9.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"first moment"** about a plane through the origin, perpendicular to the x-axis, is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} 2\pi xy\delta s,$$

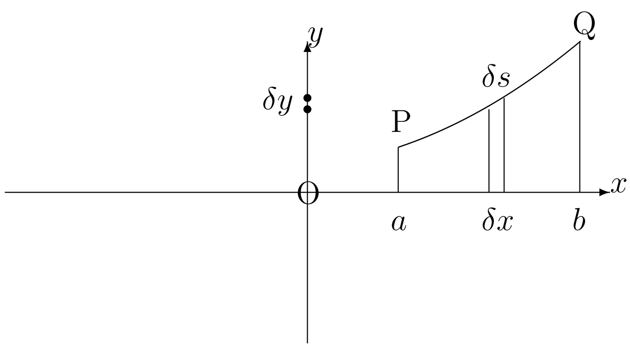
where x is the perpendicular distance, from the plane of moments, of the thin band, with surface area $2\pi y\delta s$, so generated.

13.9.2 INTEGRATION FORMULAE FOR FIRST MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x;$$

so that, for the surface of revolution of the arc about the x-axis, the first moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x y\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\delta x = \int_a^b 2\pi x y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} 2\pi xy\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the first moment about the plane through the origin, perpendicular to the x-axis is given by

$$\text{First Moment} = \pm\int_{t_1}^{t_2} 2\pi xy\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

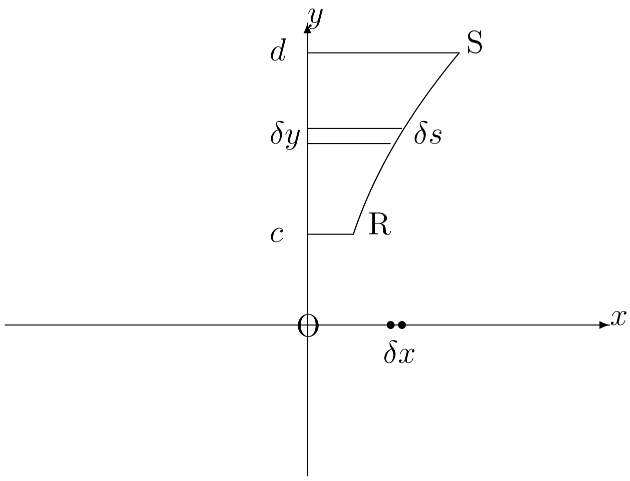
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the first moment about a plane through the origin, perpendicular to the y-axis is given by

$$\int_c^d 2\pi yx\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the first moment about a plane through the origin, perpendicular to the y-axis, is given by

$$\text{First moment} = \pm \int_{t_1}^{t_2} 2\pi yx\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

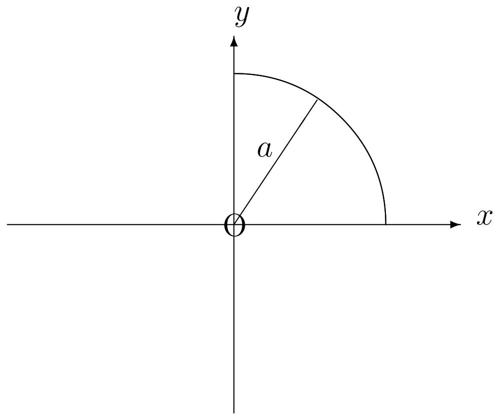
EXAMPLES

1. Determine the first moment about a plane through the origin, perpendicular to the x-axis, for the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.
The first moment about the specified plane is therefore given by

$$\int_0^a 2\pi xy\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi xy\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

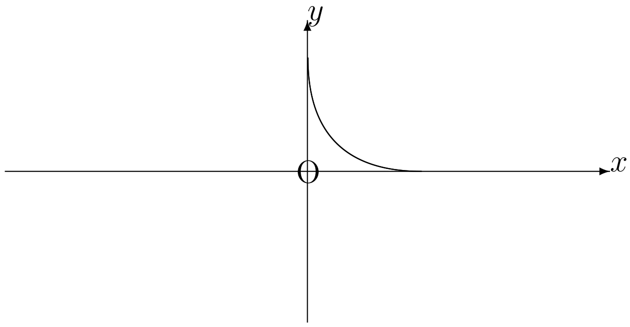
But $x^2 + y^2 = a^2$, and so the first moment becomes

$$\int_0^a 2\pi ax\ \mathrm{d}x = [\pi ax^2]_0^a = \pi a^3.$$

2. Determine the first moments about planes through the origin, (a) perpendicular to the x-axis and (b) perpendicular to the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\mathrm{cos}^3\theta, \quad y = a\mathrm{sin}^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the first moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi xy\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^2\cos^3\theta\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^3\cos^4\theta\sin^4\theta\ \mathrm{d}\theta.$$

Using $2\sin\theta\cos\theta \equiv \sin 2\theta$, the integral reduces to

$$\frac{3\pi a^3}{8}\int_0^{\frac{\pi}{2}} \sin^4 2\theta\ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7, gives

$$\frac{3\pi a^3}{32}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos 4\theta + \frac{1+\cos 8\theta}{2}\right)\ \mathrm{d}\theta = \frac{3\pi a^3}{32}\left[\frac{3\theta}{2} - \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{9\pi a^3}{128}.$$

By symmetry, or by direct integration, the first moment about a plane through the origin, perpendicular to the y-axis is also $\frac{9\pi a^3}{128}$.

13.9.3 THE CENTROID OF A SURFACE OF REVOLUTION

Having calculated the first moment of a surface of revolution about a plane through the origin, perpendicular to the x-axis, it is possible to determine a point, $(\overline{x}, 0)$, on the x-axis with the property that the first moment is given by $S\overline{x}$, where S is the total surface area.

The point is called the **"centroid"** or the **"geometric centre"** of the surface of revolution and, for the surface of revolution of the arc of the curve whose equation is $y = f(x)$, between $x = a$ and $x = b$, the value of $\overline{x}$ is given by

$$\overline{x} = \frac{\int_a^b 2\pi xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}{\int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x} = \frac{\int_a^b xy\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x}{\int_a^b y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x}.$$

Note:
The centroid effectively tries to concentrate the whole surface at a single point for the purposes of considering first moments. In practice, it corresponds to the position of the centre of mass of a thin sheet, for example, with uniform density.

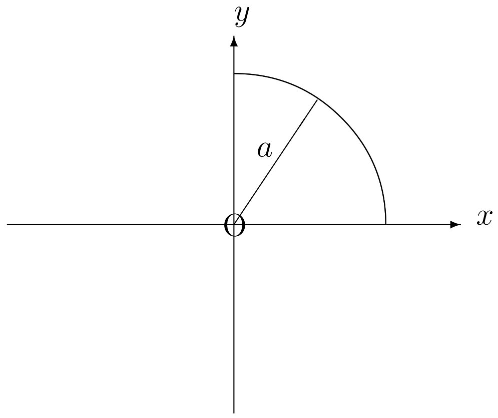
EXAMPLES

1. Determine the position of the centroid of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 of Section 13.9.2, we know that the first moment of the surface about a plane through the origin, perpendicular to the the x-axis is equal to πa^3.

Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

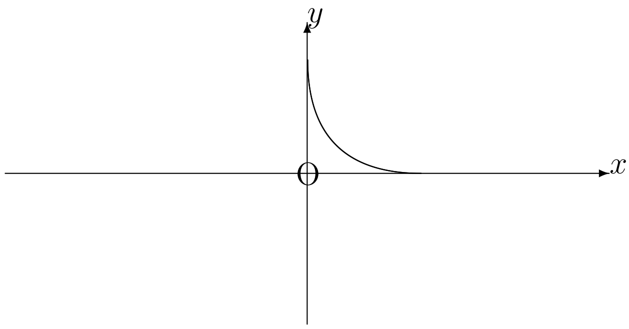
which implies that

$$\overline{x} = \frac{\pi a^3}{2\pi a^2} = \frac{a}{2}.$$

2. Determine the position of the centroid of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 of Section 13.9.2 that that the first moment of the surface about a plane through the origin, perpendicular to the x-axis is equal to $\frac{9\pi a^3}{128}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a \sin^3\theta . 3a\cos\theta\sin\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 3a^2 \sin^4\theta\cos\theta \, \mathrm{d}\theta = 3\pi a^2 \left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$\overline{x} = \frac{15a}{128}.$$

13.9.4 EXERCISES

1. Determine the first moment, about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(3, 4)$.

2. Determine the first moment about a plane through the origin, perpendicular to the x-axis, of the surface of revolution (about the x-axis) of the arc of the curve whose equation is

$$y^2 = 4x,$$

lying between $x = 0$ and $x = 1$.

3. Determine the first moment about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose equation is

$$y^2 = 4(x - 1),$$

lying between $y = 2$ and $y = 4$.

4. Determine the first moment, about a plane through the origin, perpendicular to the y-axis, of the surface of revolution (about the y-axis) of the arc of the curve whose parametricc equations are

$$x = 2\cos t, \quad y = 3\sin t,$$

joining the point $(2, 0)$ to the point $(0, 3)$.

5. Determine the position of the centroid of a hollow right-circular cone with height h.

6. For the curve whose equation is

$$9y^2 = x(3 - x)^2,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - x}{2\sqrt{x}}.$$

Hence, show that the centroid of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis lies at the point $\left(\frac{5}{4}, 0\right)$.

13.9.5 ANSWERS TO EXERCISES

1.

$$40\pi.$$

2.

$$4\pi\left[\frac{12\sqrt{2}}{5}-\frac{4}{15}\right]\simeq 39.3$$

3.

$$\left[\frac{8\pi}{5}\left(1+\frac{y^2}{4}\right)^{\frac{5}{2}}\right]_2^4\simeq 41.98$$

4.

$$\left[-\frac{4\pi}{5}\left(4+5\cos^2 t\right)^{\frac{3}{2}}\right]_0^{\frac{\pi}{2}}\simeq 47.75$$

5. Along the central axis, at a distance of $\frac{2h}{3}$ from the vertex.

6.

$$\text{First moment}=\frac{15\pi}{4}\quad\text{Surface Area}=3\pi.$$

“JUST THE MATHS”

UNIT NUMBER

13.10

INTEGRATION APPLICATIONS 10
(Second moments of an arc)

by

A.J.Hobson

UNIT 13.10 - INTEGRATION APPLICATIONS 10

SECOND MOMENTS OF AN ARC

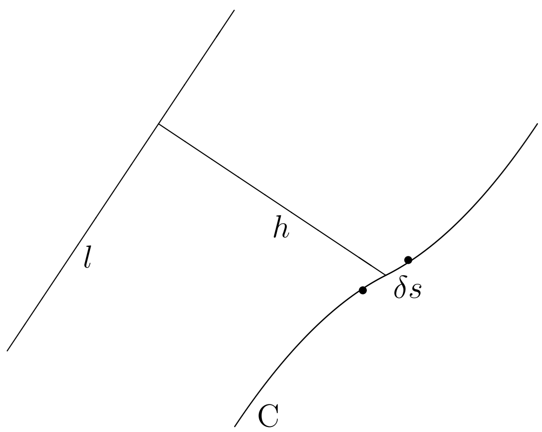
13.10.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then the **"second moment"** of C about a fixed line, l, in the plane of C is given by

$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} h^2 \delta s,$$

where h is the perpendicular distance, from l, of the element with length δs.

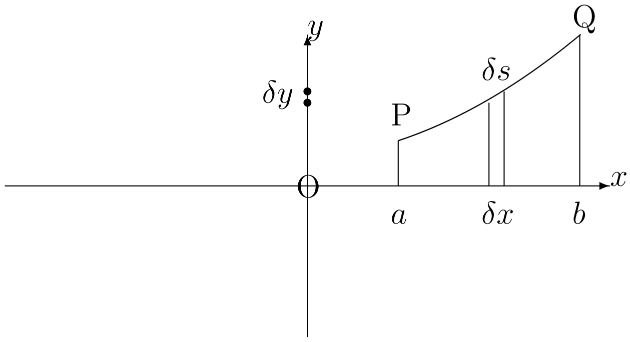


13.10.2 THE SECOND MOMENT OF AN ARC ABOUT THE Y-AXIS

Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

The second moment of each element about the y-axis is x^2 times the length of the element; that is, $x^2\delta s$, implying that the total second moment of the arc about the y-axis is given by

$$\lim_{\delta s\to 0}\sum_{\mathrm{C}} x^2\delta s.$$

But, by Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2+(\delta y)^2} = \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x,$$

so that the second moment of arc becomes

$$\lim_{\delta x\to 0}\sum_{x=a}^{x=b} x^2\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b x^2\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t),\quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the y-axis is given by

$$\pm\int_{t_1}^{t_2} x^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

13.10.3 THE SECOND MOMENT OF AN ARC ABOUT THE X-AXIS

(a) For an arc whose equation is

$$y = f(x),$$

contained between $x = a$ and $x = b$, the second moment about the x-axis will be

$$\int_a^b y^2\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, the second moment of the arc about the x-axis is given by

$$\pm\int_{t_1}^{t_2} y^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

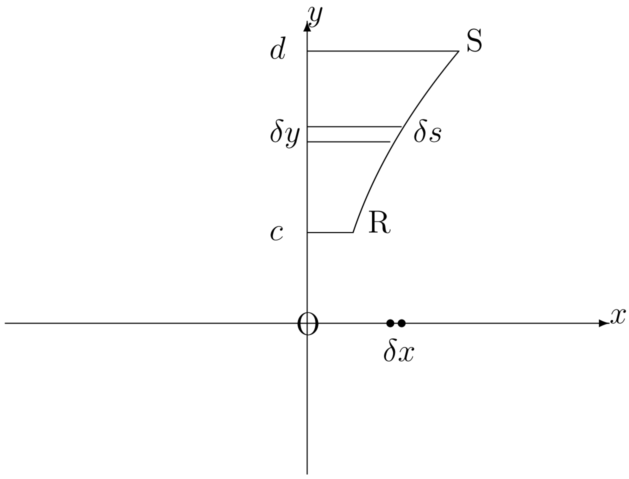
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in section 13.10.2 so that the second moment about the x-axis is given by

$$\int_c^d y^2 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y.$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then, using the same principles as in Unit 13.4, we may conclude that the second moment of the arc about the x-axis is given by

$$\pm \int_{t_1}^{t_2} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative and where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$.

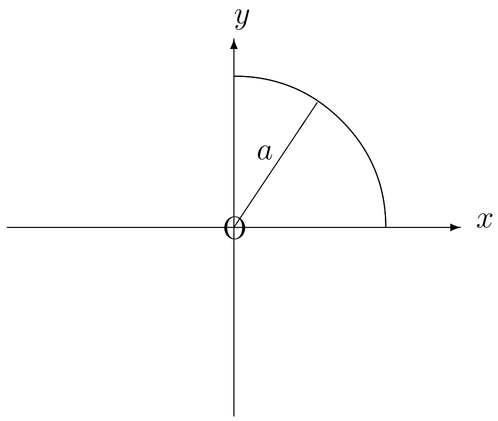
EXAMPLES

1. Determine the second moments about the x-axis and the y-axis of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the y-axis is therefore given by

$$\int_0^a x^2\sqrt{1 + \frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a \frac{x^2}{y}\sqrt{x^2 + y^2}\ \mathrm{d}x.$$

But $x^2 + y^2 = a^2$ and, hence,

$$\text{second moment} = \int_0^a \frac{ax^2}{y}\mathrm{d}x.$$

Making the substitution $x = a\sin u$ gives

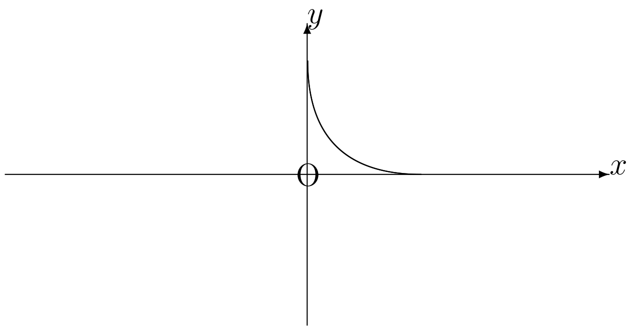
$$\text{second moment} = \int_0^{\frac{\pi}{2}} a^3\sin^2 u \ \mathrm{d}u = a^3\int_0^{\frac{\pi}{2}} \frac{1-\cos 2u}{2}\ \mathrm{d}u = a^3\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^3}{4}.$$

By symmetry, the second moment about the x-axis will also be $\frac{\pi a^3}{4}$.

2. Determine the second moments about the x-axis and the y-axis of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the y-axis is given by

$$-\int_{\frac{\pi}{2}}^0 x^2\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta + \sin^2\theta \equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} a^2\cos^6\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \cos^7\theta \sin\theta \ \mathrm{d}\theta$$

$$= 3a^2 \left[-\frac{\cos^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8}.$$

Similarly, the second moment about the x-axis is given by

$$\int_0^{\frac{\pi}{2}} y^2 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} a^2\sin^6\theta.(3a\cos\theta\sin\theta) \ \mathrm{d}\theta$$

$$= 3a^3 \int_0^{\frac{\pi}{2}} \sin^7\theta \cos\theta \ \mathrm{d}\theta = 3a^3 \left[\frac{\sin^8\theta}{8} \right]_0^{\frac{\pi}{2}} = \frac{3a^3}{8},$$

though, again, this second result could be deduced, by symmetry, from the first.

13.10.4 THE RADIUS OF GYRATION OF AN ARC

Having calculated the second moment of an arc about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by sk^2, where s is the total length of the arc.

We simply divide the value of the second moment by s in order to obtain the value of k^2 and, hence, the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole arc at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

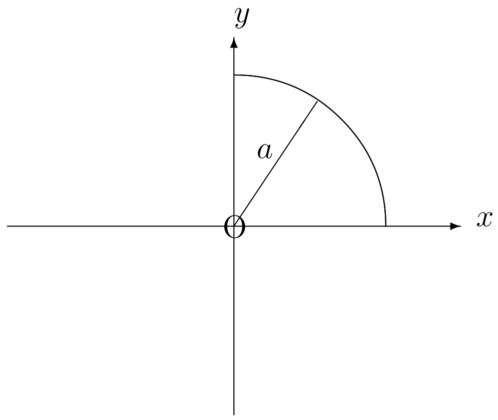
EXAMPLES

1. Determine the radius of gyration, about the y-axis, of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From Example 1 in Section 13.10.3, we know that the Second Moment of the arc about the y-axis is equal to $\frac{\pi a^3}{4}$.

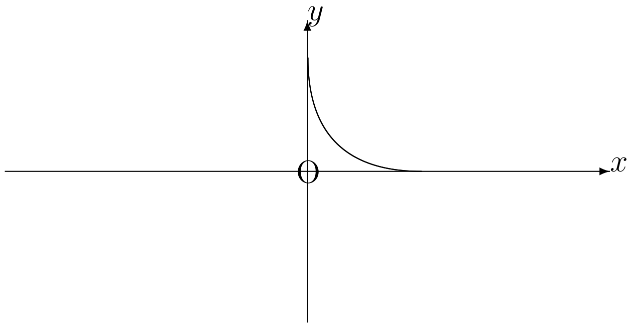
Also, the length of the arc is $\frac{\pi a}{2}$, which implies that the radius of gyration is

$$\sqrt{\frac{\pi a^3}{4} \times \frac{2}{\pi a}} = \frac{a}{\sqrt{2}}.$$

2. Determine the radius of gyration, about the y-axis, of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



From Example 2 in Section 13.10.3, we know that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$$

and that the second moment of the arc about the y-axis is equal to $\frac{3a^3}{8}$.
Also, the length of the arc is given by

$$-\int_{\frac{\pi}{2}}^{a} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta.$$

This simplifies to

$$3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{3a}{2}.$$

Thus, the radius of gyration is

$$\sqrt{\frac{3a^3}{8}\times\frac{2}{3a}} = \frac{a}{2}.$$

13.10.5 EXERCISES

1. Determine the second moments about (a) the x-axis and (b) the y-axis of the straight line segment with equation

$$y = 2x + 1,$$

lying between $x = 0$ and $x = 3$.

2. Determine the second moment about the y-axis of the first-quadrant arc of the curve whose equation is

$$25y^2 = 4x^5,$$

lying between $x = 0$ and $x = 2$.

3. Determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y = e^x,$$

lying between $x = 0.1$ and $x = 0.5$.

4. Given that

$$\int \sqrt{x^2 - 1}\ \mathrm{d}x = \frac{1}{2}\left(x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})\right) + C,$$

determine, correct to two places of decimals, the second moment, about the x-axis, of the arc of the curve whose equation is

$$y^2 = 8x,$$

lying between $x = 0$ and $x = 1$.

5. Verify, using integration, that the radius of gyration, about the y-axis, of the straight line segment defined by the equation

$$y = 3x + 2,$$

from $x = 0$ to $x = 1$ is $\frac{1}{\sqrt{3}}$.

6. Determine the radius of gyration about the x-axis of the arc of the circle given parametrically by

$$x = 5\cos\theta, \quad y = 5\sin\theta,$$

from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

7. For the curve whose equation is

$$9y^2 = x(3-x)^2,$$

show that

$$\frac{dy}{dx} = \frac{1-x}{2\sqrt{x}}.$$

Hence, determine, correct to three significant figures, the radius of gyration, about the y-axis, of the first quadrant arch of this curve.

13.10.6 ANSWERS TO EXERCISES

1.

$$\text{(a)} \ \frac{9\sqrt{5}}{2} \quad \text{(b)} \ 12\sqrt{5}.$$

2.

$$\frac{52}{9} \simeq 5.78$$

3.

$$1.29$$

4.

$$8.59$$

5.

$$\text{Second moment} = \frac{\sqrt{10}}{3} \quad \text{Length} = \sqrt{10}.$$

6.

$$k = \sqrt{\frac{125(\pi-2)}{10\pi}} \simeq 2.13$$

7.

$$k \simeq 1.68$$

"JUST THE MATHS"

UNIT NUMBER

13.11

INTEGRATION APPLICATIONS 11
(Second moments of an area (A))

by

A.J.Hobson

UNIT 13.11 - INTEGRATION APPLICATIONS 11

SECOND MOMENTS OF AN AREA (A)

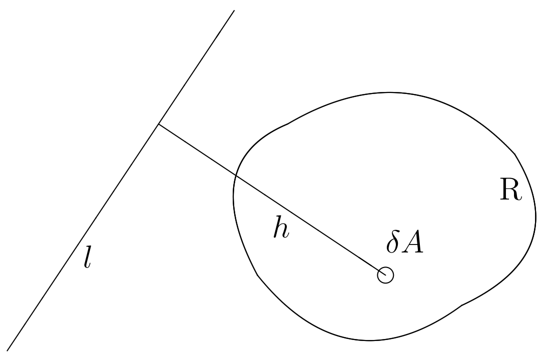
13.11.1 INTRODUCTION

Suppose that R denotes a region (with area A) of the xy-plane in cartesian co-ordinates, and suppose that δA is the area of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, **not necessarily in the plane of** R, is given by

$$\lim_{\delta A \to 0} \sum_{\text{R}} h^2 \delta A,$$

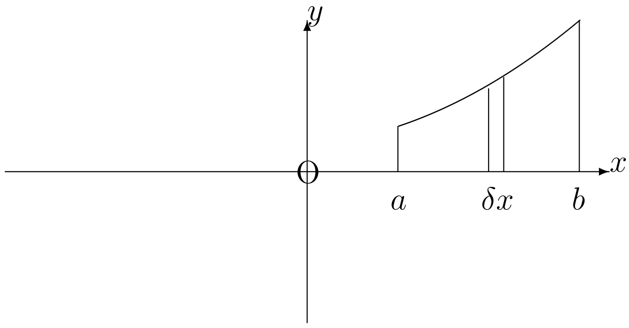
where h is the perpendicular distance from l of the element with area, δA.



13.11.2 THE SECOND MOMENT OF AN AREA ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The region may be divided up into small elements by using a network consisting of neighbouring lines parallel to the y-axis and neighbouring lines parallel to the x-axis.

But all of the elements in a narrow 'strip', of width δx and height y (parallel to the y-axis), have the same perpendicular distance, x, from the y-axis.

Hence the second moment of this strip about the y-axis is x^2 times the area of the strip; that is, $x^2(y\delta x)$, implying that the total second moment of the region about the y-axis is given by

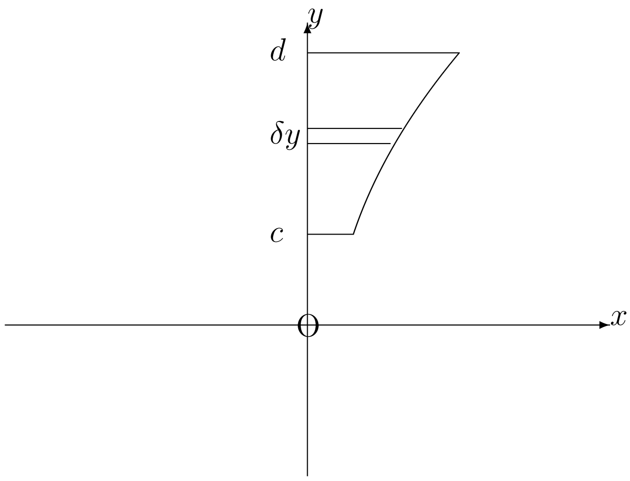
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} x^2 y \delta x = \int_a^b x^2 y \, \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for a region of the first quadrant, bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

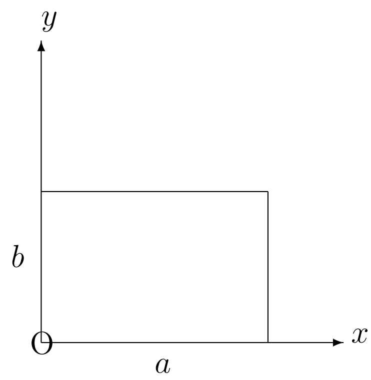
$$\int_c^d y^2 x \, \mathrm{d}y.$$



EXAMPLES

1. Determine the second moment of a rectangular region with sides of lengths, a and b, about the side of length b.

   Solution



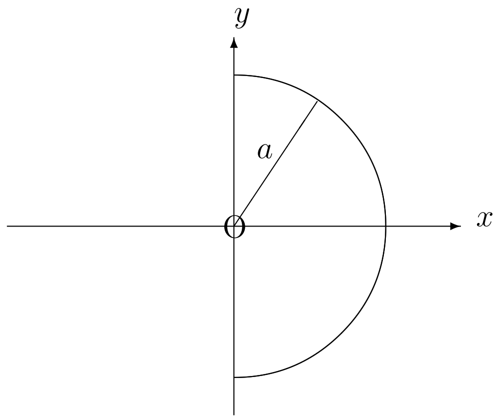
   The second moment about the y-axis is given by

$$\int_0^a x^2 b \, \mathrm{d}x = \left[\frac{x^3 b}{3}\right]_0^a = \frac{1}{3}a^3 b.$$

2. Determine the second moment about the y-axis of the semi-circular region, bounded in the first and fourth quadrants, by the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2.$$

Solution



Since there will be equal contributions from the upper and lower halves of the region, the second moment about the y-axis is given by

$$2\int_0^a x^2\sqrt{a^2 - x^2}\ \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

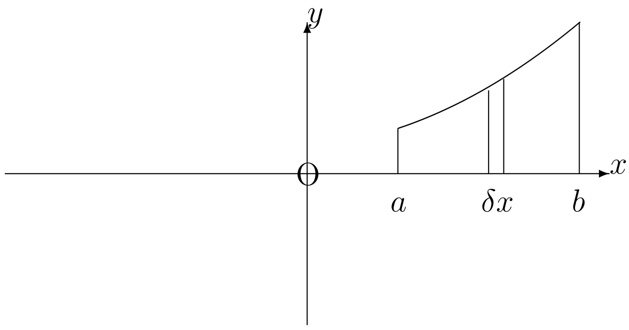
This simplifies to

$$2a^4\int_0^{\frac{\pi}{2}} \frac{\sin^2 2\theta}{4}\ \mathrm{d}\theta = \frac{a^4}{2}\int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2}\ \mathrm{d}\theta$$

$$= \frac{a^4}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}.$$

13.11.3 THE SECOND MOMENT OF AN AREA ABOUT THE X-AXIS

In the first example of the previous section, a formula was established for the second moment of a rectangular region about one of its sides. This result may now be used to determine the second moment about the x-axis, of a region enclosed, in the first quadrant, by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is $y = f(x)$.



If a narrow strip of width δx and height y is regarded, approximately, as a rectangle, its second moment about the x-axis is $\frac{1}{3}y^3\delta x$. Hence the second moment of the whole region about the x-axis is given by

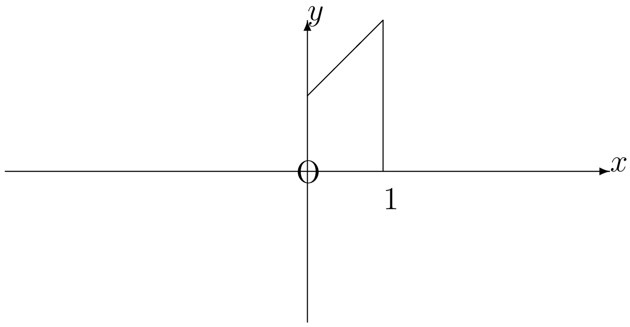
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{1}{3}y^3\delta x = \int_a^b \frac{1}{3}y^3 \, \mathrm{d}x.$$

EXAMPLES

1. Determine the second moment about the x-axis of the region bounded, in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{1}{3}(x+1)^3 \, \mathrm{d}x$$

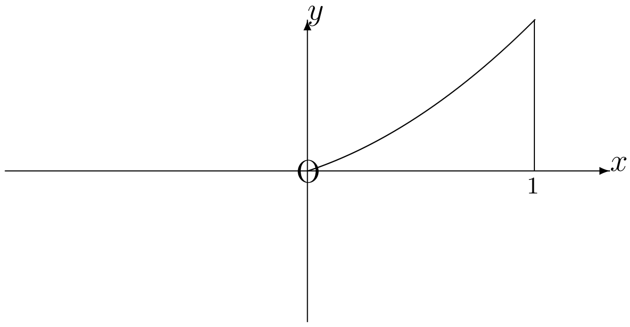
$$= \frac{1}{3}\int_0^1 (x^3 + 3x^2 + 3x + 1) \, \mathrm{d}x = \frac{1}{3}\left[\frac{x^4}{4} + x^3 + +\frac{3x^2}{2} + x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4} + 1 + \frac{3}{2} + 1\right) = \frac{5}{4}.$$

2. Determine the second moment about the x-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis, the line $x = 1$ and the curve

$$y = xe^x.$$

Solution



$$\text{Second moment } = \int_0^1 \frac{1}{3}x^3e^{3x}\ \mathrm{d}x$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \int_0^1 x^2e^{3x}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \int_0^1 2x\frac{e^{3x}}{3}\ \mathrm{d}x\right)$$

$$= \frac{1}{3}\left(\left[x^3\frac{e^{3x}}{3}\right]_0^1 - \left[x^2\frac{e^{3x}}{3}\right]_0^1 + \frac{2xe^{3x}}{9} - \frac{2}{3}\int_0^1 \frac{e^{3x}}{3}\ \mathrm{d}x\right).$$

That is,

$$\frac{1}{3}\left[x^3\frac{e^{3x}}{3} - x^2\frac{e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}\right]_0^1 = \frac{4e^3+2}{81} \simeq 1.02$$

Note:
The Second Moment of an area about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a thin plate with uniform density, ρ, the moment of inertia is ρ times the second moment of area, since multiplication by ρ, of elements of area, converts them into elements of mass.

13.11.7 EXERCISES

Determine the second moment of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is

$$y = 1 - 2x^2.$$

Axis: The y-axis.

2. Bounded in the first quadrant by the x-axis and the curve whose equation is

$$y = \sin x.$$

Axis: The x-axis.

3. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The x-axis

4. Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The y-axis.

13.11.8 ANSWERS TO EXERCISES

1.

$$\frac{\sqrt{2}}{30}.$$

2.

$$\frac{4}{9}.$$

3.

$$0.055, \text{ approximately.}$$

4.

$$0.083, \text{ approximately.}$$

"JUST THE MATHS"

UNIT NUMBER

13.12

INTEGRATION APPLICATIONS 12
(Second moments of an area (B))

by

A.J.Hobson

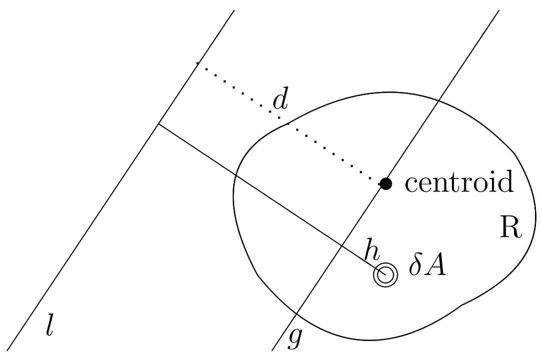
UNIT 13.12 - INTEGRATION APPLICATIONS 12

SECOND MOMENTS OF AN AREA (B)

13.12.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis, in the same plane as R and having a perpendicular distance of d from the first axis.



We have

$$M_l = \sum_{\mathrm{R}} (h+d)^2\delta A = \sum_{\mathrm{R}} (h^2 + 2hd + d^2).$$

That is,

$$M_l = \sum_{\mathrm{R}} h^2\delta A + 2d\sum_{\mathrm{R}} h\delta A + d^2\sum_{\mathrm{R}} \delta A = M_g + Ad^2,$$

since the summation, $\sum_{\mathrm{R}} h\delta A$, is the first moment about the an axis through the centroid and therefore zero; (see Unit 13.7, section 13.7.4).

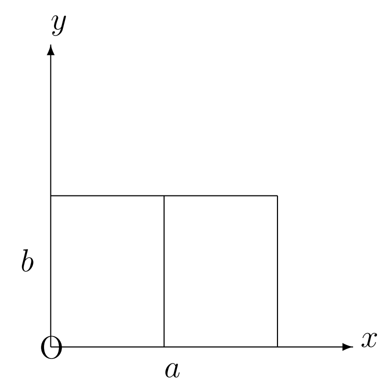
The Parallel Axis Theorem states that

$$M_l = M_g + Ad^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region about an axis through its centroid, parallel to one side.

   Solution

   For a rectangular region with sides of length a and b, the second moment about the side of length b is $\frac{a^3b}{3}$ from Example 1 in the previous Unit, section 13.11.2.

   The perpendicular distance between the two axes is then $\frac{a}{2}$, so that the required second moment, M_g is given by

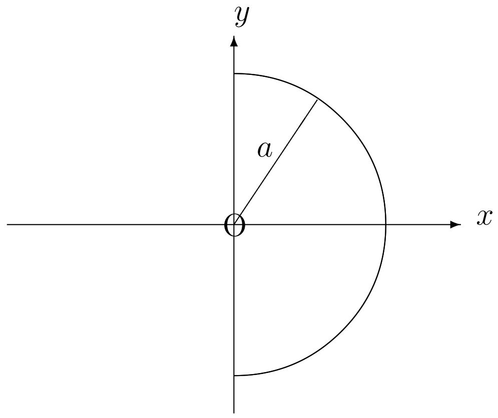
   $$\frac{a^3b}{3} = M_g + ab\left(\frac{a}{2}\right)^2 = M_g + \frac{a^3b}{4}$$

   Hence,

   $$M_g = \frac{a^3b}{12}.$$

2. Determine the second moment of a semi-circular region about an axis through its centroid, parallel to its diameter.

Solution



The second moment of the semi-circular region about its diameter is $\frac{\pi a^4}{8}$, from Example 2 in the previous Unit, section 13.11.2.

Also the position of the centroid, from Example 2 in Unit 13.7, section 13.7.4, is a distance of $\frac{4a}{3\pi}$ from the diameter, along the radius which perpendicular to it.

Hence,

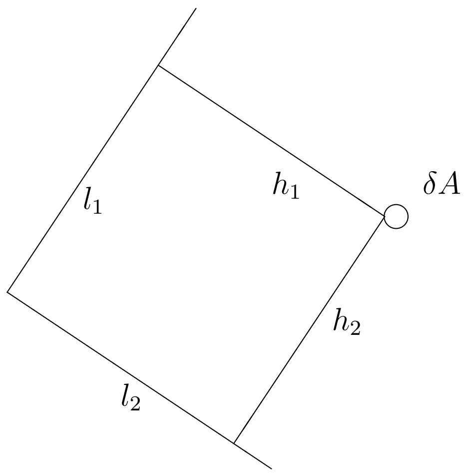
$$\frac{\pi a^4}{8} = M_g + \frac{\pi a^2}{2}.\left(\frac{4a}{3\pi}\right)^2 = M_g + \frac{8a^4}{9\pi^2}.$$

That is,

$$M_g = \frac{\pi a^4}{8} - \frac{8a^4}{9\pi^2}.$$

13.12.2 THE PERPENDICULAR AXIS THEOREM

Suppose l_1 and l_2 are two straight lines, at right-angles to each other, in the plane of a region R with area A and suppose h_1 and h_2 are the perpendicular distances from these two lines, respectively, of an element δA in R.



The second moment about l_1 is given by

$$M_1 = \sum_R h_1^2 \delta A$$

and the second moment about l_2 is given by

$$M_2 = \sum_R h_2^2 \delta A.$$

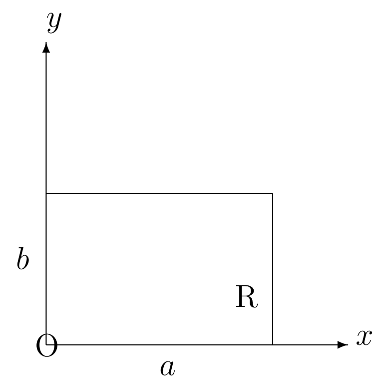
Adding these two together gives the second moment about an axis, perpendicular to the plane of R and passing through the point of intersection of l_1 and l_2. This is because the square of the perpendicular distance, h_3, of δA from this new axis is given, from Pythagoras's Theorem, by

$$h_3^2 = h_1^2 + h_2^2.$$

EXAMPLES

1. Determine the second moment of a rectangular region, R, with sides of length a and b, about an axis through one corner, perpendicular to the plane of R.

Solution

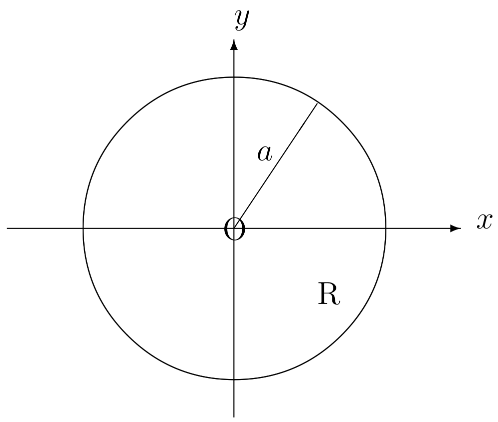


Using Example 1 in the previous Unit, section 13.11.2, the required second moment is

$$\frac{1}{3}a^3b + \frac{1}{3}b^3a = \frac{1}{3}ab(a^2 + b^2).$$

2. Determine the second moment of a circular region, R, with radius a, about an axis through its centre, perpendicular to the plane of R.

Solution



The second moment of R about a diameter is, from Example 2 in the previous Unit, section 13.11.2, equal to $\frac{\pi a^4}{4}$; that is, twice the value of the second moment of a semi-circular region about its diameter.

The required second moment is thus

$$\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}.$$

13.12.3 THE RADIUS OF GYRATION OF AN AREA

Having calculated the second moment of a two dimensional region about a certain axis it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Ak^2, where A is the total area of the region.

We simply divide the value of the second moment by A in order to obtain the value of k^2 and hence the value of k.

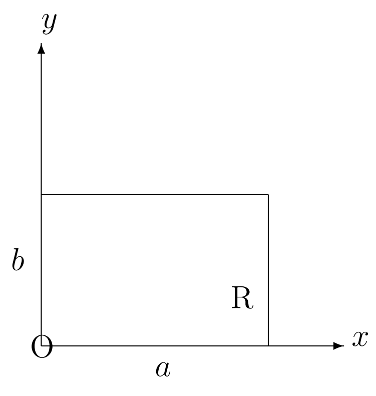
The value of k is called the **"radius of gyration"** of the given region about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole area at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a rectangular region, R, with sides of lengths a and b about an axis through one corner, perpendicular to the plane of R.

   Solution



Using Example 1 from the previous section, the second moment is

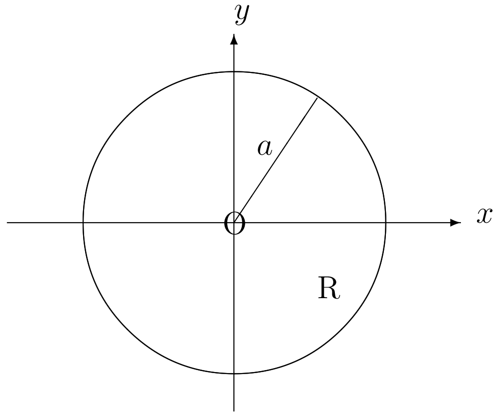
$$\frac{1}{3}ab(a^2+b^2)$$

and, since the area itself is ab, we obtain

$$k=\sqrt{a^2+b^2}.$$

2. Determine the radius of gyration of a circular region, R, about an axis through its centre, perpendicular to the plane of R.

Solution



From Example 2 in the previous section, the second moment about the given axis is $\frac{\pi a^4}{2}$ and, since the area itself is πa^2, we obtain

$$k=\frac{a}{\sqrt{2}}.$$

13.12.4 EXERCISES

Determine the radius of gyration of each of the following regions of the xy-plane about the axis specified:

1. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$.
   Axis: Through the point $\left(\frac{a}{2}, \frac{b}{2}\right)$, perpendicular to the xy-plane.
2. Bounded in the first quadrant by the x-axis, the y-axis and the lines $x = a$, $y = b$.
   Axis: The line $x = 4a$.
3. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$x^2 + y^2 = a^2.$$
   Axis: Through the origin, perpendicular to the xy-plane.
4. Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$x^2 + y^2 = a^2.$$
   Axis: The line $x = a$.

13.12.5 ANSWERS TO EXERCISES

1.
$$\frac{1}{12}\left(a^2 + b^2\right).$$

2.
$$\frac{7a}{\sqrt{3}}.$$

3.
$$\frac{a}{\sqrt{2}}.$$

4.
$$\frac{a\sqrt{5}}{2}.$$

"JUST THE MATHS"

UNIT NUMBER

13.13

INTEGRATION APPLICATIONS 13
(Second moments of a volume (A))

by

A.J.Hobson

UNIT 13.13 - INTEGRATION APPLICATIONS 13

SECOND MOMENTS OF A VOLUME (A)

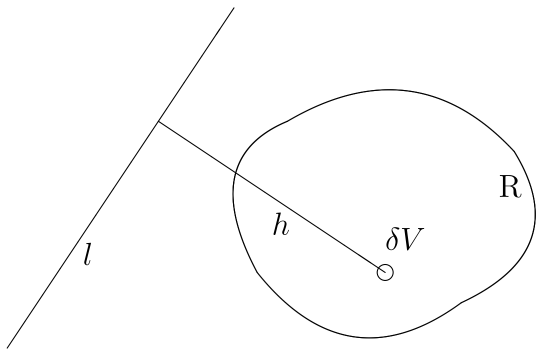
13.13.1 INTRODUCTION

Suppose that R denotes a region (with volume V) in space and suppose that δV is the volume of a small element of this region.

Then the **"second moment"** of R about a fixed line, l, is given by

$$\lim_{\delta V \to 0} \sum_{\mathrm{R}} h^2 \delta V,$$

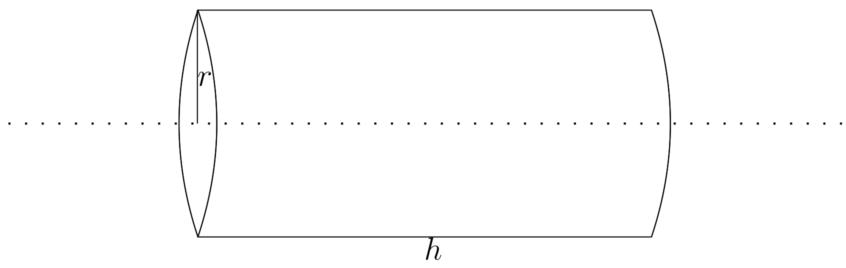
where h is the perpendicular distance from l of the element with volume δV.



EXAMPLE

Determine the second moment, about its own axis, of a solid right-circular cylinder with height, h, and radius, a.

Solution



In a thin cylindrical shell with internal radius, r, and thickness, δr, all of the elements of volume have the same perpendicular distance, r, from the axis of moments.

Hence the second moment of this shell will be the product of its volume and r^2; that is, $r^2(2\pi rh\delta r)$.

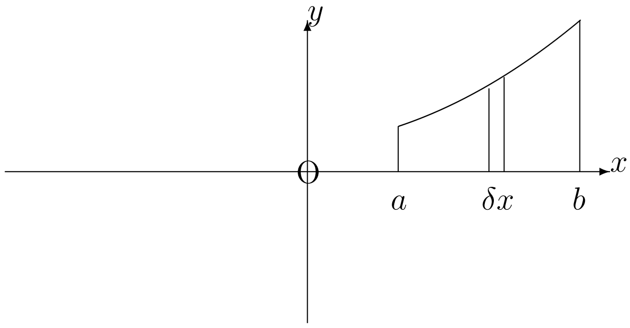
The total second moment is therefore given by

$$\lim_{\delta r\to 0}\sum_{r=0}^{r=a} r^2(2\pi rh\delta r) = \int_0^a 2\pi hr^3\ \mathrm{d}r = \frac{\pi a^4 h}{2}.$$

13.13.2 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE Y-AXIS

Let us consider a region in the first quadrant of the xy-plane, bounded by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution of a narrow 'strip', of width, δx, and height, y, (parallel to the y-axis), is a cylindrical 'shell', of internal radius x, height, y, and thickness, δx.

Hence, from the example in the previous section, its second moment about the y-axis is $2\pi x^3 y\delta x$.

Thus, the total second moment about the y-axis is given by

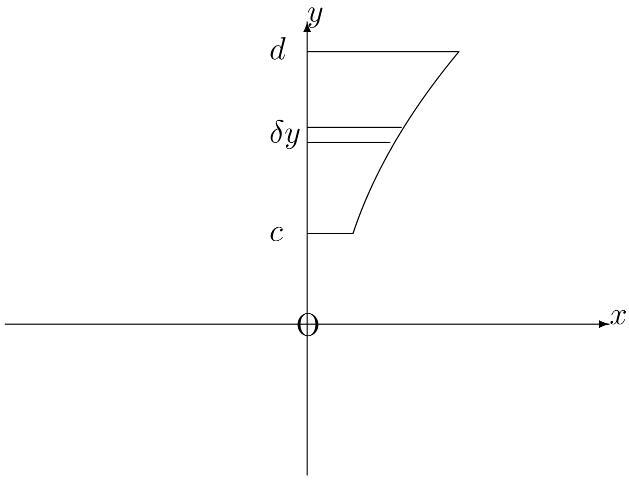
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi x^3 y\delta x = \int_a^b 2\pi x^3 y \, \mathrm{d}x.$$

Note:
Second moments about the x-axis will be discussed mainly in the next section of this Unit; but we note that, for the volume of revolution, about the x-axis, of a region in the first quadrant bounded by the y-axis, the lines $y = c$, $y = d$ and the curve whose equation is

$$x = g(y),$$

we may reverse the roles of x and y so that the second moment about the x-axis is given by

$$\int_c^d 2\pi y^3 x \, \mathrm{d}y.$$



EXAMPLE

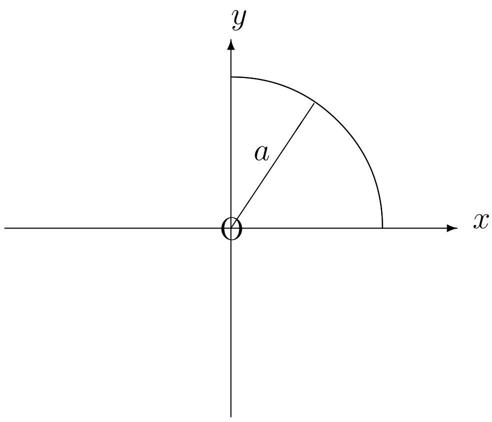
Determine the second moment, about a diameter, of a solid sphere with radius a.

Solution

We may consider, first, the volume of revolution about the y-axis of the region, bounded in the first quadrant, by the x-axis, the y-axis and the circle whose equation is

$$x^2 + y^2 = a^2,$$

then double the result obtained.



The total second moment is given by

$$2\int_0^a 2\pi x^3\sqrt{a^2-x^2}\ \mathrm{d}x = 4\pi\int_0^{\frac{\pi}{2}} a^3\sin^3\theta.a\cos\theta.a\cos\theta\mathrm{d}\theta,$$

if we substitute $x = a\sin\theta$.

This simplifies to

$$4\pi a^5\int_0^{\frac{\pi}{2}} \sin^3\theta\ \cos^2\theta\mathrm{d}\theta = 4\pi\int_0^{\frac{\pi}{2}} \left(\cos^2\theta - \cos^4\theta\right)\sin\theta\mathrm{d}\theta,$$

if we make use of the trigonometric identity

$$\sin^2\theta \equiv 1 - \cos^2\theta.$$

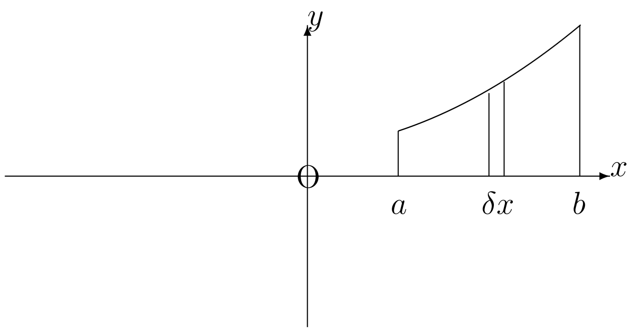
The total second moment is now given by

$$4\pi a^5\left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}} = 4\pi a^5\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8\pi a^5}{15}.$$

13.13.3 THE SECOND MOMENT OF A VOLUME OF REVOLUTION ABOUT THE X-AXIS

In the introduction to this Unit, a formula was established for the second moment of a solid right-circular cylinder about its own axis. This result may now be used to determine the second moment about the x-axis for the volume of revolution about this axis of a region enclosed in the first quadrant by the x-axis, the lines $x = a$, $x = b$ and the curve whose equation is

$$y = f(x).$$



The volume of revolution about the x-axis of a narrow strip, of width δx and height y, is a cylindrical 'disc' whose second moment about the x-axis is $\frac{\pi y^4 \delta x}{2}$. Hence the second moment of the whole region about the x-axis is given by

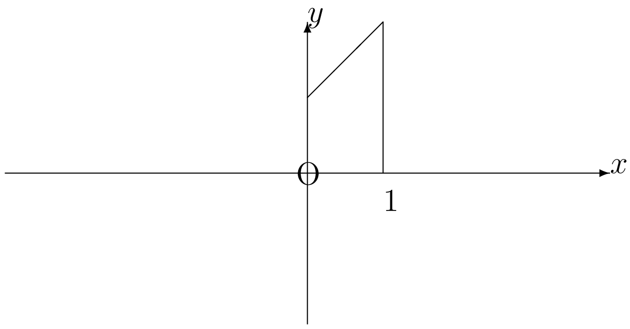
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \frac{\pi y^4}{2} \delta x = \int_a^b \frac{\pi y^4}{2} \, \mathrm{d}x.$$

EXAMPLE

Determine the second moment about the x-axis for the volume of revolution about this axis of the region bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is

$$y = x + 1.$$

Solution



$$\text{Second moment} = \int_0^1 \frac{\pi(x+1)^4}{2}\,\mathrm{d}x = \left[\pi\frac{(x+1)^4}{10}\right]_0^1 = \frac{31\pi}{10}.$$

Note:
The second moment of a volume about a certain axis is closely related to its **"moment of inertia"** about that axis. In fact, for a solid, with uniform density, ρ, the Moment of Inertia is ρ times the second moment of volume, since multiplication by ρ of elements of volume converts them into elements of mass.

13.13.4 EXERCISES

1. Determine the second moment about a diameter of a circular disc with small thickness, t, and radius, a.
2. Determine the second moment, about the axis specified, for the volume of revolution of each of the following regions of the xy-plane about this axis:
   (a) Bounded in the first quadrant by the x-axis, the y-axis and the curve whose equation is
   $$y = 1 - 2x^2.$$
   Axis: The y-axis.
   (b) Bounded in the first quadrant by the x-axis and the curve whose equation is
   $$y^2 = \sin x.$$
   Axis: The x-axis.

(c) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The x-axis

(d) Bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the curve whose equation is

$$y = e^{-2x}.$$

Axis: The y-axis.

13.13.5 ANSWERS TO EXERCISES

1.

$$\frac{\pi a^4 t}{4}.$$

2. (a)

$$\frac{\pi}{24}.$$

(b)

$$\frac{\pi^2}{4}.$$

(c)

$$0.196, \text{ approximately.}$$

(d)

$$0.337, \text{ approximately.}$$

"JUST THE MATHS"

UNIT NUMBER

13.14

INTEGRATION APPLICATIONS 14
(Second moments of a volume (B))

by

A.J.Hobson

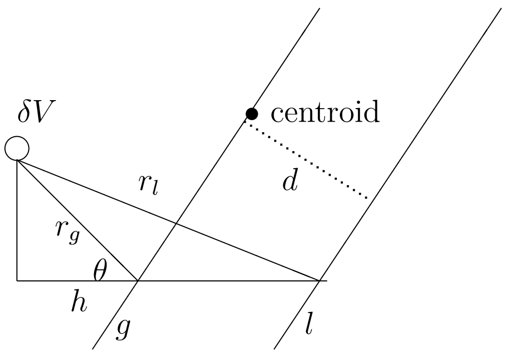
UNIT 13.14 - INTEGRATION APPLICATIONS 14

SECOND MOMENTS OF A VOLUME (B)

13.14.1 THE PARALLEL AXIS THEOREM

Suppose that M_g denotes the second moment of a given region, R, about an axis, g, through its centroid.

Suppose also that M_l denotes the second moment of R about an axis, l, which is parallel to the first axis and has a perpendicular distance of d from the first axis.



In the above **three-dimensional** diagram, we have

$$M_l = \sum_{\mathrm{R}} r_l^2 \delta V \text{ and } M_g = \sum_{\mathrm{R}} r_g^2 \delta V.$$

But, from the Cosine Rule,

$$r_l^2 = r_g^2 + d^2 - 2r_g d\cos(180^\circ - \theta) = r_g^2 + d^2 + 2r_g d\cos\theta.$$

Hence,

$$r_l^2 = r_g^2 + d^2 + 2dh$$

and so

$$\sum_R r_l^2 \delta V = \sum_R r_g^2 \delta V + \sum_R d^2 \delta V + 2d \sum_R h \delta V.$$

Finally, the expression

$$\sum_R h \delta V$$

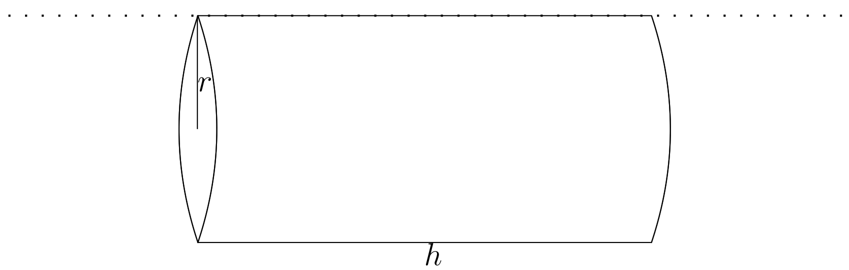
represents the first moment of R about a plane through the centroid, which is perpendicular to the plane containing l and g. Such first moment will be zero and hence,

$$M_l = M_g + Vd^2.$$

EXAMPLE

Determine the second moment of a solid right-circular cylinder about one of its generators (that is, a line in the surface, parallel to the central axis).

Solution



The second moment of the cylinder about the central axis was shown, in Unit 13.13, section 13.13.2, to be $\frac{\pi a^4 h}{2}$; and, since this axis and the generator are a distance a apart, the required second moment is given by

$$\frac{\pi a^4 h}{2} + (\pi a^2 h)a^2 = \frac{3\pi a^4 h}{2}.$$

13.14.2 THE RADIUS OF GYRATION OF A VOLUME

Having calculated the second moment of a three-dimensional region about a certain axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Vk^2, where V is the total volume of the region.

We simply divide the value of the second moment by V in order to obtain the value of k^2 and hence, the value of k.

The value of k is called the **"radius of gyration"** of the given region about the given axis.

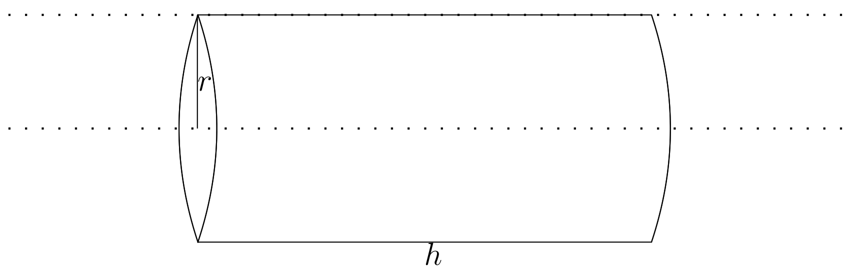
Note:

The radius of gyration effectively tries to concentrate the whole volume at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

EXAMPLES

1. Determine the radius of gyration of a solid right-circular cylinder with height, h, and radius, a, about (a) its own axis and (b) one of its generators.

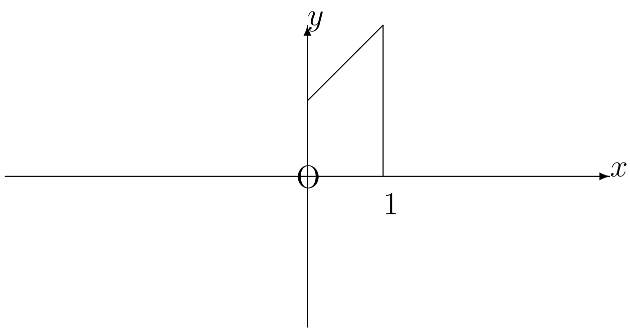
   Solution

   Using earlier examples, together with the volume, $V = \pi a^2 h$, the required radii of gyration are (a) $\sqrt{\frac{\pi a^4 h}{2} \div \pi a^2 h} = \frac{a}{\sqrt{2}}$ and (b) $\sqrt{\frac{3\pi a^4 h}{2} \div \pi a^2 h} = a\sqrt{\frac{3}{2}}$.

2. Determine the radius of gyration of the volume of revolution about the x-axis of the region, bounded in the first quadrant by the x-axis, the y-axis, the line $x = 1$ and the line whose equation is
$$y = x + 1.$$

Solution



From Unit 13.13, section 13.13.3, the second moment about the given axis is $\frac{31\pi}{10}$. The volume itself is given by

$$\int_0^1 \pi(x+1)^2 \, \mathrm{d}x = \left[\pi\frac{(x+1)^3}{3}\right]_0^1 = \frac{7\pi}{3}.$$

Hence,

$$k^2 = \frac{31\pi}{10} \times \frac{3}{7\pi} = \frac{93}{70}.$$

That is,

$$k = \sqrt{\frac{93}{70}} \simeq 1.15$$

13.14.3 EXERCISES

1. Determine the radius of gyration of a hollow cylinder with internal radius, a, and external radius, b, about
   (a) its central axis;
   (b) a generator lying in its outer surface.
2. Determine the radius of gyration of a solid hemisphere, with radius a, about
   (a) its base-diameter;
   (b) an axis through its centroid, parallel to its base-diameter.
3. For a solid right-circular cylinder with height, h, and radius, a, determine the radius of gyration about
   (a) a diameter of one end;
   (b) an axis through the centroid, perpendicular to the axis of the cylinder.
4. For a solid right-circular cone with height, h, and base-radius, a, determine the radius of gyration about
   (a) the axis of the cone;
   (b) a line through the vertex, perpendicular to the axis of the cone;
   (c) a line through the centroid, perpendicular to the axis of the cone.

13.14.4 ANSWERS TO EXERCISES

1. (a)

$$\sqrt{\frac{a^2+b^2}{2}};$$

(b)

$$\sqrt{\frac{3b^2+a^2}{2}}.$$

2. (a)

$$a\sqrt{\frac{2}{5}};$$

(b)

$$a\sqrt{\frac{173}{320}}.$$

3. (a)

$$\sqrt{\frac{3a^2+4h^2}{12}};$$

(b)

$$\sqrt{\frac{3a^2+7h^2}{12}}.$$

4. (a)

$$a\sqrt{\frac{3}{10}};$$

(b)

$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{5}};$$

(c)

$$\sqrt{\frac{3a^2}{20}+\frac{3h^2}{80}}.$$

"JUST THE MATHS"

UNIT NUMBER

13.15

INTEGRATION APPLICATIONS 15
(Second moments of a surface of revolution)

by

A.J.Hobson

UNIT 13.15 - INTEGRATION APPLICATIONS 15

SECOND MOMENTS OF A SURFACE OF REVOLUTION

13.15.1 INTRODUCTION

Suppose that C denotes an arc (with length s) in the xy-plane of cartesian co-ordinates, and suppose that δs is the length of a small element of this arc.

Then, for the surface obtained when the arc is rotated through 2π radians about the x-axis, the **"second moment"** about the x-axis, is given by

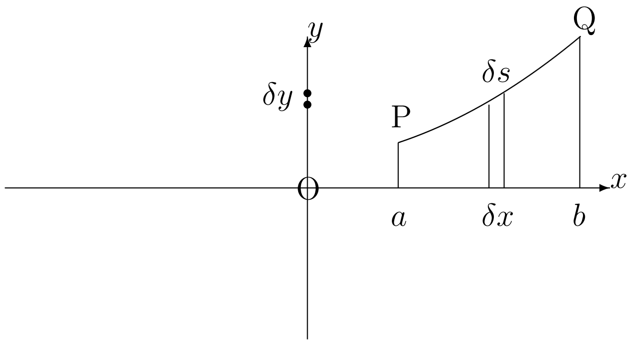
$$\lim_{\delta s \to 0} \sum_{\mathrm{C}} y^2 .2\pi y \delta s.$$

13.15.2 INTEGRATION FORMULAE FOR SECOND MOMENTS

(a) Let us consider an arc of the curve whose equation is

$$y = f(x),$$

joining two points, P and Q, at $x = a$ and $x = b$, respectively.



The arc may be divided up into small elements of typical length, δs, by using neighbouring points along the arc, separated by typical distances of δx (parallel to the x-axis) and δy (parallel to the y-axis).

From Pythagoras' Theorem,

$$\delta s \simeq \sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

so that, for the surface of revolution of the arc about the x-axis, the second moment becomes

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} 2\pi y^3 \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x = \int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\;\mathrm{d}x.$$

Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Hence,

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

provided that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive on the arc being considered. If not, then the above line needs to be prefixed by a negative sign.

From the technique of integration by substitution,

$$\int_a^b 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\;\mathrm{d}x = \int_{t_1}^{t_2} 2\pi y^3 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.\frac{\mathrm{d}x}{\mathrm{d}t}\;\mathrm{d}t,$$

where $t = t_1$ when $x = a$ and $t = t_2$ when $x = b$.

We may conclude that the second moment about the plane through the origin, perpendicular to the x-axis, is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi y^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$$

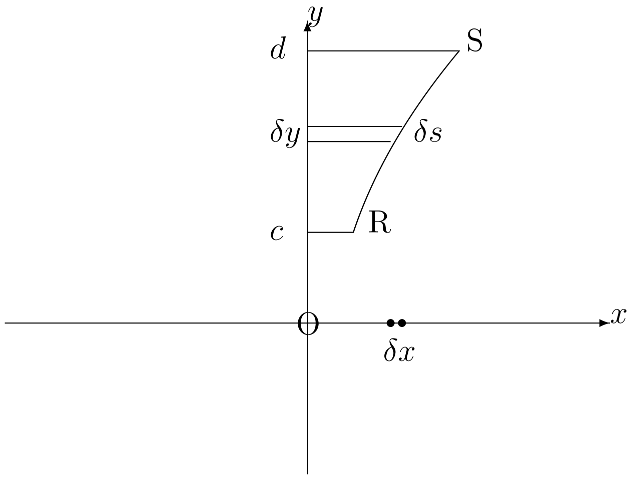
according as $\frac{\mathrm{d}x}{\mathrm{d}t}$ is positive or negative.

(b) For an arc whose equation is

$$x = g(y),$$

contained between $y = c$ and $y = d$, we may reverse the roles of x and y in the previous section so that the second moment about the y-axis is given by

$$\int_c^d 2\pi x^3 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\ \mathrm{d}y$$



Note:
If the curve is given parametrically by

$$x = x(t), \quad y = y(t),$$

where $t = t_1$ when $y = c$ and $t = t_2$ when $y = d$, then the second moment about the y-axis is given by

$$\text{second moment} = \pm \int_{t_1}^{t_2} 2\pi x^3 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t,$$

according as $\frac{\mathrm{d}y}{\mathrm{d}t}$ is positive or negative.

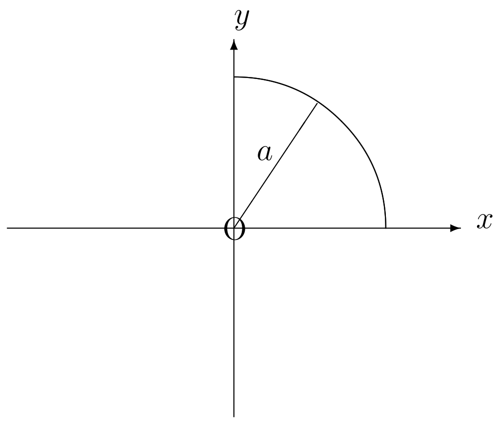
EXAMPLES

1. Determine the second moment about the x-axis of the hemispherical surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



Using implicit differentiation, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and, hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

The second moment about the x-axis is therefore given by

$$\int_0^a 2\pi y^3\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi y^3\sqrt{\frac{x^2+y^2}{y^2}}\ \mathrm{d}x.$$

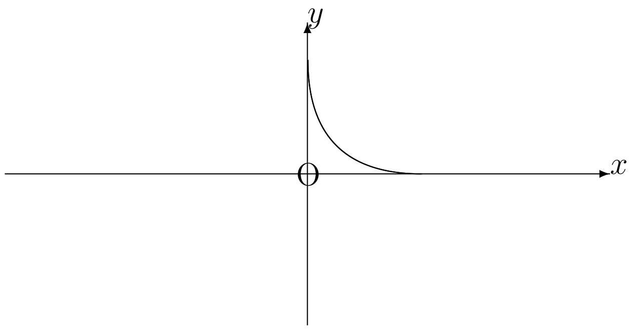
But $x^2+y^2=a^2$, and so the second moment becomes

$$\int_0^a 2\pi a(a^2-x^2)\ \mathrm{d}x = 2\pi a\left[a^2x-\frac{x^3}{3}\right]_0^a = \frac{4\pi a^4}{3}.$$

2. Determine the second moment about the axis of revolution, when the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

is rotated through 2π radians about (a) the x-axis and (b) the y-axis.
Solution



(a) Firstly, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta \ \text{ and }\ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta.$$

Hence, the second moment about the x-axis is given by

$$-\int_{\frac{\pi}{2}}^0 2\pi y^3\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\ \mathrm{d}\theta,$$

which, on using $\cos^2\theta+\sin^2\theta\equiv 1$, becomes

$$\int_0^{\frac{\pi}{2}} 2\pi a^3 \sin^{27}\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 6\pi a^4 \sin^{28}\theta\cos\theta\ \mathrm{d}\theta$$

$$= 6\pi a^4 \int_0^{\frac{\pi}{2}} \sin^{28}\theta\cos\theta\ \mathrm{d}\theta,$$

which, by the methods of Unit 12.7 gives

$$6\pi a^4 \left[\frac{\sin^{29}\theta}{29}\right]_0^{\frac{\pi}{2}} = \frac{6\pi a^4}{29}.$$

(b) By symmetry, or by direct integration, the second moment about the y-axis is also $\frac{6\pi a^4}{29}$.

13.15.3 THE RADIUS OF GYRATION OF A SURFACE OF REVOLUTION

Having calculated the second moment of a surface of revolution about a specified axis, it is possible to determine a positive value, k, with the property that the second moment about the axis is given by Sk^2, where S is the total surface area of revolution.

We simply divide the value of the second moment by S in order to obtain the value of k^2 and hence the value of k.

The value of k is called the **"radius of gyration"** of the given arc about the given axis.

Note:
The radius of gyration effectively tries to concentrate the whole surface at a single point for the purposes of considering second moments; but, unlike a centroid, this point has no specific location.

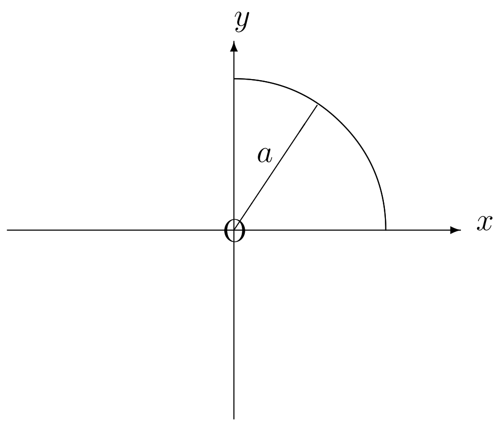
EXAMPLES

1. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the arc of the circle whose equation is

$$x^2 + y^2 = a^2,$$

lying in the first quadrant.

Solution



From an Example 1 in section 13.15.2, we know that the second moment of the surface about the x-axis is equal to $\frac{4\pi a^4}{3}$.
Also, the total surface area is

$$\int_0^a 2\pi y\sqrt{1+\frac{x^2}{y^2}}\ \mathrm{d}x = \int_0^a 2\pi a\ \mathrm{d}x = 2\pi a^2,$$

which implies that

$$k^2 = \frac{4\pi a^4}{3} \times \frac{1}{2\pi a^2} \quad = \frac{2a^2}{3}.$$

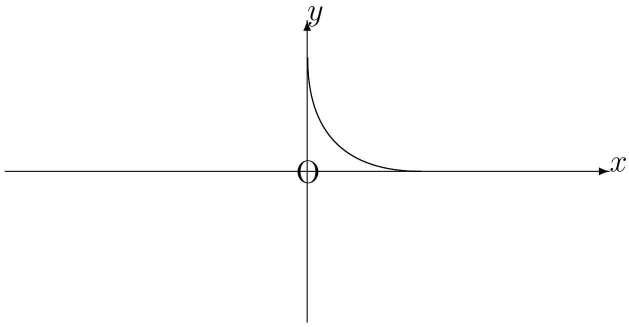
The radius of gyration is thus given by

$$k = a\sqrt{\frac{2}{3}}.$$

2. Determine the radius of gyration about the x-axis of the surface of revolution (about the x-axis) of the first quadrant arc of the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Solution



We know from Example 2 in section 13.15.2, that that the second moment of the surface about the x-axis is equal to $\frac{6\pi a^4}{29}$.
Also, the total surface area is given by

$$-\int_{\frac{\pi}{2}}^{0} 2\pi a\sin^3\theta.3a\cos\theta\sin\theta\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 3a^2\sin^4\theta\cos\theta\ \mathrm{d}\theta = 3\pi a^2\left[\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}} = \frac{3\pi a^2}{5}.$$

Thus,

$$k^2 = \frac{6\pi a^4}{29} \times \frac{5}{3\pi a^2} = \frac{10a^2}{29}.$$

13.15.4 EXERCISES

1. Determine the second moment, about the x-axis, of the surface of revolution (about the x-axis) of the straight-line segment joining the origin to the point $(2, 3)$.
2. Determine the second moment about the x-axis, of the surface of revolution (about the x-axis) of the first quadrant arc of the curve whose equation is $y^2 = 4x$, lying between $x = 0$ and $x = 1$.
3. Determine, correct to two places of decimals, the second moment about the y-axis, of the surface of revolution (about the y-axis) of the first quadrant arc of the curve whose equation is $3y = x^3$, lying between $x = 1$ and $x = 2$.

4. Determine, correct to two places of decimals, the second moment, about the y-axis, of the surface of revolution (about the y-axis) of the arc of the circle given parametrically by
$$x = 2\cos t, \quad y = 2\sin t,$$
joining the point $(\sqrt{2}, \sqrt{2})$ to the point $(0, 2)$.

5. Determine the radius of gyration of a hollow right-circular cone with maximum radius, a, about its central axis.

6. For the curve whose equation is $9y^2 = x(3-x)^2$, show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-x}{2\sqrt{x}}.$$
Hence, show that the radius of gyration about the y-axis of the surface obtained when the first quadrant arch of this curve is rotated through 2π radians about the x-axis is 4, correct to the nearest whole number.

13.15.5 ANSWERS TO EXERCISES

1.
$$\frac{\pi 27\sqrt{13}}{2}.$$

2.
$$\frac{32\pi}{5}[4-\sqrt{2}] \simeq 51.99$$

3.
$$70.44$$

4.
$$0.73$$

5.
$$k = \frac{a}{\sqrt{2}}.$$

6.
$$\text{Second moment} \simeq 139.92, \quad \text{surface area} \simeq 9.42$$

"JUST THE MATHS"

UNIT NUMBER

13.16

INTEGRATION APPLICATIONS 16
(Centres of pressure)

by

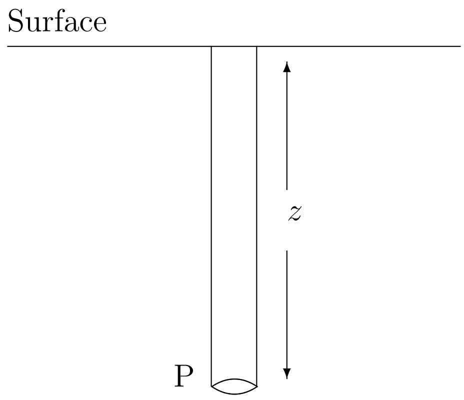
A.J.Hobson

UNIT 13.16 - INTEGRATION APPLICATIONS 16

CENTRES OF PRESSURE

13.16.1 THE PRESSURE AT A POINT IN A LIQUID

In the following diagram, we consider the pressure in a liquid at a point, P, whose depth below the surface of the liquid is z.



Ignoring atmospheric pressure, the pressure, p, at P is measured as the thrust acting upon unit area and is due to the weight of the column of liquid with height z above it.

Hence,

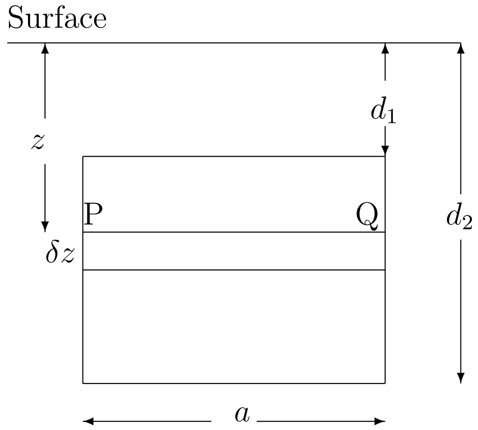
$$p = wz$$

where w is the weight, per unit volume, of the liquid.

Note:
The pressure at P is directly proportional to the depth of P below the surface; and we shall assume that the pressure acts equally in all directions at P.

13.16.2 THE PRESSURE ON AN IMMERSED PLATE

We now consider a rectangular plate, with dimensions a and $(d_2 - d_1)$, immersed vertically in a liquid as shown below.



For a thin strip, PQ, of width, δz, at a depth, z, below the surface of the liquid, the thrust on PQ will be the pressure at P multiplied by the area of the strip; that is, $wz \times a\delta z$.

The total thrust on the whole plate will therefore be

$$\sum_{z=d_1}^{z=d_2} waz\delta z.$$

Allowing δz to tend to zero, the total thrust becomes

$$\int_{d_1}^{d_2} waz \ \mathrm{d}z = \left[\frac{waz^2}{2}\right]_{d_1}^{d_2} = \frac{wa}{2}(d_2^2 - d_1^2).$$

This may be written

$$wa(d_2 - d_1)\left(\frac{d_2 + d_1}{2}\right),$$

where, in this form, $a(d_2 - d_1)$ is the area of the plate and $\frac{d_2+d_1}{2}$ is the depth of the centroid of the plate.

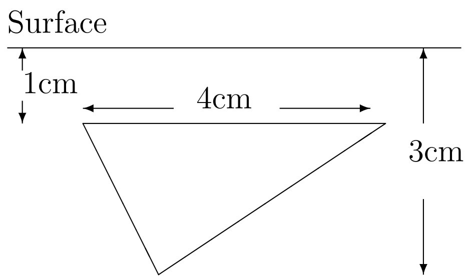
We conclude that

$$\text{Total Thrust} = \text{Area of Plate} \times \text{Pressure at the Centroid.}$$

Note:
It may be shown that this result holds whatever the shape of the plate is; and even when the plate is not vertical.

EXAMPLES

1. A triangular plate is immersed vertically in a liquid for which the weight per unit volume is w. The dimensions of the plate and its position in the liquid is shown in the following diagram:



Determine the total thrust on the plate as a multiple of w.

Solution

The area of the plate is given by

$$\text{Area} = \frac{1}{2} \times 4 \times 2 = 4\text{cm}^2$$

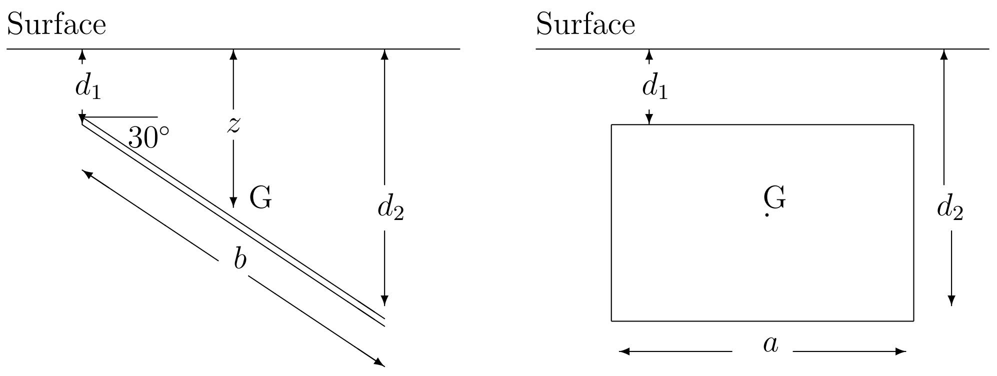
The centroid of the plate, which is at a distance from its horizontal side equal to one third of its perpendicular height, will lie at a depth of

$$\left(1 + \frac{1}{3} \times 2\right) \text{cms.} = \frac{5}{3}\text{cms.}$$

Hence, the pressure at the centroid is $\frac{5w}{3}$ and we conclude that

$$\text{Total thrust} = 4 \times \frac{5w}{3} = \frac{20w}{3}.$$

2. The following diagram shows a rectangular plate immersed in a liquid for which the weight per unit volume is w; and the plate is inclined at $30°$ to the horizontal:



Determine the total thrust on the plate as a multiple of w.

Solution

The depth, z, of the centroid, G, of the plate is given by

$$z = d_1 + \frac{b}{2}\sin 30° = d_1 + \frac{b}{4}.$$

Hence, the pressure, p, at G is given by

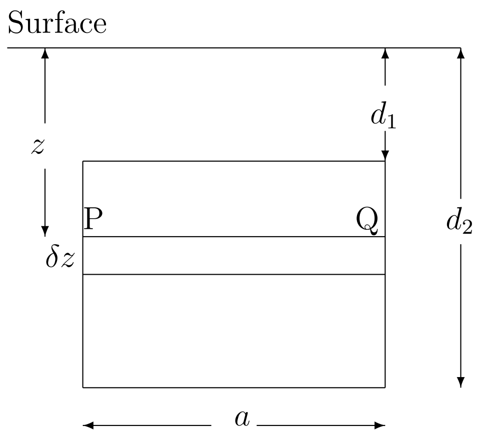
$$p = \left(d_1 + \frac{b}{4}\right)w;$$

and, since the area of the plate is ab, we obtain

$$\text{Total thrust} = ab\left(d_1 + \frac{b}{4}\right)w.$$

13.16.3 THE DEPTH OF THE CENTRE OF PRESSURE

In this section, we consider again an earlier diagram for a rectangular plate, immersed vertically in a liquid whose weight per unit volume is w.



We have already seen that the total thrust on the plate is

$$\int_{d_1}^{d_2} waz \text{ d}z = w\int_{d_1}^{d_2} az \text{ d}z$$

and is the resultant of varying thrusts, acting according to depth, at each level of the plate.

But, by taking first moments of these thrusts about the line in which the plane of the plate intersects the surface of the liquid, we may determine a particular depth at which the total thrust may be considered to act.

This depth is called **"the depth of the centre of pressure"**.

The Calculation

In the diagram, the thrust on the strip PQ is $waz\delta z$ and its first moment about the line in the surface is $waz^2\delta z$ so that the sum of the first moments on all such strips is given by

$$\sum_{z=d_1}^{z=d_2} waz^2\delta z = w\int_{d_1}^{d_2} az^2 \text{ d}z$$

where the definite integral is, in fact, the <u>second</u> moment of the plate about the line in the surface.

Next, we define the depth, C_p, of the centre of pressure to be such that

$$\text{Total thrust} \times C_p = \text{sum of first moments of strips like PQ.}$$

That is,

$$w\int_{d_1}^{d_2} az \ \mathrm{d}z \times C_p = w\int_{d_1}^{d_2} az^2 \ \mathrm{d}z$$

and, hence,

$$C_p = \frac{\int_{d_1}^{d_2} az^2 \ \mathrm{d}z}{\int_{d_1}^{d_2} az \ \mathrm{d}z},$$

which may be interpreted as

$$C_p = \frac{Ak^2}{A\overline{z}} = \frac{k^2}{\overline{z}},$$

where A is the area of the plate, k is the radius of gyration of the plate about the line in the surface of the liquid and $\overline{z}$ is the depth of the centroid of the plate.

Notes:

(i) It may be shown that the formula

$$C_p = \frac{k^2}{\overline{z}}$$

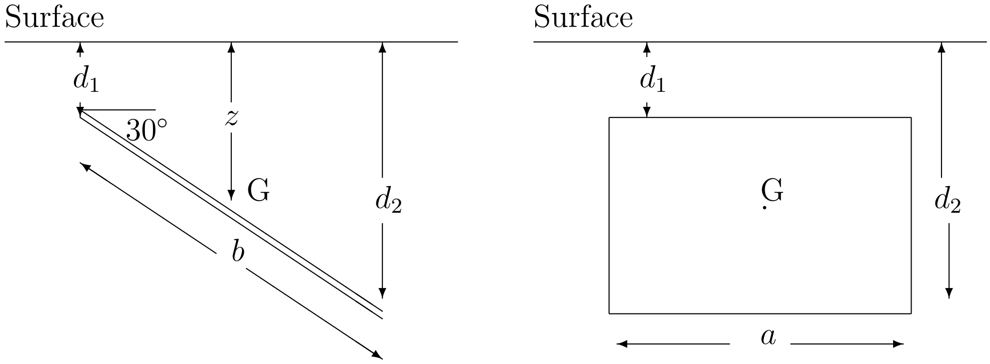
holds for any shape of plate immersed at any angle.

(ii) The phrase, "centre of pressure" suggests a particular point at which the total thrust is considered to act; but this is simply for convenience. The calculation is only for the depth of the centre of pressure.

EXAMPLE

Determine the depth of the centre of pressure for the second example of the previous section.

Solution



The depth of the centroid is

$$d_1 + \frac{b}{4}$$

and the square of the radius of gyration of the plate about an axis through the centroid, parallel to the side with length a is $\frac{a^2}{12}$.

Furthermore, the perpendicular distance between this axis and the line of intersection of the plane of the plate with the surface of the liquid is

$$\frac{b}{2} + \frac{d_1}{\sin 30^\circ} = \frac{b}{2} + 2d_1.$$

Hence, the square of the radius of gyration of the plate about the line in the surface is

$$\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2,$$

using the Theorem of Parallel Axes.

Finally, the depth of the centre of pressure is given by

$$C_p = \frac{\frac{a^2}{12} + \left(\frac{b}{2} + 2d_1\right)^2}{d_1 + \frac{b}{4}}.$$

13.16.4 EXERCISES

1. A thin equilateral triangular plate is immersed vertically in a liquid for which the weight per unit volume is w, with one edge on the surface. If the length of each side is a, determine the total thrust on the plate.

2. A thin plate is bounded by the arc of a parabola and a straight line segment of length 1.2m perpendicular to the axis of symmetry of the parabola, this axis being of length 0.4m.

   If the plate is immersed vertically in a liquid with the straight edge on the surface, determine the total thrust on the plate in the form lw, where w is the weight per unit volume of the liquid and l is in decimals, correct to two places.

3. A thin rectangular plate, with sides of length 10cm and 20cm is immersed in a liquid so that the sides of length 10cm are horizontal and the sides of length 20cm are incline at 55° to the horizontal. If the uppermost side of the plate is at a depth of 13cm, determine the total thrust on then plate in the form lw, where w is the mass per unit volume of the liquid.

4. A thin circular plate, with diameter 0.5m is immersed vertically in a tank of liquid so that the uppermost point on its circumference is 2m below the surface. Determine the depth of the centre of pressure. correct to two places of decimals.

5. A thin plate is in the form of a trapezium with parallel sides of length 1m and 2.5m, a distance of 0.75m apart, and the remaining two sides inclined equally to either one of the parallel sides.

   If the plate is immersed vertically in water with the side of length 2.5m on the surface, calculate the depth of the centre of pressure, correct to two places of decimals.

13.16.5 ANSWERS TO EXERCISES

1.

$$\text{Total \ thrust} = \frac{wa^3}{8}.$$

2.

$$\text{Total \ Thrust} = 5.12w.$$

3.

$$C_p \simeq 2.26\text{m}.$$

4.

$$C_p \simeq 0.46\text{m}.$$

“JUST THE MATHS”

UNIT NUMBER

14.1

PARTIAL DIFFERENTIATION 1
(Partial derivatives of the first order)

by

A.J.Hobson

UNIT 14.1 - PARTIAL DIFFERENTIATION 1 - PARTIAL DERIVATIVES OF THE FIRST ORDER

14.1.1 FUNCTIONS OF SEVERAL VARIABLES

In most scientific problems, it is likely that a variable quantity under investigation will depend (for its values), not only on **one** other variable quantity, but on **several** other variable quantities.

The type of notation used may be indicated by examples such as the following:

1. $$z = f(x, y),$$

   which means that the variable, z, depends (for its values) on two variables, x and y.

2. $$w = F(x, y, z),$$

   which means that the variable, w, depends (for its values) on three variables, x, y and z.

   Normally, the variables on the right-hand side of examples like those above may be chosen independently of one another and, as such, are called the **"independent variables"**. By contrast, the variable on the left-hand side is called the **"dependent variable"**.

Notes:

(i) Some relationships between several variables are not stated as an **explicit** formula for one of the variables in terms of the others.

An illustration of this type would be $x^2 + y^2 + z^2 = 16$.

In such cases, it may be necessary to specify separately which is the dependent variable.

(ii) The variables on the right-hand side of an explicit formula, giving a dependent variable in terms of them, may not actually be independent of one another. This would occur if those variables were already, themselves, dependent on a quantity not specifically mentioned in the formula.

For example, in the formula

$$z = xy^2 + \sin(x - y),$$

suppose it is also known that $x = t - 1$ and $y = 3t + 2$.

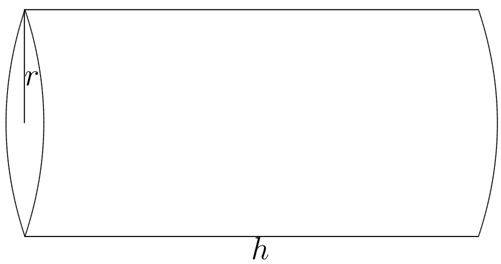
Then the variables x and y are not independent of each other. In fact, by eliminating t, we obtain

$$y = 3(x + 1) + 2 = 3x + 5.$$

14.1.2 THE DEFINITION OF A PARTIAL DERIVATIVE

ILLUSTRATION

Consider the formulae for the volume, V, and the surface area, S, of a solid right-circular cylinder with radius, r, and height, h.



The relevant formulae are

$$V = \pi r^2 h \quad \text{and} \quad S = 2\pi r^2 + 2\pi r h,$$

so that both V and S are functions of the two variables, r and h.

But suppose it were possible for r to be held constant while h is allowed to vary. Then the corresponding rates of increase of V and S with respect to h are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r \text{ const.}} = \pi r^2$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}h}\right]_{r\ \text{const.}} = 2\pi r.$$

These two expressions are called the **"partial derivatives of V and S with respect to h"**.

Similarly, suppose it were possible for h to be held constant while r is allowed to vary. Then the corresponding rates of increase of V and S with respect to r are given by

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h\ \text{const.}} = 2\pi rh$$

and

$$\left[\frac{\mathrm{d}S}{\mathrm{d}r}\right]_{h\ \text{const.}} = 4\pi r + 2\pi h.$$

These two expressions are called the **"partial derivatives of V and S with respect to r"**.

THE NOTATION FOR PARTIAL DERIVATIVES

In the defining illustration above, the notation used for the partial derivatives of V and S was an adaptation of the notation for what will, in future, be referred to as **ordinary** derivatives.

It was, however, rather cumbersome; and the more standard notation which uses the symbol ∂ rather than d is indicated by restating the earlier results as

$$\frac{\partial V}{\partial h} = \pi r^2, \quad \frac{\partial S}{\partial h} = 2\pi r$$

and

$$\frac{\partial V}{\partial r} = 2\pi rh, \quad \frac{\partial S}{\partial r} = 4\pi r + 2\pi h.$$

In this notation, it is understood that each independent variable (except the one with respect to which we are differentiating) is held constant.

EXAMPLES

1. Determine the partial derivatives of the following functions with respect to each of the independent variables:

   (a)

$$z = (x^2 + 3y)^5;$$

   Solution

$$\frac{\partial z}{\partial x} = 5(x^2 + 3y)^4.2x = 10x(x^2 + 3y)^4$$

   and

$$\frac{\partial z}{\partial y} = 5(x^2 + 3y)^4.3 = 15(x^2 + 3y)^4.$$

   (b)

$$w = ze^{3x-7y};$$

   Solution

$$\frac{\partial w}{\partial x} = 3ze^{3x-7y},$$

$$\frac{\partial w}{\partial y} = -7ze^{3x-7y},$$

   and

$$\frac{\partial w}{\partial z} = e^{3x-7y}.$$

   (c)

$$z = x\sin(2x^2 + 5y).$$

   Solution

$$\frac{\partial z}{\partial x} = \sin(2x^2 + 5y) + 4x^2\cos(2x^2 + 5y)$$

   and

$$\frac{\partial z}{\partial y} = 5x\cos(2x^2 + 5y).$$

2. If

$$z = f(x^2 + y^2),$$

show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

Solution

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2)$$

and

$$\frac{\partial z}{\partial y} = 2yf'(x^2 + y^2).$$

Hence,

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$$

3. Given the formula

$$\cos(x + 2z) + 3y^2 + 2xyz = 0$$

as an implicit relationship between two independent variables x and y and a dependent variable z, determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x, y and z.

Solution

Differentiating the formula partially with respect to x and y in turn, we obtain

$$-\sin(x + 2z).\left(1 + 2\frac{\partial z}{\partial x}\right) + 2y\left(x\frac{\partial z}{\partial x} + y\right) = 0$$

and

$$-\sin(x + 2z).2\frac{\partial z}{\partial y} + 6y + 2x\left(y\frac{\partial z}{\partial y} + z\right) = 0,$$

respectively.

Thus,

$$\frac{\partial z}{\partial x} = \frac{\sin(x + 2z) - 2y^2}{2yx - 2\sin(x + 2z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{2xz + 6y}{2\sin(x + 2z) - 2xy} = \frac{xz + 3y}{\sin(x + 2z) - xy}.$$

14.1.3 EXERCISES

1. Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in the following cases:

   (a)
   $$z = 2x^2 - 4xy + y^3;$$

   (b)
   $$z = \cos(5x - 3y);$$

   (c)
   $$z = e^{x^2+2y^2};$$

   (d)
   $$z = x\sin(y - x).$$

2. If
   $$z = (x + y)\ln\left(\frac{x}{y}\right),$$
   show that
   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

3. Determine $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ in the following cases:

   (a)
   $$w = x^5 + 3xyz + z^2;$$

   (b)
   $$w = ze^{2x-3y};$$

   (c)
   $$w = \sin(x^2 - yz).$$

14.1.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\partial z}{\partial x} = 4x - 4y \quad \text{and} \quad \frac{\partial z}{\partial y} = -4x + 3y^2;$$

(b)

$$\frac{\partial z}{\partial x} = -5\sin(5x - 3y) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\sin(5x - 3y);$$

(c)

$$\frac{\partial z}{\partial x} = 2xe^{x^2+2y^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = 4ye^{x^2+2y^2};$$

(d)

$$\frac{\partial z}{\partial x} = \sin(y - x) - x\cos(y - x) \quad \text{and} \quad \frac{\partial z}{\partial y} = x\cos(y - x).$$

2.

$$\frac{\partial z}{\partial x} = \ln\left(\frac{x}{y}\right) + \frac{x+y}{x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \ln\left(\frac{x}{y}\right) - \frac{x+y}{y}.$$

3. (a)

$$\frac{\partial w}{\partial x} = 5x^4 + 3yz, \quad \frac{\partial w}{\partial y} = 3xz, \quad \frac{\partial w}{\partial z} = 3xy + 2z;$$

(b)

$$\frac{\partial w}{\partial x} = 2ze^{2x-3y}, \quad \frac{\partial w}{\partial y} = -3ze^{2x-3y}, \frac{\partial w}{\partial z} = e^{2x-3y};$$

(c)

$$\frac{\partial w}{\partial x} = 2x\cos(x^2 - yz), \quad \frac{\partial w}{\partial y} = -z\cos(x^2 - yz), \quad \frac{\partial w}{\partial z} = -y\cos(x^2 - yz).$$

"JUST THE MATHS"

UNIT NUMBER

14.2

PARTIAL DIFFERENTIATION 2
(Partial derivatives of order higher than one)

by

A.J.Hobson

UNIT 14.2 - PARTIAL DIFFERENTIATION 2

PARTIAL DERIVATIVES OF THE SECOND AND HIGHER ORDERS

14.2.1 STANDARD NOTATIONS AND THEIR MEANINGS

In Unit 14.1, the partial derivatives encountered are known as partial derivatives of the **first order**; that is, the dependent variable was differentiated only **once** with respect to each independent variable.

But a partial derivative will, in general contain **all** of the independent variables, suggesting that we may need to differentiate again with respect to **any** of those variables.

For example, in the case where a variable, z, is a function of two independent variables, x and y, the possible partial derivatives of the second order are

(i)

$$\frac{\partial^2 z}{\partial x^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right);$$

(ii)

$$\frac{\partial^2 z}{\partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right);$$

(iii)

$$\frac{\partial^2 z}{\partial x \partial y}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right);$$

(iv)

$$\frac{\partial^2 z}{\partial y \partial x}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

The last two can be shown to give the same result for all elementary functions likely to be encountered in science and engineering.

Note:

Occasionally, it may be necessary to use partial derivatives of order higher than two, as illustrated, for example, by

$$\frac{\partial^3 z}{\partial x \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]$$

and

$$\frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]\right).$$

EXAMPLES

Determine all the first and second order partial derivatives of the following functions:

1.
$$z = 7x^3 - 5x^2y + 6y^3.$$

Solution

$$\tfrac{\partial z}{\partial x} = 21x^2 - 10xy; \qquad \tfrac{\partial z}{\partial y} = -5x^2 + 18y^2;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 42x - 10y; \qquad \tfrac{\partial^2 z}{\partial y^2} = 36y;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = -10x; \qquad \tfrac{\partial^2 z}{\partial x \partial y} = -10x.$$

2.
$$z = y \sin x + x \cos y.$$

Solution

$$\tfrac{\partial z}{\partial x} = y \cos x + \cos y; \qquad \tfrac{\partial z}{\partial y} = \sin x - x \sin y;$$

$$\tfrac{\partial^2 z}{\partial x^2} = -y \sin x; \qquad \tfrac{\partial^2 z}{\partial y^2} = -x \cos y;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = \cos x - \sin y; \qquad \tfrac{\partial^2 z}{\partial x \partial y} = \cos x - \sin y.$$

3.

$$z = e^{xy}(2x - y).$$

Solution

$$\begin{aligned} \frac{\partial z}{\partial x} &= e^{xy}[y(2x-y)+2] \\ &= e^{xy}[2xy - y^2 + 2]; \end{aligned} \qquad \begin{aligned} \frac{\partial z}{\partial y} &= e^{xy}[x(2x-y)-1] \\ &= e^{xy}[2x^2 - xy - 1]; \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= e^{xy}[y(2xy - y^2 + 2) + 2y] \\ &= e^{xy}[2xy^2 - y^3 + 4y]; \end{aligned} \qquad \begin{aligned} \frac{\partial^2 z}{\partial y^2} &= e^{xy}[x(2x^2 - xy - 1) - x] \\ &= e^{xy}[2x^3 - x^2y - 2x]; \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y \partial x} &= e^{xy}[x(2xy - y^2 + 2) + 2x - 2y] \\ &= e^{xy}[2x^2y - xy^2 + 4x - 2y]; \end{aligned} \qquad \begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= e^{xy}[y(2x^2 - xy - 1) + 4x - y] \\ &= e^{xy}[2x^2y - xy^2 + 4x - 2y]. \end{aligned}$$

14.2.2 EXERCISES

1. Determine all the first and second order partial derivatives of the following functions:

   (a)

   $$z = 5x^2y^3 - 7x^3y^5;$$

   (b)

   $$z = x^4 \sin 3y.$$

2. Determine all the first and second order partial derivatives of the function

   $$w \equiv z^2 e^{xy} + x\cos(y^2 z).$$

3. If

   $$z = (x+y)\ln\left(\frac{x}{y}\right),$$

   show that

   $$x^2\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial y^2}.$$

4. If
$$z = f(x+ay) + F(x-ay),$$
show that
$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial y^2}.$$

14.2.3 ANSWERS TO EXERCISES

1. (a) The required partial derivatives are as follows:

$$\tfrac{\partial z}{\partial x} = 10xy^3 - 21x^2y^5; \qquad \tfrac{\partial z}{\partial y} = 15x^2y^2 - 35x^3y^4;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 10y^3 - 42xy^5; \qquad \tfrac{\partial^2 z}{\partial y^2} = 30x^2y - 140x^3y^3;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = 30xy^2 - 105x^2y^4; \quad \tfrac{\partial^2 z}{\partial x \partial y} = 30xy^2 - 105x^2y^4.$$

(b) The required partial derivatives are as follows:

$$\tfrac{\partial z}{\partial x} = 4x^3 \sin 3y; \qquad \tfrac{\partial z}{\partial y} = 3x^4 \cos 3y;$$

$$\tfrac{\partial^2 z}{\partial x^2} = 12x^2 \sin 3y; \qquad \tfrac{\partial^2 z}{\partial y^2} = -9x^4 \sin 3y;$$

$$\tfrac{\partial^2 z}{\partial y \partial x} = 12x^3 \cos 3y; \qquad \tfrac{\partial^2 z}{\partial x \partial y} = 12x^3 \cos 3y.$$

2. The required partial derivatives are as follows:

$$\frac{\partial w}{\partial x} = yz^2e^{xy} + \cos(y^2z); \quad \frac{\partial w}{\partial y} = z^2xe^{xy} - 2xyz\sin(y^2z); \quad \frac{\partial w}{\partial z} = 2ze^{xy} - xy^2\sin(y^2z);$$

$$\frac{\partial^2 w}{\partial x^2} = y^2z^2e^{xy}; \quad \frac{\partial^2 w}{\partial y^2} = z^2x^2e^{xy} - 2xz\sin(y^2z) + 4xy^2z^2\cos(y^2z); \quad \frac{\partial^2 w}{\partial z^2} = 2e^{xy} - xy^4\cos(y^2z);$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = z^2 e^{xy} + z^2 xy e^{xy} - 2yz \sin(y^2 z);$$

$$\frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial y} = 2zx e^{xy} - 2xy \sin(y^2 z) - 2xy^3 z \cos(y^2 z);$$

$$\frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial z} = 2zy e^{xy} - y^2 \sin(y^2 z).$$

3.

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x} - \frac{y}{x^2} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{y} + \frac{x}{y^2}.$$

4.

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ay) + F''(x - ay) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 F''(x - ay).$$

"JUST THE MATHS"

UNIT NUMBER

14.3

PARTIAL DIFFERENTIATION 3
(Small increments and small errors)

by

A.J.Hobson

UNIT 14.3 - PARTIAL DIFFERENTIATION 3

SMALL INCREMENTS AND SMALL ERRORS

14.3.1 FUNCTIONS OF ONE INDEPENDENT VARIABLE - A RECAP

For functions of **one** independent variable, a discussion of small increments and small errors has already taken place in Unit 11.6.

It was established that, if a dependent variable, y, is related to an independent variable, x, by means of the formula

$$y = f(x),$$

then

(a) The **increment**, δy, in y, due to an increment of δx, in x is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x;$$

and, in much the same way,

(b) The **error**, δy, in y, due to an error of δx in x, is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

14.3.2 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE

Let us consider, first, a function, z, of two independent variables, x and y, given by the formula

$$z = f(x, y).$$

If x is subject to a small increment (or a small error) of δx, while y remains constant, then the corrresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x.$$

Similarly, if y is subject to a small increment (or a small error) of δy, while x remains constant, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial y}\delta y.$$

It seems reasonable to assume, therefore, that, when x is subject to a small increment (or a small error) of δx **and** y is subject to a small increment (or a small error) of δy, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

It may be shown that, to the first order of approximation, this is indeed true.

Notes:

(i) To prove more rigorously that the above result is true, use would have to be made of the result known as **"Taylor's Theorem"** for a function of two independent variables.

In the present case, where $z = f(x, y)$, it would give

$$f(x+\delta x, y+\delta y) = f(x, y) + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right) + \left(\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\delta x \delta y + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right) + \ldots,$$

which shows that

$$\delta z = f(x+\delta x, y+\delta y) - f(x, y) \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

to the first order of approximation.

(ii) The formula for a function of two independent variables may be extended to functions of a greater number of independent variables by simply adding further appropriate terms to the right hand side.

For example, if

$$w = F(x, y, z),$$

then

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

EXAMPLES

1. A rectangle has sides of length $x$cms. and $y$cms.

   Determine, approximately, in terms of x and y, the increment in the area, A, of the rectangle when x and y are subject to increments of δx and δy, respectively.

   Solution

   The area, A, is given by

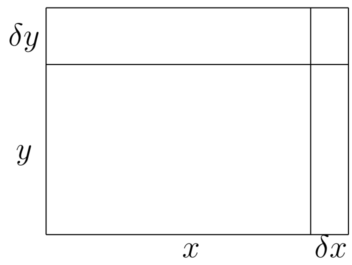
   $$A = xy,$$

   so that

   $$\delta A \simeq \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y = y\delta x + x\delta y.$$

   Note:

   The exact value of δA may be seen in the following diagram:



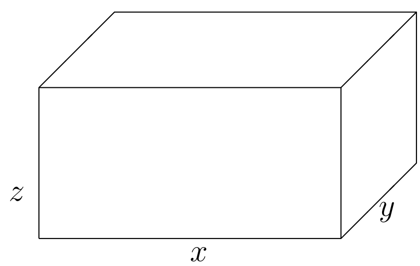
The difference between the approximate value and the exact value is represented by the area of the small rectangle having sides δxcms. and δycms.

2. In measuring a rectangular block of wood, the dimensions were found to be 10cms., 12cms and 20cms. with a possible error of $\pm 0.05 cms.$ in each.

   Calculate, approximately, the greatest possible error in the surface area, S, of the block and the percentage error so caused.

   Solution

   First, we may denote the lengths of the edges of the block by x, y and z.



The surface area, S, is given by

$$S = 2(xy + yz + zx),$$

which has the value 1120cms^2 when $x = 10$cms., $y = 12$cms. and $z = 20$cms.

Also,

$$\delta S \simeq \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z,$$

which gives

$$\delta S \simeq 2(y+z)\delta x + 2(x+z)\delta y + 2(y+x)\delta z;$$

and, on substituting $x = 10$, $y = 12$, $z = 20$, $\delta x = \pm 0.05$, $\delta y = \pm 0.05$ and $\delta z = \pm 0.05$, we obtain

$$\delta S \simeq \pm 2(12+20)(0.05) \pm 2(10+20)(0.05) \pm 2(12+10)(0.05).$$

The greatest error will occur when all the terms of the above expression have the same sign. Hence, the greatest error is given by

$$\delta S_{\max} \simeq \pm 8.4\text{cms.}^2;$$

and, since the originally calculated value was 1120, this represents a percentage error of approximately

$$\pm\frac{8.4}{1120} \times 100 = \pm 0.75$$

3. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

We have

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

That is,

$$\delta w \simeq \frac{3x^2 z}{y^4}\delta x - \frac{4x^3 z}{y^5}\delta y + \frac{x^3}{y^4}\delta z,$$

where

$$\delta x = -\frac{3x}{100}, \quad \delta y = \frac{y}{100} \quad \text{and} \quad \delta z = \frac{2z}{100}.$$

Thus,

$$\delta w \simeq \frac{x^3 z}{y^4}\left[-\frac{9}{100} - \frac{4}{100} + \frac{2}{100}\right] = -\frac{11w}{100}.$$

The percentage error in w is given approximately by

$$\frac{\delta w}{w} \times 100 = -11.$$

That is, w is too small by approximately 11%.

14.3.3 THE LOGARITHMIC METHOD

In this section we consider again examples where it is required to calculate either a percentage increment or a percentage error.

We may conveniently use logarithms if the right hand side of the formula for the dependent variable involves a product, a quotient, or a combination of these two in which the independent variables are separated. This would be so, for instance, in the final example of the previous section.

The method is to take the natural logarithms of both sides of the equation before considering any partial derivatives; and we illustrate this, firstly, for a function of **two** independent variables.

Suppose that

$$z = f(x, y)$$

where $f(x, y)$ is the type of function described above.

Then,

$$\ln z = \ln f(x, y);$$

and, if we temporarily replace $\ln z$ by w, we have a new formula

$$w = \ln f(x, y).$$

The increment (or the error) in w, when x and y are subject to increments (or errors) of δx and δy respectively, is given by

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y.$$

That is,

$$\delta w \simeq \frac{1}{f(x,y)}\frac{\partial f}{\partial x}\delta x + \frac{1}{f(x,y)}\frac{\partial f}{\partial y}\delta y = \frac{1}{f(x,y)}\left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right].$$

In other words,

$$\delta w \simeq \frac{1}{z}\left[\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right].$$

We conclude that

$$\delta w \simeq \frac{\delta z}{z},$$

which means that the fractional increment (or error) in z approximates to the actual increment (or error) in $\ln z$. Multiplication by 100 will, of course, convert the fractional increment (or error) into a percentage.

Note:
The logarthmic method will apply equally well to a function of more than two independent variables where it takes the form of a product, a quotient, or a combination of these two.

EXAMPLES

1. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

Taking the natural logarithm of both sides of the given formula,

$$\ln w = 3\ln x + \ln z - 4\ln y,$$

giving

$$\frac{\delta w}{w} \simeq 3\frac{\delta x}{x} + \frac{\delta z}{z} - 4\frac{\delta y}{y},$$

where

$$\frac{\delta x}{x} = -\frac{3}{100}, \quad \frac{\delta y}{y} = \frac{1}{100} \quad \text{and} \quad \frac{\delta z}{z} = \frac{2}{100}.$$

Hence,

$$\frac{\delta w}{w} \times 100 = -9 + 2 - 4 = -13.$$

Thus, w is too small by approximately 11%, as before.

2. In the formula,

$$w = \sqrt{\frac{x^3}{y}},$$

x is subjected to an increase of 2%. Calculate, approximately, the percentage change needed in y to ensure that w remains unchanged.

Solution

Taking the natural logarithm of both sides of the formula,

$$\ln w = \frac{1}{2}[3\ln x - \ln y].$$

Hence,

$$\frac{\delta w}{w} \simeq \frac{1}{2}\left[3\frac{\delta x}{x} - \frac{\delta y}{y}\right],$$

where $\frac{\delta x}{x} = 0.02$, and we require that $\delta w = 0$.
Thus,

$$0 = \frac{1}{2}\left[0.06 - \frac{\delta y}{y}\right],$$

giving

$$\frac{\delta y}{y} = 0.06,$$

which means that y must be approximately 6% too large.

14.3.4 EXERCISES

1. A triangle is such that two of its sides (of length 6cms. and 8cms.) are at right-angles to each other.

   Calculate, approximately, the change in the length of the hypotenuse of the triangle when the shorter side is lengthened by 0.25cms. and the longer side is shortened by 0.125cms.

2. Two sides of a triangle are measured as $x = 150$cms. and $y = 200$cms. while the angle included between them is measured as $\theta = 60°$. Calculate the area of the triangle.

   If there are possible errors of ± 0.2cms. in the measurement of the sides and $\pm 1°$ in the angle, determine, approximately, the maximum possible error in the calculated area of the triangle.

   State your answers correct to the nearest whole number.

   (**Hint** use the formula, Area $= \frac{1}{2}xy\sin\theta$).

3. Given that the volume of a segment of a sphere is $\frac{1}{6}x(x^2 + 3y^2)$ where x is the height and y is the radius of the base, obtain, in terms of x and y, the percentage error in the volume when x is too large by 1% and y is too small by 0.5%.

4. If

   $$z = kx^{0.01}y^{0.08},$$

   where k is a constant, calculate, approximately, the percentage change in z when x is increased by 2% and y is decreased by 1%.

5. If
$$w = \frac{5xy^4}{z^3},$$
calculate, approximately, the maximum percentage error in w if x, y and z are subject to errors of $\pm 3\%$, $\pm 2.5\%$ and $\pm 4\%$, respectively.

6. If
$$w = 2xyz^{-\frac{1}{2}},$$
where x and z are subject to errors of 0.2%, calculate, approximately, the percentage error in y which results in w being without error.

14.3.5 ANSWERS TO EXERCISES

1. 0.05cms.
2. 12990cms.^2 and 161cms.^2
3. $\frac{3x^2}{x^2+3y^2}$.
4. z decreases by 0.06%.
5. 25%.
6. -0.1%.

"JUST THE MATHS"

UNIT NUMBER

14.4

PARTIAL DIFFERENTIATION 4
(Exact differentials)

by

A.J.Hobson

UNIT 14.4 - PARTIAL DIFFERENTIATION 4

EXACT DIFFERENTIALS

14.4.1 TOTAL DIFFERENTIALS

In Unit 14.3, use was made of expressions of the form,

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

as an approximation for the increment (or error), δf, in the function, $f(x, y, ...)$, when x, y etc. are subject to increments (or errors) of δx, δy etc., respectively.

The expression may be called the **"total differential"** of $f(x, y, ...)$ and may be denoted by $\mathrm{d}f$, giving

$$\mathrm{d}f \simeq \delta f.$$

OBSERVATIONS

Consider the formula,

$$\mathrm{d}f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ \ldots$$

(a) In the special case when $f(x, y, ...) \equiv x$, we may conclude that $\mathrm{d}f = \delta x$ or, in other words,

$$\mathrm{d}x = \delta x.$$

(b) In the special case when $f(x, y, ...) \equiv y$, we may conclude that $\mathrm{d}f = \delta y$ or, in other words,

$$\mathrm{d}y = \delta y.$$

(c) Observations (a) and (b) imply that the total differential of each **independent** variable is the same as the small increment (or error) in that variable; but the total differential of the **dependent** variable is only approximately equal to the increment (or error) in that variable.

(d) All of the previous observations may be summarised by means of the formula

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \ldots$$

14.4.2 TESTING FOR EXACT DIFFERENTIALS

In general, an expression of the form

$$P(x, y, ...)\mathrm{d}x + Q(x, y, ...)\mathrm{d}y + \ldots$$

will not be the total differential of a function, $f(x, y, ...)$, unless the functions, $P(x, y, ...)$, $Q(x, y, ...)$ etc. can be identified with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc., respectively.

If this **is** possible, then the expression is known as an **"exact differential"**.

RESULTS

(i) The expression

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$$

is an exact differential if and only if

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Proof:

(a) If the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

is an exact differential, $\mathrm{d}f$, then

$$\frac{\partial f}{\partial x} \equiv P(x, y) \;\text{ and }\; \frac{\partial f}{\partial y} \equiv Q(x, y).$$

Hence, it must be true that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \left(\equiv \frac{\partial^2 f}{\partial x \partial y} \right).$$

(b) Conversely, suppose that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

We can certainly say that

$$P(x,y) \equiv \frac{\partial u}{\partial x}$$

for <u>some</u> function $u(x,y)$, since $P(x,y)$ could be integrated partially with respect to x.

But then,

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \equiv \frac{\partial^2 u}{\partial y \partial x};$$

and, on integrating partially with respect to x, we obtain

$$Q(x,y) = \frac{\partial u}{\partial y} + A(y),$$

where $A(y)$ is an **arbitrary** function of y.

Thus,

$$P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \frac{\partial u}{\partial x}\mathrm{d}x + \left(\frac{\partial u}{\partial y} + A(y) \right) \mathrm{d}y;$$

and the right-hand side is the exact differential of the function,

$$u(x,y) + \int A(y) \; \mathrm{d}y.$$

(ii) By similar reasoning, it may be shown that the expression

$$P(x,y,z)\mathrm{d}x + Q(x,y,z)\mathrm{d}y + R(x,y,z)\mathrm{d}z$$

is an exact differential, provided that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

ILLUSTRATIONS

1.

$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left[\frac{1}{2}\left(x^2 + y^2\right)\right].$$

2.

$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}[xy].$$

3.

$$y\mathrm{d}x - x\mathrm{d}y$$

is not an exact differential since

$$\frac{\partial y}{\partial y} = 1 \quad \text{and} \quad \frac{\partial(-x)}{\partial x} = -1.$$

4.

$$2\ln y\mathrm{d}x + (x+z)\mathrm{d}y + z^2\mathrm{d}z$$

is not an exact differential since

$$\frac{\partial(2\ln y)}{\partial y} = \frac{2}{y}, \quad \text{and} \quad \frac{\partial(x+z)}{\partial x} = 1.$$

14.4.3 INTEGRATION OF EXACT DIFFERENTIALS

In section 14.4.2, the second half of the proof of the condition for the expression,

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y,$$

to be an exact differential suggests, also, a method of determining which function, $f(x, y)$, it is the total differential of. The method may be illustrated by the following examples:

EXAMPLES

1. Verify that the expression,

$$(x + y\cos x)\mathrm{d}x + (1 + \sin x)\mathrm{d}y,$$

is an exact differential, and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(x + y\cos x) \equiv \frac{\partial}{\partial x}(1 + \sin x) \equiv \cos x;$$

and, hence, the expression is an exact differential.

Secondly, suppose that the expression is the total differential of the function, $f(x, y)$.

Then,

$$\frac{\partial f}{\partial x} \equiv x + y\cos x \quad ---------- (1)$$

and

$$\frac{\partial f}{\partial y} \equiv 1 + \sin x. \quad -----------(2)$$

Integrating (1) partially with respect to x gives

$$f(x, y) \equiv \frac{x^2}{2} + y\sin x + A(y),$$

where $A(y)$ is an **arbitrary** function of y only.

Substituting this result into (2) gives

$$\sin x + \frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1 + \sin x.$$

That is,

$$\frac{\mathrm{d}A}{\mathrm{d}y} \equiv 1;$$

and, hence,

$$A(y) \equiv y + \text{constant}.$$

We conclude that

$$f(x, y) \equiv \frac{x^2}{2} + y \sin x + y + \text{constant}.$$

2. Verify that the expression,

$$(yz + 2)\mathrm{d}x + (xz + 6y)\mathrm{d}y + (xy + 3z^2)\mathrm{d}z,$$

is an exact differential and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(yz + 2) \equiv \frac{\partial}{\partial x}(xz + 6y) \equiv z,$$

$$\frac{\partial}{\partial z}(xz + 6y) \equiv \frac{\partial}{\partial y}(xy + 3z^2) \equiv x,$$

and

$$\frac{\partial}{\partial x}(xy + 3z^2) \equiv \frac{\partial}{\partial z}(yz + 2) \equiv y,$$

so that the given expression is an exact differential.

Suppose it is the total differential of the function, $F(x, y, z)$.
Then,

$$\begin{aligned}
\frac{\partial F}{\partial x} &\equiv yz + 2, \quad - - - - - - - - -(1) \\
\frac{\partial F}{\partial y} &\equiv xz + 6y, \quad - - - - - - - - (2) \\
\frac{\partial F}{\partial z} &\equiv xy + 3z^2. \quad - - - - - - - - (3)
\end{aligned}$$

Integrating (1) partially with respect to x gives

$$F(x, y, z) \equiv xyz + 2x + A(y, z),$$

where $A(y, z)$ is an arbitrary function of y and z only.
Substituting this result into both (2) and (3) gives

$$\begin{aligned}
xz + \frac{\partial A}{\partial y} &\equiv xz + 6y, \\
xy + \frac{\partial A}{\partial z} &\equiv xy + 3z^2.
\end{aligned}$$

That is,

$$\begin{aligned}
\frac{\partial A}{\partial y} &\equiv 6y, \quad - - - - - - -(4) \\
\frac{\partial A}{\partial z} &\equiv 3z^2. \quad - - - - - - -(5)
\end{aligned}$$

Integrating (4) partially with respect to y gives

$$A(y, z) \equiv 3y^2 + B(z),$$

where $B(z)$ is an arbitrary function of z only.
Substituting this result into (5) gives

$$\frac{\mathrm{d}B}{\mathrm{d}z} \equiv 3z^2,$$

which implies that

$$B(z) \equiv z^3 + \text{constant.}$$

We conclude that

$$F(x, y, z) \equiv xyz + 2x + 3y^2 + z^3 + \text{constant.}$$

14.4.4 EXERCISES

1. Verify which of the following are exact differentials and integrate those which are:

   (a)

   $$5x + 12y - 9)\mathrm{d}x + (2x + 5y - 4)\mathrm{d}y;$$

   (b)

   $$12x + 5y - 9)\mathrm{d}x + (5x + 2y - 4)\mathrm{d}y;$$

   (c)

   $$(3x^2 + 2y + 1)\mathrm{d}x + (2x + 6y^2 + 2)\mathrm{d}y;$$

   (d)

   $$(y - e^x)\mathrm{d}x + x\mathrm{d}y;$$

   (e)

   $$\frac{1}{x}\mathrm{d}x - \left(\frac{y}{x^2} + 2x\right)\mathrm{d}y;$$

   (f)

   $$\cos(x + y)\mathrm{d}x + \cos(y - x)\mathrm{d}y;$$

   (g)

   $$(1 - \cos 2x)\mathrm{d}y + 2y \sin 2x\mathrm{d}x.$$

2. Verify that the expression,

   $$3x^2\mathrm{d}x + 2yz\mathrm{d}y + y^2\mathrm{d}z,$$

   is an exact differential and obtain the function of which it is the total differential.

3. Verify that the expression,

$$e^{xy}[y \sin z\mathrm{d}x + x \sin z\mathrm{d}y + \cos z\mathrm{d}z],$$

is an exact differential and obtain the function of which it is the total differential.

14.4.5 ANSWERS TO EXERCISES

1. (a) Not exact;

   (b)

$$6x^2 + 5xy - 9x + y^2 - 4y + \text{constant};$$

   (c)

$$x^3 + 2xy + x + 2y^3 + 2y + \text{constant};$$

   (d)

$$xy - e^x + \text{constant};$$

   (e) Not exact;

   (f) Not exact;

   (g)

$$y(1 - \cos 2x) + \text{constant}.$$

2.

$$x^3 + y^2 z;$$

3.

$$e^{xy} \sin z.$$

"JUST THE MATHS"

UNIT NUMBER

14.5

PARTIAL DIFFERENTIATION 5
(Partial derivatives of composite functions)

by

A.J.Hobson

UNIT 14.5 - PARTIAL DIFFERENTIATION 5

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS

14.5.1 SINGLE INDEPENDENT VARIABLES

In this Unit, we shall be concerned with functions, $f(x, y...)$, of two or more variables in which those variables are not independent, but are themselves dependent on some other variable, t.

The problem is to calculate the rate of increase (positive or negative) of such functions with respect to t.

Let us suppose that the variable, t, is subject to a small increment of δt, so that the variables $x, y...$ are subject to small increments of $\delta x,\ \delta y,\ \ldots$, respectively. Then the corresponding increment, δf, in $f(x, y...)$ is given by

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots,$$

where we note that no label other than f is being used, here, for the function of several variables. That is, it is not essential to use a specific **formula**, such as $w = f(x, y..)$.

Dividing throughout by δt gives

$$\frac{\delta f}{\delta t} \simeq \frac{\partial f}{\partial x}.\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}.\frac{\delta y}{\delta t} + \ldots$$

Allowing δt to tend to zero, we obtain the standard result for the **"total derivative"** of $f(x, y..)$ with respect to t, namely

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots$$

This rule may be referred to as the **"chain rule"**, but more advanced versions of it will appear later.

EXAMPLES

1. A point, P, is moving along the curve of intersection of the surface whose cartesian equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = z \quad \text{(a Paraboloid)}$$

and the surface whose cartesian equation is

$$x^2 + y^2 = 5 \quad \text{(a Cylinder).}$$

If x is increasing at 0.2 cms/sec, how fast is z changing when $x = 2$?

Solution

We may use the formula

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

But, from the equation of the paraboloid,

$$\frac{\partial z}{\partial x} = \frac{x}{8} \text{ and } \frac{\partial z}{\partial y} = -\frac{2y}{9};$$

and, from the equation of the cylinder,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Substituting $x = 2$ gives $y = \pm 1$ on the curve of intersection, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{2}{8}\right)(0.2) + \left(-\frac{2}{9}\right)(\pm 1)(0.2)\left(\frac{-2}{\pm 1}\right) = 0.2\left(\frac{1}{4} + \frac{4}{9}\right) = \frac{5}{36} \text{ cms/sec.}$$

2. Determine the total derivative of u with respect to t in the case when

$$u = xy + yz + zx, \quad x = e^t, \quad y = e^{-t} \text{ and } z = x + y.$$

Solution

We may use the formula

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}.\frac{\mathrm{d}z}{\mathrm{d}t},$$

where

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial u}{\partial y} = z + x, \quad \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t} = -y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = e^t - e^{-t} = x - y.$$

Hence,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (y + z)x - (z + x)y + (x + y)(x - y)$$

$$= -zy + zx + x^2 - y^2$$

$$= z(x - y) + (x - y)(x + y).$$

That is,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (x - y)(x + y + z).$$

14.5.2 SEVERAL INDEPENDENT VARIABLES

We may now extend the work of the previous section to functions, $f(x, y..)$, of two or more variables in which $x, y..$ are each dependent on two or more variables, $s, t..$

Since the function, $f(x, y..)$, is dependent on $s, t..$, we may wish to determine its **partial** derivatives with respect to any one of these (independent) variables.

The result previously established for a **single** independent variable may easily be adapted as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial s} + \ldots$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \ldots$$

Again, this is referred to as the **"chain rule"**.

EXAMPLES

1. Determine the first-order partial derivatives of z with respect to r and θ in the case when

$$z = x^2 + y^2, \quad \text{where} \quad x = r\cos\theta \quad \text{and} \quad y = r\sin 2\theta.$$

Solution

We may use the formulae

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}.$$

These give

$$\text{(i)} \quad \frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$= 2r\left(\cos^2\theta + \sin^2 2\theta\right)$$

and

$$\text{(ii)}\quad \frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$= 2r^2\left(2\cos 2\theta \sin 2\theta - \cos\theta\sin\theta\right).$$

2. Determine the first-order partial derivatives of w with respect to u, θ and ϕ in the case when

$$w = x^2 + 2y^2 + 2z^2,$$

where

$$x = u\sin\phi\cos\theta,\quad y = u\sin\phi\sin\theta \quad\text{and}\quad z = u\cos\phi.$$

Solution

We may use the formulae

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \theta}$$

and

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}.\frac{\partial z}{\partial \phi}.$$

These give

$$\text{(i)}\quad \frac{\partial w}{\partial u} = 2x\sin\phi\cos\theta + 4y\sin\phi\sin\theta + 4z\cos\phi$$

$$= 2u\sin^2\phi\cos^2\theta + 4u\sin^2\phi\sin^2\theta + 4u\cos^2\phi;$$

$$\text{(ii)} \quad \frac{\partial w}{\partial \theta} = -2xu \sin\phi \sin\theta + 4yu \sin\phi \cos\theta$$

$$= -2u^2 \sin^2\phi \sin\theta \cos\theta + 4u^2 \sin^2\phi \sin\theta \cos\theta$$

$$= 2u^2 \sin^2\phi \sin\theta \cos\theta;$$

$$\text{(iii)} \quad \frac{\partial w}{\partial \phi} = 2xu \cos\phi \cos\theta + 4yu \cos\phi \sin\theta - 4zu \sin\phi$$

$$= 2u^2 \sin\phi \cos\phi \cos^2\theta + 4u^2 \sin\phi \cos\phi \sin^2\theta - 4u^2 \sin\phi \cos\phi$$

$$= 2u^2 \sin\phi \cos\phi \left(\cos^2\theta + 2\sin^2\theta - 2\right).$$

14.5.3 EXERCISES

1. Determine the total derivative of z with respect to t in the cases when

   (a)
   $$z = x^2 + 3xy + 5y^2, \quad \text{where} \quad x = \sin t \quad \text{and} \quad y = \cos t;$$

   (b)
   $$z = \ln(x^2 + y^2), \quad \text{where} \quad x = e^{-t} \quad \text{and} \quad y = e^t;$$

   (c)
   $$z = x^2y^2 \quad \text{where} \quad x = 2t^3 \quad \text{and} \quad y = 3t^2.$$

2. If $z = f(x, y)$, show that, when y is a function of x,
   $$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x}.$$

   Hence, determine $\frac{\mathrm{d}z}{\mathrm{d}x}$ in the case when $z = xy + x^2y$ and $y = \ln x$.

3. The base radius, r, of a cone is decreasing at a rate of 0.1cms/sec while the perpendicular height, h, is increasing at a rate of 0.2cms/sec. Determine the rate at which the volume, V, is changing when $r = 2$cm and $h = 3$cm. (**Hint:** $V = (\pi r^2 h)/3$).

4. A rectangular solid has sides of lengths 3cms, 4cms and 5cms. Determine the rate of increase of the length of the diagonal of the solid if the sides are increasing at rates of $\frac{1}{3}$cms./sec, $\frac{1}{4}$cms./sec and $\frac{1}{5}$cms/sec, respectively.

5. If
$$z = (2x + 3y)^2 \quad \text{where} \quad x = r^2 - s^2 \quad \text{and} \quad y = 2rs,$$
determine, in terms of r and s the first-order partial derivatives of z with respect to r and s.

6. If
$$z = f(x, y) \quad \text{where} \quad x = e^u \cos v \quad \text{and} \quad y = e^u \sin v,$$
show that
$$\frac{\partial z}{\partial u} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial z}{\partial v} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

7. If
$$w = 5x - 3y^2 + 7z^3 \quad \text{where} \quad x = 2s + 3t, \quad y = s - t \quad \text{and} \quad z = 4s + t,$$
determine, in terms of s and t, the first order partial derivatives of w with respect to s and t.

14.5.4 ANSWERS TO EXERCISES

1. (a)
$$\frac{dz}{dt} = 3\cos 2t - 4\sin 2t;$$
(b)
$$\frac{dz}{dt} = 2\left[\frac{e^{4t} - 1}{e^{4t} + 1}\right];$$
(c)
$$\frac{dz}{dt} = 360t^9.$$

2.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^2 + 2xy + 1 + x.$$

3. The volume is decreasing at a rate of approximately 0.42 cubic centimetres per second.
4. The diagonal is increasing at a rate of approximately 0.42 centimetres per second.
5.

$$\frac{\partial z}{\partial r} = 8(r^2 - s^2 + 3rs)(2r + 3s) \quad \text{and} \quad \frac{\partial z}{\partial s} = 8(r^2 - s^2 + 3rs)(3r - 2s).$$

6. Results follow immediately from the formulae

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}.$$

7.

$$\frac{\partial w}{\partial s} = 1344s^2 + 672st + 84t^2 - 6s + 6t + 10$$

and

$$\frac{\partial w}{\partial t} = 336s^2 + 168st + 21t^2 + 6s - 6t + 15.$$

"JUST THE MATHS"

UNIT NUMBER

14.6

PARTIAL DIFFERENTIATION 6
(Implicit functions)

by

A.J.Hobson

UNIT 14.6 - PARTIAL DIFFERENTIATION 6

IMPLICIT FUNCTIONS

14.6.1 FUNCTIONS OF TWO VARIABLES

The chain rule, encountered earlier, has a convenient application to implicit relationships of the form,

$$f(x, y) = \text{constant},$$

between two independent variables, x and y.

It provides a means of determining the total derivative of y with respect to x.

Explanation

Taking x as the single independent variable, we may interpret $f(x, y)$ as a function of x and y in which both x and y are functions of x.

Differentiating both sides of the relationship, $f(x, y) = \text{constant}$, with respect to x gives

$$\frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In other words,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

EXAMPLES

1. If
$$f(x,y) \equiv x^3 + 4x^2y - 3xy + y^2 = 0,$$
determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 8xy - 3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4x^2 - 3x + 2y.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 8xy - 3y}{4x^2 - 3x + 2y}.$$

2. If
$$f(x,y) \equiv x\sin(2x - 3y) + y\cos(2x - 3y),$$
determine and expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\frac{\partial f}{\partial x} = \sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)$$

and

$$\frac{\partial f}{\partial y} = -3x\cos(2x - 3y) + \cos(2x - 3y) + 3y\sin(2x - 3y).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)}{3x\cos(2x - 3y) - \cos(2x - 3y) - 3y\sin(2x - 3y)}.$$

14.6.2 FUNCTIONS OF THREE VARIABLES

For relationships of the form,

$$f(x, y, z) = \text{constant},$$

let us suppose that x and y are independent of each other.

Then, regarding $f(x, y, z)$ as a function of x, y and z, where x, y and z are **all** functions of x and y, the chain rule gives

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0.$$

But,

$$\frac{\partial x}{\partial x} = 1 \text{ and } \frac{\partial y}{\partial x} = 0.$$

Hence,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}.\frac{\partial z}{\partial x} = 0,$$

giving

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}};$$

and, similarly,

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

EXAMPLES

1. If

$$f(x, y, z) \equiv z^2xy + zy^2x + x^2 + y^2 = 5,$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = z^2y + zy^2 + 2x,$$

$$\frac{\partial f}{\partial y} = z^2x + 2zyx + 2y$$

and

$$\frac{\partial f}{\partial z} = 2zxy + y^2x.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{z^2y + zy^2 + 2x}{2zxy + y^2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{z^2x + 2zyx + 2y}{2zxy + y^2x}.$$

2. If

$$f(x, y, z) \equiv xe^{y^2+2z},$$

determine expressions for $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}z}{\mathrm{d}y}$.

Solution

$$\frac{\partial f}{\partial x} = e^{y^2+2z},$$

$$\frac{\partial f}{\partial y} = 2yxe^{y^2+2z},$$

and

$$\frac{\partial f}{\partial z} = 2xe^{y^2+2z}.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{e^{y^2+2z}}{2xe^{y^2+2z}} = -\frac{1}{2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2yxe^{y^2+2z}}{2xe^{y^2+2z}} = -y.$$

14.6.3 EXERCISES

1. Use partial differentiation to determine expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases:

   (a)
$$x^3 + y^3 - 2x^2y = 0;$$

   (b)
$$e^x \cos y = e^y \sin x;$$

   (c)
$$\sin^2 x - 5\sin x \cos y + \tan y = 0.$$

2. If
$$x^2y + y^2z + z^2x = 10,$$
where x and y are independent, determine expressions for
$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

3. If
$$xyz - 2\sin(x^2 + y + z) + \cos(xy + z^2) = 0,$$
where x and y are independent, determine expressions for
$$\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}.$$

4. If
$$r^2 \sin\theta = (r\cos\theta - 1)z,$$
where r and θ are independent, determine expressions for
$$\frac{\partial z}{\partial r} \text{ and } \frac{\partial z}{\partial \theta}.$$

14.6.4 ANSWERS TO EXERCISES

1. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4xy - 3x^2}{3y^2 - 2x^2};$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x \cos y - e^y \cos x}{x^x \sin y + e^y \sin x};$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5\cos x \cos y - 2\sin x \cos x}{5\sin x siny + \sec^2 y}.$$

2.

$$\frac{\partial z}{\partial x} = -\frac{2xy + z^2}{y^2 + 2zx}$$

and

$$\frac{\partial z}{\partial y} = -\frac{x^2 + 2yz}{y^2 + 2zx}.$$

3.

$$\frac{\partial z}{\partial x} = -\frac{yz - 4x\cos(x^2 + y + z) - y\sin(xy + z^2)}{xy - 2\cos(x^2 + y + z) - 2z\sin(xy + z^2)}$$

and

$$\frac{\partial z}{\partial y} = -\frac{xz - 2\cos(x^2 + y + z) - x\sin(xy + z^2)}{xy - 2\cos(x^2 + y + z) - 2z\sin(xy + z^2)}.$$

4.

$$\frac{\partial z}{\partial r} = \frac{2r\sin\theta - z\cos\theta}{r\cos\theta - 1}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{r^2\cos\theta + rz\sin\theta}{r\cos\theta - 1}.$$

"JUST THE MATHS"

UNIT NUMBER

14.7

PARTIAL DIFFERENTIATION 7
(Change of independent variable)

by

A.J.Hobson

UNIT 14.7 - PARTIAL DIFFERENTIATON 7

CHANGE OF INDEPENDENT VARIABLE

14.7.1 ILLUSTRATIONS OF THE METHOD

In the theory of **"partial differential equations"** (that is, equations which involve partial derivatives) it is sometimes required to express a given equation in terms of a new set of independent variables. This would be necessary, for example, in changing a discussion from one geometrical reference system to another. The method is an application of the chain rule for partial derivatives and we illustrate it with examples.

EXAMPLES

1. Express, in plane polar co-ordinates, r and θ, the following partial differential equations:

   (a)
   $$\frac{\partial V}{\partial x} + 5\frac{\partial V}{\partial y} = 1;$$

   (b)
   $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

   Solution

   Both differential equations involve a function, $V(x, y)$, where
   $$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

   Hence,
   $$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial r},$$

   or
   $$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cos\theta + \frac{\partial V}{\partial y}\sin\theta$$

   and
   $$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}.\frac{\partial y}{\partial \theta},$$

or

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x} r\sin\theta + \frac{\partial V}{\partial y} r\cos\theta.$$

Now, we may eliminate, first $\frac{\partial V}{\partial y}$, and then $\frac{\partial V}{\partial x}$ to obtain

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$

Hence, differential equation, (a), becomes

$$(\cos\theta + 5\sin\theta)\frac{\partial V}{\partial r} + \left(\frac{5\cos\theta}{r} - \sin\theta\right)\frac{\partial V}{\partial \theta} = 1.$$

In order to find the <u>second</u>-order derivatives of V with respect to x and y, it is necessary to write the formulae for the first-order derivatives in the form

$$\frac{\partial}{\partial x}[V] = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)[V]$$

and

$$\frac{\partial}{\partial y}[V] = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)[V].$$

From these, we obtain

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right),$$

which gives

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Adding these together gives the differential equation, (b), in the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

2. Express the differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(a) in cylindrical polar co-ordinates,

and

(b) in spherical polar co-ordinates.

Solution

(a) Using

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z,$$

we may use the results of the previous example to give

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(b) Using

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta, \quad \text{and} \quad z = u\cos\phi,$$

we could write out three formulae for $\frac{\partial V}{\partial u}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}$ and then solve for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; but this is complicated.

However, the result in part (a) provides a shorter method as follows:

Cylindrical polar co-ordinates are expressible in terms of spherical polar co-ordinates by the formulae

$$z = u\cos\phi, \quad r = u\sin\phi, \quad \theta = \theta.$$

Hence, by using the previous example with z, r, θ in place of x, y, z respectively and u, ϕ in place of r, θ, respectively, we obtain

$$\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial r^2}=\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}.$$

Therefore, to complete the conversion we need only to consider $\frac{\partial V}{\partial r}$; and, by using r, u, ϕ in place of y, r, θ, respectively, the previous formula for $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial r}=\sin\phi\frac{\partial V}{\partial u}+\frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}.$$

The given differential equation thus becomes

$$\frac{\partial^2 V}{\partial u^2}+\frac{1}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{1}{u^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{u\sin\phi}\left[\sin\phi\frac{\partial V}{\partial u}+\frac{\cos\phi}{u}\frac{\partial V}{\partial \phi}\right]=0.$$

That is,

$$\frac{\partial^2 V}{\partial u^2}+\frac{2}{u}\frac{\partial V}{\partial u}+\frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\cot\phi}{u^2}\frac{\partial V}{\partial \phi}+\frac{1}{u^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}=0.$$

14.7.2 EXERCISES

1. Express the partial differential equation,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial x}=0,$$

in plane polar co-ordinates, r and θ, where

$$x=r\cos\theta \quad \text{and} \quad y=r\sin\theta.$$

2. Express the differential equation,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=\frac{\partial V}{\partial z},$$

in spherical polar co-ordinates u, θ and ϕ, where

$$x=u\sin\phi\cos\theta, \quad y=u\sin\phi\sin\theta \quad \text{and} \quad z=u\cos\phi.$$

3. A function $\phi(x,t)$ satisfies the partial differential equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{k^2}\frac{\partial^2\phi}{\partial t^2},$$

where k is a constant.

Express this equation in terms of new independent variables, u and v, where

$$x = \frac{1}{2}(u+v) \text{ and } t = \frac{1}{2k}(u-v).$$

4. A function $\theta(x,y)$ satisfies the partial differential equation,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Express this equation in terms of new independent variables, s and t, where

$$x = \ln u \text{ and } y = \ln v.$$

Determine, also, an expression for $\frac{\partial^2\theta}{\partial x\partial y}$ in terms of θ, u and v.

14.7.3 ANSWERS TO EXERCISES

1.

$$\frac{\partial V}{\partial r} = 0.$$

2.

$$(\sin\phi - \cos\phi)\frac{\partial V}{\partial u} - \frac{\cos\phi + \sin\phi}{u}\frac{\partial V}{\partial \phi} = 0.$$

3.

$$\frac{\partial^2\phi}{\partial u\partial v} = 0.$$

4.

$$u^2\frac{\partial^2\theta}{\partial x^2} + v^2\frac{\partial^2\theta}{\partial v^2} + u\frac{\partial\theta}{\partial u} + v\frac{\partial\theta}{\partial v} = 0,$$

and

$$\frac{\partial^2\theta}{\partial x\partial y} = uv\frac{\partial^2\theta}{\partial u\partial v}.$$

"JUST THE MATHS"

UNIT NUMBER

14.8

PARTIAL DIFFERENTIATION 8
(Dependent and independent functions)

by

A.J.Hobson

UNIT 14.8 - PARTIAL DIFFERENTIATION 8

DEPENDENT AND INDEPENDENT FUNCTIONS

14.8.1 THE JACOBIAN

Suppose that

$$u \equiv u(x,y) \quad \text{and} \quad v \equiv v(x,y)$$

are two functions of two independent variables, x and y; then, in general, it is not possible to express u solely in terms of v, nor v solely in terms of u.

However, on occasions, it may be possible, as the following illustrations demonstrate:

ILLUSTRATIONS

1. If

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y},$$

then

$$u \equiv 1 + \frac{x}{y} \quad \text{and} \quad v \equiv \frac{x}{y} - 1,$$

which gives

$$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$$

Hence,

$$u \equiv 1 + \frac{1}{v+1} \quad \text{and} \quad v \equiv \frac{1}{u-1} - 1.$$

2. If

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2,$$

then

$$v \equiv u^2 \quad \text{and} \quad u \equiv \pm\sqrt{v}.$$

If u and v are **not** connected by an identical relationship, they are said to be **"independent functions"**.

THEOREM

Two functions, $u(x,y)$ and $v(x,y)$, are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \not\equiv 0.$$

Proof:

We prove an equivalent statement, namely that $u(x,y)$ and $v(x,y)$ are dependent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

(a) Suppose that v is dependent on u by virtue of the relationship

$$v \equiv v(u).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we shall establish that the determinant, J, is identically equal to zero.

We have

$$\frac{\partial v}{\partial x} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial y}.$$

Thus,

$$\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} \equiv \frac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}$$

or

$$\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}.\frac{\partial u}{\partial y} \equiv 0,$$

which means that the determinant, J, is identically equal to zero.

(b) Secondly, let us suppose that

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0,$$

and attempt to prove that $u(x, y)$ and $v(x, y)$ are dependent.

In theory, we could express v in terms of u and x by eliminating y between $u(x, y)$ and $v(x, y)$.

We shall assume that

$$v \equiv A(u, x).$$

By expressing $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we may show that $A(u, x)$ does not contain x.

We have

$$\left(\frac{\partial v}{\partial x}\right)_y \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x.$$

Hence, if the determinant, J, is identically equal to zero, we may say that

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u & \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x \end{vmatrix} \equiv 0;$$

and, on expansion, this gives

$$\left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial A}{\partial x}\right)_u \equiv 0.$$

If the first of these two is equal to zero, then u contains only x and, hence, x could be expressed in terms of u, giving v as a function of u only. If the second is equal to zero, then A contains no x's and, again, v is a function of u only.

Notes:

(i) The determinant

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

may also be denoted by

$$\frac{\partial(u,v)}{\partial(x,y)}$$

and is called the **"Jacobian determinant"** or simply the **"Jacobian"** of u and v with respect to x and y.

(ii) Similar Jacobian determinants may be used to test for the dependence or independence of three functions of three variables, four functions of four variables, and so on.

For example, the three functions

$$u \equiv u(x,y,z), \quad v \equiv v(x,y,z) \text{ and } w \equiv w(x,y,z)$$

are independent if and only if

$$J \equiv \frac{\partial(u,v,w)}{\partial(x,y,z)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \not\equiv 0.$$

ILLUSTRATIONS

1.

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y}$$

are **not** independent, since

$$J \equiv \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \equiv \frac{1}{xy} - \frac{1}{xy} \equiv 0$$

2.

$$u \equiv x+y \quad \text{and} \quad v \equiv x^2+2xy+y^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 1 & 1 \\ 2x+2y & 2x+2y \end{vmatrix} \equiv 0.$$

3.

$$u \equiv x^2+2y \quad \text{and} \quad v \equiv xy$$

are independent, since

$$J \equiv \begin{vmatrix} 2x & 2 \\ y & x \end{vmatrix} \equiv 2x^2 - 2y \not\equiv 0.$$

4.

$$u \equiv x^2-2y+z, \quad v \equiv x+3y^2-2z, \quad \text{and} \quad w \equiv 5x+y+z^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 2x & -2 & 1 \\ 1 & 6y & -2 \\ 5 & 1 & 2z \end{vmatrix} \equiv 24xyz + 4x - 30y + 4z + 25 \not\equiv 0.$$

14.8.2 EXERCISES

1. Determine which of the following pairs of functions are independent:

   (a)
   $$u \equiv x\cos y \quad \text{and} \quad v \equiv x\sin y;$$

   (b)
   $$u \equiv x+y \quad \text{and} \quad v \equiv \frac{y}{x+y};$$

   (c)
   $$u \equiv x-2y \quad \text{and} \quad v \equiv x^2+4y^2-4xy+3x-6y;$$

   (d)
   $$u \equiv x+2y \quad \text{and} \quad v \equiv x^2-y^2+2xy-x.$$

2. Show that
   $$u \equiv x+y+z, \quad v \equiv x^2+y^2+z^2-2xy-2yz-2zx$$
   and
   $$w \equiv x^3+y^3+z^3-3xyz$$
   are dependent.
   Show also that w may be expressed as a linear combination of u^3 and uv.

3. Given that
   $$x+y+z \equiv u, \quad y+z \equiv uv \quad \text{and} \quad z \equiv uvw,$$
   express x and y in terms of u, v and w.
   Hence, show that
   $$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv u^2 v.$$

14.8.3 ANSWERS TO EXERCISES

1. (a) Independent, since $J \equiv x$;
   (b) Independent, since $J \equiv 1/(x+y)$;
   (c) Dependent, since $J \equiv 0$;
   (d) Independent, since $J \equiv 2 - 2x - 6y$.
2. $$w \equiv \frac{1}{4}\left[u^3 + 3uv\right].$$
3. $$x \equiv u - uv \quad \text{and} \quad y \equiv uv - uvw.$$

"JUST THE MATHS"

UNIT NUMBER

14.9

PARTIAL DIFFERENTIATION 9
(Taylor's series)
for
(Functions of several variables)

by

A.J.Hobson

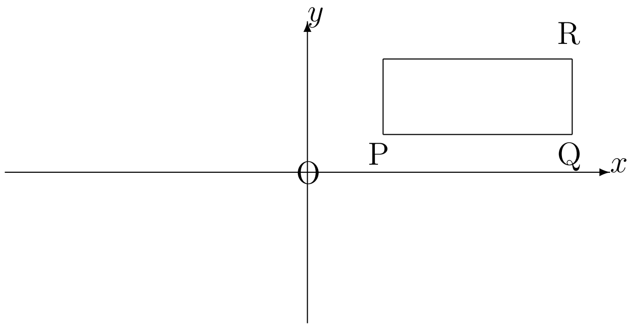
UNIT 14.9 - PARTIAL DIFFERENTIATION 9

TAYLOR'S SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

14.9.1 THE THEORY AND FORMULA

Initially, we shall consider a function, $f(x, y)$, of **two** independent variables, x, y, and obtain a formula for $f(x+h, y+k)$ in terms of $f(x, y)$ and its partial derivatives.

Suppose that P,Q and R denote the points with cartesian co-ordinates, (x, y), $(x+h, y)$ and $(x+h, y+k)$, respectively.



(a) As we move in a straight line from P to Q, y remains constant so that $f(x, y)$ behaves as a function of x only.

Hence, by Taylor's theorem for one independent variable,

$$f(x+h, y) = f(x, y) + f_x(x, y) + \frac{h^2}{2!} f_{xx}(x, y) + \ldots,$$

where $f_x(x, y)$ and $f_{xx}(x, y)$ mean $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ respectively, with similar notations encountered in what follows.

In abbreviated notation,

$$f(\mathrm{Q}) = f(\mathrm{P}) + h f_x(\mathrm{P}) + \frac{h^2}{2!} f_{xx}(\mathrm{P}) + \ \ldots$$

(b) As we move in a straight line from Q to R, x remains constant so that $f(x, y)$ behaves as a function of y only.

Hence,

$$f(x+h, y+k) = f(x+h, y) + kf_x(x+h, y) + \frac{k^2}{2!}f_{xx}(x+h, y) + \ldots;$$

or, in abbreviated notation,

$$f(\text{R}) = f(\text{Q}) + kf_y(\text{Q}) + \frac{k^2}{2!}f_{yy}(\text{Q}) + \ldots$$

(c) From the result in (a)

$$f_y(\text{Q}) = f_y(\text{P}) + hf_{yx}(\text{P}) + \frac{h^2}{2!}f_{yxx}(\text{P}) + \ldots$$

and

$$f_{yy}(\text{Q}) = f_{yy}(\text{P}) + hf_{yyx}(\text{P}) + \frac{h^2}{2!}f_{yyxx}(\text{Q}) + \ldots$$

(d) Substituting the results into (b) gives

$$f(\text{R}) = f(\text{P}) + hf_x(\text{P}) + kf_y(\text{P}) + \frac{1}{2!}\left[h^2 f_{xx}(\text{P}) + 2hkf_{yx}(\text{P}) + k^2 f_{yy}(\text{P})\right] + \ldots$$

It may be shown that the complete result can be written as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \ \ldots$$

Notes:

(i) The equivalent of this result for a function of three variables would be

$$f(x+h, y+k, z+l) = f(x,y,z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(x,y,z) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x,y,z) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^3 f(x,y,z) + \ \ldots$$

(ii) Alternative versions of Taylor's theorem may be obtained by interchanging $x, y, z...$ with $h, k, l....$

For example,

$$f(x+h, y+k) = f(h,k) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

(iii) Replacing x with $x - h$ and y with $y - k$ in (ii) gives the formula,

$$f(x,y) = f(h,k) + \left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

This is called the **"Taylor expansion of $f(x,y)$ about the point (a,b)"**

(iv) A special case of Taylor's series (for two independent variables) is obtained by putting $h = 0$ and $k = 0$ in (ii) to give

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \ldots$$

This is called a **"MacLaurin's series"** but is also the Taylor expansion of $f(x, y)$ about the point $(0, 0)$.

EXAMPLE

Determine the Taylor series expansion of the function $f\left(x + 1, y + \frac{\pi}{3}\right)$ in ascending powers of x and y when

$$f(x, y) \equiv \sin xy,$$

neglecting terms of degree higher than two.

Solution

We use the result that

$$f\left(x + 1, y + \frac{\pi}{3}\right) = f\left(1, \frac{\pi}{3}\right) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f\left(1, \frac{\pi}{3}\right) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f\left(1, \frac{\pi}{3}\right) + \ldots,$$

in which the first term on the right has value $\sqrt{3}/2$.

The partial derivatives required are as follows:

$$\frac{\partial f}{\partial x} \equiv y\cos xy \quad \text{giving} \quad -\frac{\pi}{6} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3};$$

$$\frac{\partial f}{\partial y} \equiv x\cos xy \quad \text{giving} \quad \frac{1}{2} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x^2} \equiv -y^2\sin xy \quad \text{giving} \quad -\frac{\pi^2\sqrt{3}}{18} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \cos xy - xy \sin xy \quad \text{giving} \quad \frac{1}{2} - \frac{\pi\sqrt{3}}{6} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial y^2} \equiv -x^2 \sin xy \quad \text{giving} \quad -\frac{\sqrt{3}}{2} \quad \text{at} \quad x = 1, \quad y = \frac{\pi}{3}.$$

Neglecting terms of degree higher than two, we have

$$\sin xy = \frac{\sqrt{3}}{2} + \frac{\pi}{6}x + \frac{1}{2}y - \frac{\sqrt{3}\pi^2}{36}x^2 + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)xy - \frac{\sqrt{3}}{4}y^2 + \ldots$$

14.9.2 EXERCISES

1. If $f(x, y) \equiv x^3 - 3xy^2$, show that

$$f(2 + h, 1 + k) = 2 + 9h - 12k + 6\left(h^2 - hk - k^2\right) + h^3 - 3hk^2.$$

2. If $f(x, y) \equiv \sin x \cosh y$, evaluate all the partial derivatives of $f(x, y)$ up to order five at the point, $(x, y) = (0, 0)$, and, hence, show that

$$\sin x \cosh y = x - \frac{1}{6}\left(x^3 - 3xy^2\right) + \frac{1}{120}\left(x^5 - 10x^3y^2 + 5xy^4\right) + \ldots$$

3. If z is a function of two independent variables, x and y, where $y \equiv z - x \sin z$, evaluate all the partial derivatives of $z(x, y)$ up to order three at the point, $(x, y) = (0, 0)$, and, hence, show that

$$z(x, y) = y + xy + x^2y + \ldots$$

"JUST THE MATHS"

UNIT NUMBER

14.10

PARTIAL DIFFERENTIATION 10
(Stationary values)
for
(Functions of two variables)

by

A.J.Hobson

UNIT 14.10 - PARTIAL DIFFERENTIATION 10

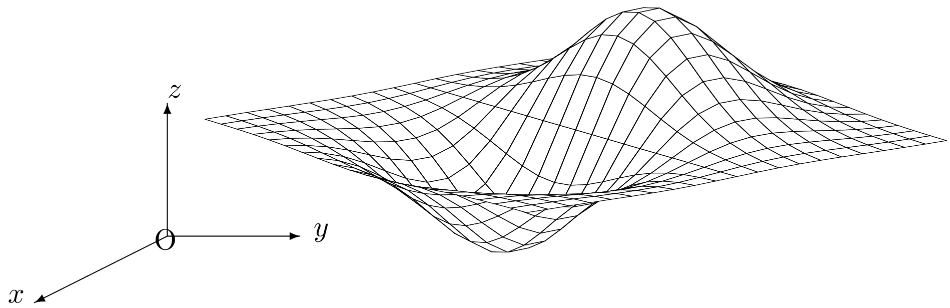
STATIONARY VALUES FOR FUNCTIONS OF TWO VARIABLES

14.10.1 INTRODUCTION

If $f(x,y)$ is a function of the two independent variables, x and y, then the equation,

$$z = f(x,y),$$

will normally represent some surface in space, referred to cartesian axes, Ox, Oy and Oz.



DEFINITION 1

The **"stationary points"**, on a surface whose equation is $z = f(x,y)$, are defined to be the points for which

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0.$$

DEFINITION 2

The function, $z = f(x,y)$, is said to have a **"local maximum"** at a point, P(x_0, y_0, z_0), if z_0 is larger than the z co-ordinates of all other points on the surface, with equation $z = f(x,y)$, in the neighbourhood of P.

DEFINITION 3

The function $z = f(x,y)$ is said to have a **"local minimum"** at a point, P(x_0, y_0, z_0), if z_0 is smaller than the z co-ordinates of all other points on the surface, with equation $z = f(x,y)$, in the neighbourhood of P.

Note:
At a stationary point, $P(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, each of the planes, $x = x_0$ and $y = y_0$, intersect the surface in a curve which has a stationary point at P.

14.10.2 SUFFICIENT CONDITIONS FOR MAXIMA AND MINIMA

A complete explanation of the conditions for a function, $z = f(x, y)$, to have a local maximum or a local minimum at a particular point require the use of Taylor's theorem for two variables.

At this stage, we state the standard set of sufficient conditions without proof.

(a) Sufficient conditions for a local maximum

A point, $P(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local maximum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} < 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} < 0$$

and

$$\frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

(b) Sufficient conditions for a local minimum

A point, $P(x_0, y_0, z_0)$, on the surface with equation $z = f(x, y)$, is a local minimum if

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial^2 z}{\partial x^2} > 0 \ \text{ or } \ \frac{\partial^2 z}{\partial y^2} > 0$$

and

$$\frac{\partial^2 z}{\partial x^2} . \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0,$$

when $x = x_0$ and $y = y_0$.

Notes:

(i) If $\frac{\partial^2 z}{\partial x^2}$ is positive (or negative) and also $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, then $\frac{\partial^2 z}{\partial y^2}$ is automatically positive (or negative).

(ii) If it turns out that $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ is **negative** at P, we have what is called a **"saddle-point"**, irrespective of what $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ themselves are.

(iii) The values of z at the local maxima and local minima of the function, $z = f(x, y)$, may also be called the **"extreme values"** of the function, $f(x, y)$.

EXAMPLES

1. Determine the extreme values and the co-ordinates of any saddle-points of the function,

$$z = x^3 + x^2 - xy + y^2 + 4.$$

Solution

(i) First, we determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x - y \quad \text{and} \quad \frac{\partial z}{\partial y} = -x + 2y.$$

(ii) Secondly, we solve the equations $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ for x and y.

$$\begin{aligned} 3x^2 + 2x - y &= 0, - - - - - - - - - - - (1) \\ -x + 2y &= 0. - - - - - - - - - - - (2) \end{aligned}$$

Substituting equation (2) into equation (1) gives

$$3x^2 + 2x - \frac{1}{2}x = 0.$$

That is,

$$6x^2 + 3x = 0 \quad \text{or} \quad 3x(2x + 1) = 0.$$

Hence, $x = 0$ or $x = -\frac{1}{2}$, with corresponding values, $y = 0$, $z = 4$ and $y = -\frac{1}{4}$, $z = -\frac{65}{16}$, respectively.

The stationary points are thus $(0, 0, 4)$ and $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$.

(iii) Thirdly, we evaluate $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ at each stationary point.

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

(a) At the point $(0, 0, 4)$,

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} > 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 3 > 0$$

and, therefore, the point, $(0, 0, 4)$, is a local minimum, with z having a corresponding extreme value of 4.

(b) At the point $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{65}{16}\right)$,

$$\frac{\partial^2 z}{\partial x^2} = -1, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -1.$$

Hence,

$$\frac{\partial^2 z}{\partial x^2} < 0, \quad \frac{\partial^2 z}{\partial y^2} > 0, \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -3 < 0$$

and, therefore, the point, $\left(-\frac{1}{2}, -\frac{1}{4}, -\frac{65}{16}\right)$, is a saddle-point.

2. Determine the stationary points of the function,

$$z = 2x^3 + 6xy^2 - 3y^3 - 150x,$$

and determine their nature.

Solution

Following the same steps as in the previous example, we have

$$\frac{\partial z}{\partial x} = 6x^2 + 6y^2 - 150 \quad \text{and} \quad \frac{\partial z}{\partial y} = 12xy - 9y^2.$$

Hence, the stationary points occur where x and y are the solutions of the simultaneous equations,

$$\begin{aligned} x^2 + y^2 &= 25, - - - - - - - - - (1) \\ y(4x - 3y) &= 0. - - - - - - - - - - (2) \end{aligned}$$

From the second equation, $y = 0$ or $4x = 3y$.

Putting $y = 0$ in the first equation gives $x = \pm 5$ and, with these values of x and y, we obtain stationary points at $(5, 0, -500)$ and $(-5, 0, 500)$.

Putting $x = \frac{3}{4}y$ into the first equation gives $y = \pm 4$, $x = \pm 3$ and, with these values of x and y, we obtain stationary points at $(3, 4, -300)$ and $(-3, -4, 300)$.

To classify the stationary points we require

$$\frac{\partial^2 z}{\partial x^2} = 12x, \quad \frac{\partial^2 z}{\partial y^2} = 12x - 18y, \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = 12y,$$

and the conclusions are given in the following table:

Point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$	Nature
$(5, 0, -500)$	60	60	0	positive	minimum
$(-5, 0, 500)$	-60	-60	0	positive	maximum
$(3, 4, -300)$	36	-36	48	negative	saddle-point
$(-3, -4, 300)$	-36	36	-48	negative	saddle-point

Note:

The conditions used in the examples above are only **sufficient** conditions; that is, if the conditions are satisfied, we may make a conclusion. But it may be shown that there are stationary points which do **not** satisfy the conditions.

Outline proof of the sufficient conditions

From Taylor's theorem for two variables,

$$f(a+h, b+k)-f(a, b) = hf_x(a, b)+kf_y(a, b)+\frac{1}{2}\left(h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)\right)+\ldots,$$

where h and k are small compared with a and b, f_x means $\frac{\partial f}{\partial x}$, f_y means $\frac{\partial f}{\partial y}$, f_{xx} means $\frac{\partial^2 f}{\partial x^2}$, f_{yy} means $\frac{\partial^2 f}{\partial y^2}$ and f_{xy} means $\frac{\partial^2 f}{\partial x \partial y}$.

If $f_x(a, b) = 0$ and $f_y(a, b) = 0$, the conditions for a local minimum at the point $(a, b, f(a, b))$ will be satisfied when the second term on the right-hand side is positive; and the conditions for a local maximum at this point are satisfied when the second term on the right is negative.

We assume, here, that later terms of the Taylor series expansion are negligible.

Also, it may be shown that a quadratic expression of the form

$$Lh^2 + 2Mhk + Nk^2$$

is positive when $L > 0$ or $N > 0$ and $LN - M^2 > 0$; but negative when $L < 0$ or $N < 0$ and $LN - M^2 > 0$.

If it happens that $LN - M^2 < 0$, then it may be shown that the quadratic expression may take both positive and negative values.

Finally, replacing L, M and N by $f_{xx}(a, b)$, $f_{yy}(a, b)$ and $f_{xy}(a, b)$ respectively, the sufficient conditions for local maxima, local minima and saddle-points follow.

14.10.3 EXERCISES

1. Show that the function,

$$z = 3x^3 - y^3 - 4x + 3y,$$

has a local minimum value when $x = \frac{2}{3}$, $y = -1$ and calculate this minimum value. What other stationary points are there, and what is their nature ?

2. Determine the smallest value of the function,

$$z = 2x^2 + y^2 - 4x + 8y.$$

3. Show that the function,

$$z = 2x^2y^2 + x^2 + 4y^2 - 12xy,$$

has three stationary points and determine their nature.

4. Investigate the local extreme values of the function,

$$z = x^3 + y^3 + 9(x^2 + y^2) + 12xy.$$

5. Discuss the stationary points of the following functions and, where possible, determine their nature:

   (a)

$$z = x^2 - 2xy + y^2;$$

   (b)

$$z = xy.$$

Note:
It will not be possible to use all of the standard conditions; and a geometrical argument will be necessary.

14.10.4 ANSWERS TO EXERCISES

1. $\left(\frac{2}{3}, -1, -\frac{34}{9}\right)$ is a local minimum;
   $\left(-\frac{2}{3}, 1, \frac{34}{9}\right)$ is a local maximum;
   $\left(\frac{2}{3}, 1, \frac{2}{9}\right)$ is a saddle-point;
   $\left(-\frac{2}{3}, -1, -\frac{2}{9}\right)$ is a saddle-point.

2. The smallest value is -18, since there is a single local minimum at the point $(1, -4, -18)$.

3. $(0, 0, 0)$ is a saddle-point;
   $(2, 1, -8)$ is a local minimum; (**Hint:** try $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$)
   $(-2, -1, -8)$ is a local minimum.

4. $(0, 0, 0)$ is a local minimum;
   $(-10, -10, 1000)$ is a local maximum; (**Hint**: try $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$)
   $(-4, 2, 28)$ is a saddle-point;
   $(2, -4, 28)$ is a saddle-point.

5. (a) Points $(\alpha, \alpha, 0)$ are such that $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$; but $\frac{\partial^2 z}{\partial x^2}.\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$. In fact the surface is a "parabolic cylinder" which contains the straight line $x = y, \ z = 0$ and is symmetrical about the plane $x = y$.

   (b) $(0, 0, 0)$ is a saddle-point since z may have both positive and negative values in the neighbourhood of this point.

“JUST THE MATHS”

UNIT NUMBER

14.11

PARTIAL DIFFERENTIATION 11
(Constrained maxima and minima)

by

A.J.Hobson

UNIT 14.11 - PARTIAL DIFFERENTIATION 11

CONSTRAINED MAXIMA AND MINIMA

Having discussed the determination of local maxima and local minima for a function, $f(x, y, ...)$, of several independent variables, we shall now consider that an additional constraint is imposed in the form of a relationship, $g(x, y, ...) = 0$.

This would occur, for example, if we wished to construct a container with the largest possible volume for a fixed value of the surface area.

14.11.1 THE SUBSTITUTION METHOD

The following examples illustrate a technique which may be used in elementary cases:

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$f(x, y) \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

In this kind of example, it is possible to eliminate either x or y by using the constraint. If we eliminate x, for instance, we may write $f(x, y)$ as a function, $F(y)$, of y only.

In fact,

$$f(x, y) \equiv F(y) \equiv 3(1 - 2y)^2 + 2y^2 \equiv 3 - 12y + 14y^2.$$

Using the principles of maxima and minima for functions of a single independent variable, we have

$$F'(y) \equiv 28y - 12 \quad \text{and} \quad F'' \equiv 28$$

and, hence, a local minimum occurs when $y = 3/7$ and hence, $x = 1/7$.

The corresponding local minimum value of $f(x, y)$ is

$$3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{21}{49} = \frac{3}{7}.$$

2. Determine any local maxima or local minima of the function,

$$f(x, y, z) \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Eliminating x, we may write $f(x, y, z)$ as a function, $F(y, z)$, of y and z only.
In fact,

$$f(x, y, z) \equiv F(y, z) \equiv (1 - 2y - 3z)^2 + y^2 + z^2.$$

That is,

$$F(y, z) \equiv 1 - 4y - 6z + 12yz + 5y^2 + 10z^2.$$

Using the principles of maxima and minima for functions of two independent variables we have,

$$\frac{\partial F}{\partial y} \equiv -4 + 12z + 10y \text{ and } \frac{\partial F}{\partial z} \equiv -6 + 12y + 20z,$$

and a stationary value will occur when these are both equal to zero.
Thus,

$$\begin{aligned} 5y + 6z &= 2, \\ 6y + 10z &= 3, \end{aligned}$$

which give $y = 1/7$ and $z = 3/14$, on solving simultaneously.
The corresponding value of x is $1/14$, which gives a stationary value, for $f(x, y, z)$, of $14/(14)^2 = \frac{1}{14}$.
Also, we have

$$\frac{\partial^2 F}{\partial y^2} \equiv 10 > 0, \quad \frac{\partial^2 F}{\partial z^2} \equiv 20 > 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial y \partial z} \equiv 12,$$

which means that

$$\frac{\partial^2 F}{\partial y^2} . \frac{\partial^2 F}{\partial z^2} - \left(\frac{\partial^2 F}{\partial y \partial z}\right)^2 = 200 - 144 > 0.$$

Hence there is a local minimum value, $\frac{1}{14}$, of $x^2+y^2+z^2$, subject to the constraint that $x+2y+3z=1$, at the point where

$$x=\frac{1}{14}, \quad y=\frac{1}{7}, \quad \text{and} \quad z=\frac{3}{14}.$$

Note:
Geometrically, this example is calculating the square of the shortest distance from the origin onto the plane whose equation is $x+2y+3z=1$.

14.11.2 THE METHOD OF LAGRANGE MULTIPLIERS

In determining the local maxima and local minima of a function, $f(x,y,...)$, subject to the constraint that $g(x,y,...)=0$, it may be inconvenient (or even impossible) to eliminate one of the variables, $x, y, ...$

An alternative method may be illustrated by means of the following steps for a function of two independent variables:

(a) Suppose that the function, $z\equiv f(x,y)$, is subject to the constraint that $g(x,y)=0$.

Then, since z is effectively a function of x only, its stationary values will be determined by the equation

$$\frac{\mathrm{d}z}{\mathrm{d}x}=0.$$

(b) From Unit 14.5 (Exercise 2), the total derivative of $z\equiv f(x,y)$ with respect to x, when x and y are not independent of each other, is given by the formula,

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(c) From the constraint that $g(x,y)=0$, the process used in **(b)** gives

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

and, hence, for all points on the surface with equation, $g(x,y)=0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

Thus, throughout the surface with equation, $g(x, y) = 0$,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial f}{\partial x} - \left(\frac{\partial f}{\partial y}\right)\frac{\left(\frac{\partial g}{\partial x}\right)}{\left(\frac{\partial g}{\partial y}\right)}.$$

(d) Stationary values of z, subject to the constraint that $g(x, y) = 0$, will, therefore, occur when

$$\frac{\partial f}{\partial x}.\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}.\frac{\partial g}{\partial x} = 0.$$

But this may be interpreted as the condition that the two equations,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0,$$

should have a common solution for λ.

(e) Suppose that

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y).$$

Then $\phi(x, y, \lambda)$ would have stationary values whenever its first order partial derivatives with respect to x, y and λ were equal to zero.

In other words,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0, \quad \text{and} \quad g(x, y) = 0.$$

Conclusion

The stationary values of the function, $z \equiv f(x, y)$, subject to the constraint that $g(x, y) = 0$, occur at the points for which the function

$$\phi(x, y, \lambda) \equiv f(x, y) + \lambda g(x, y)$$

has stationary values.

The number, λ, is called a **"Lagrange multiplier"**.

Notes:

(i) In order to determine the nature of the stationary values of z, it will usually be necessary to examine the geometrical conditions in the neighbourhood of the stationary points.

(ii) The Lagrange multiplier method may also be applied to functions of three or more independent variables.

EXAMPLES

1. Determine any local maxima or local minima of the function,

$$z \equiv 3x^2 + 2y^2,$$

subject to the constraint that $x + 2y - 1 = 0$.

Solution

Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x^2 + 2y^2 + \lambda(x + 2y - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 6x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 4y + 2\lambda \;\text{ and }\; \frac{\partial \phi}{\partial \lambda} \equiv x + 2y - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 6x + \lambda &= 0, \\ 2y + \lambda &= 0. \end{aligned}$$

Eliminating λ shows that $6x - 2y = 0$, or $y = 3x$; and, if we substitute this into the constraint, we obtain $7x - 1 = 0$.

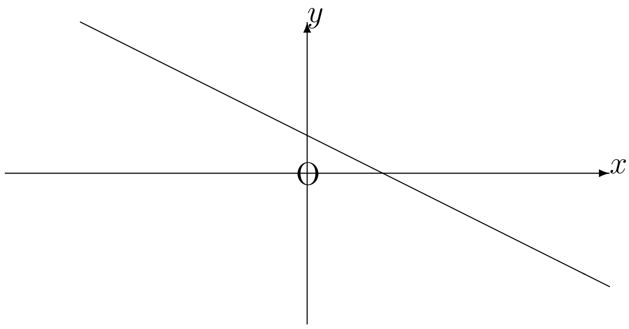
Hence,

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad \lambda = -\frac{6}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{7}, \quad y = \frac{3}{7} \quad \text{and} \quad z = 3\left(\frac{1}{7}\right)^2 + 2\left(\frac{3}{7}\right)^2 = \frac{3}{7}.$$

Finally, the geometrical conditions imply that the stationary value of z occurs at a point on the straight line whose equation is $x + 2y - 1 = 0$.



The stationary point is, in fact, a **minimum** value of z, since the function, $3x^2 + 2y^2$, has values larger than $3/7 \simeq 0.429$ at any point either side of the point, $(1/7, 3/7) = (0.14, 0.43)$, on the line whose equation is $x + 2y - 1 = 0$.

For example, at the points, $0.12, 0.44)$ and $(0.16, 0.42)$, on the line, the values of z are 0.4304 and 0.4296, respectively.

2. Determine the maximum and minimum values of the function, $z \equiv 3x + 4y$, subject to the constraint that $x^2 + y^2 = 1$.

   Solution

   Firstly, we write

$$\phi(x, y, \lambda) \equiv 3x + 4y + \lambda(x^2 + y^2 - 1).$$

Then,

$$\frac{\partial\phi}{\partial x} \equiv 3 + 2\lambda x, \quad \frac{\partial\phi}{\partial y} \equiv 4 + 2\lambda y \quad \text{and} \quad \frac{\partial\phi}{\partial\lambda} \equiv x^2 + y^2 - 1.$$

The third of these is already equal to zero; but we equate the first two to zero, giving

$$\begin{aligned} 3 + 2\lambda x &= 0, \\ 2 + \lambda y &= 0. \end{aligned}$$

Thus,

$$x = -\frac{3}{2\lambda} \quad \text{and} \quad y = \frac{2}{\lambda},$$

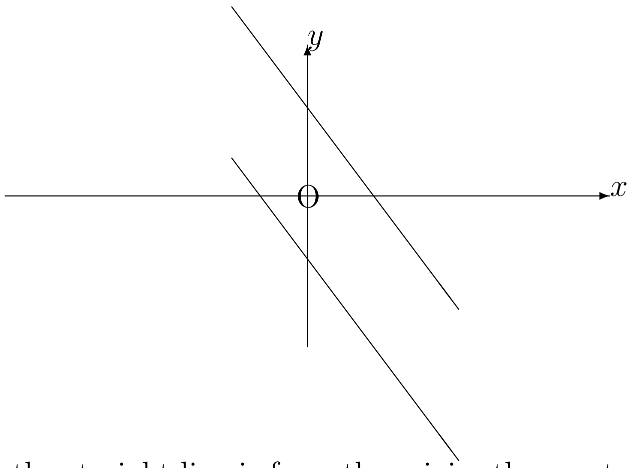
which we may substitute into the constraint to give

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1.$$

That is,

$$9 + 16 = 4\lambda^2 \quad \text{and} \quad \text{hence} \quad \lambda = \pm\frac{5}{2}.$$

We may deduce that $x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$, giving stationary values, ± 5, of z.
Finally, the geometrical conditions suggest that we consider a straight line with equation $3x + 4y = c$ (a constant) moving across the circle with equation $x^2 + y^2 = 1$.



The further the straight line is from the origin, the greater is the value of the constant, c.

The maximum and minimum values of $3x+4y$, subject to the constraint that $x^2+y^2=1$ will occur where the straight line touches the circle; and we have shown that these are the points, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. Determine any local maxima or local minima of the function,

$$w \equiv x^2 + y^2 + z^2,$$

subject to the constraint that $x + 2y + 3z = 1$.

Solution

Firstly, we write

$$\phi(x, y, z, \lambda) \equiv x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 1).$$

Then,

$$\frac{\partial \phi}{\partial x} \equiv 2x + \lambda, \quad \frac{\partial \phi}{\partial y} \equiv 2y + 2\lambda, \quad \frac{\partial \phi}{\partial z} \equiv 2z + 3\lambda \text{ and } \frac{\partial \phi}{\partial \lambda} \equiv x + 2y + 3z - 1.$$

The fourth of these is already equal to zero; but we equate the first three to zero, giving

$$\begin{aligned} 2x + \lambda &= 0, \\ y + \lambda &= 0, \\ 2z + 3\lambda &= 0. \end{aligned}$$

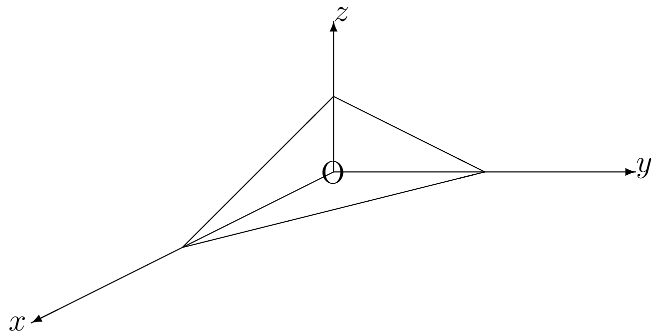
Eliminating λ shows that $2x - y = 0$, or $y = 2x$, and $6x - 2z = 0$, or $z = 3x$. Substituting these into the constraint gives $14x = 1$.
Hence,

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \quad \text{and} \quad \lambda = -\frac{1}{7}.$$

A single stationary point therefore occurs at the point where

$$x = \frac{1}{14}, \quad y = \frac{1}{7}, \quad z = \frac{3}{14} \quad \text{and} \quad w = \left(\frac{1}{14}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{3}{14}\right)^2 = \frac{1}{14}.$$

Finally, the geometrical conditions imply that the stationary value of w occurs at a point on the plane whose equation is $x + 2y + 3z = 1$.



The stationary point must give a **minimum** value of w since the function, $x^2+y^2+z^2$, represents the square of the distance of a point, (x, y, z), from the origin; and, if the point is constrained to lie on a plane, this distance is bound to have a minimum value.

14.11.3 EXERCISES

1. In the following exercises, use both the substitution method and the Lagrange multiplier method:

   (a) Determine the minimum value of the function,

   $$z \equiv x^2 + y^2,$$

   subject to the constraint that $x + y = 1$.

   (b) Determine the maximum value of the function,

   $$z \equiv xy,$$

   subject to the constraint that $x + y = 15$.

(c) Determine the maximum value of the function,

$$z \equiv x^2 + 3xy - 5y^2,$$

subject to the constraint that $2x + 3y = 6$.

2. In the following exercises, use the Lagrange multiplier method:

   (a) Determine the maximum and minimum values of the function,

   $$w \equiv x - 2y + 5z,$$

   subject to the constraint that $x^2 + y^2 + z^2 = 30$.

   (b) If $x > 0$, $y > 0$ and $z > 0$, determine the maximum value of the function,

   $$w \equiv xyz,$$

   subject to the constraint that $x + y + z^2 = 16$.

   (c) Determine the maximum value of the function,

   $$w \equiv 8x^2 + 4yz - 16z + 600,$$

   subject to the constraint that $4x^2 + y^2 + 4z^2 = 16$.

14.11.4 ANSWERS TO EXERCISES

1. (a) The minimum value is $z = 1/2$, and occurs when $x = y = 1/2$;

   (b) The maximum value is $z \simeq 56.25$, and occurs when $x = y = 15/2$;

   (c) The maximum value is $z = 9$, and occurs when $x = 3$ and $y = 0$.

2. (a) The maximum value is 30, and occurs when $x = 1$, $y = -2$ and $z = 5$;
   The minimum value is -30, and occurs when $x = -1$, $y = 2$ and $z = -5$;

   (b) The maximum value is

   $$\frac{4096}{25\sqrt{5}} \simeq 73.27,$$

   and occurs when $x = 32/\sqrt{5}$, $y = 32/\sqrt{5}$ and $z = 4/\sqrt{5}$;

   (c) The maximum value is approximately 613.86, and occurs when $x = 0$, $y = -2$ and $z = \sqrt{3}$.

"JUST THE MATHS"

UNIT NUMBER

14.12

PARTIAL DIFFERENTIATION 12
(The principle of least squares)

by

A.J.Hobson

UNIT 14.12 - PARTIAL DIFFERENTIATION 12

THE PRINCIPLE OF LEAST SQUARES

14.12.1 THE NORMAL EQUATIONS

Suppose two variables, x and y, are known to obey a **"straight line law"**, of the form $y = a + bx$, where a and b are constants to be found.

Suppose also that, in an experiment to test this law, we obtain n pairs of values, (x_i, y_i), where i = 1,2,3,...,n.

If the values x_i are **assigned** values, they are likely to be free from error, whereas the **observed** values, y_i, will be subject to experimental error.

The principle underlying the straight line of **"best fit"** is that, in its most likely position, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

The Calculation

The y-deviation, ϵ_i, of the point, (x_i, y_i), is given by

$$\epsilon_i = y_i - (a + bx_i).$$

Hence,

$$\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} [y_i - (a + bx_i]^2 = P \ \text{ say.}$$

Regarding P as a function of a and b, it will be a minimum when

$$\frac{\partial P}{\partial a} = 0, \ \ \frac{\partial P}{\partial b} = 0, \ \ \frac{\partial^2 P}{\partial a^2} > 0 \ \text{ or } \ \frac{\partial^2 P}{\partial b^2} > 0, \ \text{ and } \ \frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

For these conditions, we have

$$\frac{\partial P}{\partial a} = -2\sum_{i=1}^{n} [y_i - (a + bx_i] \ \text{ and } \ \frac{\partial P}{\partial b} = -2\sum_{i=1}^{n} x_i[y_i + bx_i],$$

and these will be zero when

$$\sum_{i=1}^{n} [y_i - (a + bx_i] = 0 \quad - - - (1)$$

and

$$\sum_{i=1}^{n} x_i[y_i + bx_i] = 0 \quad - - - (2).$$

From (1),

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - \sum_{i=1}^{n} bx_i = 0.$$

That is,

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad - - - (3).$$

From (2),

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \quad - - - (4).$$

The statements (3) and (4) are two simultaneous equations which may be solved for a and b.

They are called the **"normal equations"**

A simpler notation for the normal equations is

$$\Sigma\, y = na + b\Sigma\, x;$$

$$\Sigma xy = a\Sigma x + b\Sigma\, x^2.$$

By eliminating a and b in turn, we obtain the solutions

$$a = \frac{\Sigma\ x^2.\Sigma\ y - \Sigma x.\Sigma\ xy}{n\Sigma\ x^2 - (\Sigma\ x)^2} \quad \text{and} \quad b = \frac{n\Sigma\ xy - \Sigma\ x.\Sigma\ y}{n\Sigma\ x^2 - (\Sigma\ x)^2}.$$

With these values of a and b, the straight line with equation, $y = a + bx$, is called the **"regression line of y on x"**.

Note:
To verify that the y-deviations from the regression line have indeed been minimised, we also need the results that

$$\frac{\partial^2 P}{\partial a^2} = \sum_{i=1}^{n} 2 = 2n, \quad \frac{\partial^2 P}{\partial b^2} = \sum_{i=1}^{n} 2x_i^2, \quad \text{and} \quad \frac{\partial^2 P}{\partial a \partial b} = \sum_{i=1}^{n} 2x_i.$$

The first two of these are clearly positive; and it may be shown that

$$\frac{\partial^2 P}{\partial a^2}.\frac{\partial^2 P}{\partial b^2} - \left(\frac{\partial^2 P}{\partial a \partial b}\right)^2 > 0.$$

EXAMPLE

Determine the equation of the regression line of y on x for the following data, which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x thus has equation $y = a + bx$ where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus,

$$y = 0.176x - 0.645$$

14.12.2 SIMPLIFIED CALCULATION OF REGRESSION LINES

A simpler method of determining the regression line of y on x for a given set of data, is to consider a temporary change of origin to the point $(\overline{x}, \overline{y})$, where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\ y}{n} = a + b\frac{\Sigma\ x}{n}.$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y, is associated with the formulae

$$X = x - \overline{x} \quad \text{and} \quad Y = y - \overline{y}$$

and, in this system of reference, the regression line will pass through the origin.

Its equation is therefore

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\ X = \Sigma\ (x - \overline{x}) = \Sigma\ x - \Sigma\ \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\ Y = \Sigma\ (y - \overline{y}) = \Sigma\ y - \Sigma\ \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\ XY}{\Sigma\ X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x});$$

though, there may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	-0.5	-0.84	0.42	0.25
42	6.30	-3.5	-1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	-3.5	-0.38	1.33	12.25
35	5.90	-10.5	-1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	-5.5	-1.17	6.435	30.25
39	6.55	-6.5	-0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5).$$

That is,

$$y = 0.176x - 0.638$$

14.12.3 EXERCISES

1. For the following tables, determine the regression line of y on x, assuming that $y = a + bx$.

(a)

x	0	2	3	5	6
y	6	-1	-3	-10	-16

(b)

x	0	20	40	60	80
y	54	65	75	85	96

(c)

x	1	3	5	10	12
y	58	55	40	37	22

2. To determine the relation between the normal stress and the shear resistance of soil, a shear-box experiment was performed, giving the following results:

Normal Stress, x p.s.i.	11	13	15	17	19	21
Shear Stress, y p.s.i.	15.2	17.7	19.3	21.5	23.9	25.4

   If $y = a + bx$, determine the regression line of y on x.

3. Fuel consumption, y miles per gallon, at speeds of x miles per hour, is given by the following table:

x	20	30	40	50	60	70	80	90
y	18.3	18.8	19.1	19.3	19.5	19.7	19.8	20.0

   Assuming that

$$y = a + \frac{b}{x},$$

   determine the most probable values of a and b.

14.12.4 ANSWERS TO EXERCISES

1. (a)

$$y = 6.46 - 3.52x;$$

   (b)

$$y = 54.20 + 0.52x;$$

   (c)

$$y = 60.78 - 2.97x.$$

2.

$$y = 4.09 + 1.03x.$$

3.

$$a \simeq -42 \quad \text{and} \quad b \simeq 20.$$

"JUST THE MATHS"

UNIT NUMBER

15.1

ORDINARY DIFFERENTIAL EQUATIONS 1 (First order equations (A))

by

A.J.Hobson

UNIT 15.1 - ORDINARY DIFFERENTIAL EQUATIONS 1

FIRST ORDER EQUATIONS (A)

15.1.1 INTRODUCTION AND DEFINITIONS

1. An **ordinary differential equation** is a relationship between an independent variable (such as x), a dependent variable (such as y) and one or more ordinary derivatives of y with respect to x.

   There is no discussion, in Units 15, of **partial** differential equations, which involve partial derivatives (see Units 14). Hence, in what follows, we shall refer simply to "differential equations".

   For example,

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-2x}, \quad x\frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

   are differential equations.

2. The **"order"** of a differential equation is the order of the highest derivative which appears in it.

3. The **"general solution"** of a differential equation is the most general algebraic relationship between the dependent and independent variables which satisfies the differential equation.

   Such a solution will not contain any derivatives; but we shall see that it will contain one or more arbitrary constants (the number of these constants being equal to the order of the equation). The solution need not be an explicit formula for one of the variables in terms of the other.

4. A **"boundary condition"** is a numerical condition which must be obeyed by the solution. It usually amounts to the substitution of particular values of the dependent and independent variables into the general solution.

5. An **"initial condition"** is a boundary condition in which the independent variable takes the value zero.

6. A **"particular solution"** (or **"particular integral"**) is a solution which contains no arbitrary constants.

   Particular solutions are usually the result of appplying a boundary condition to a general solution.

15.1.2 EXACT EQUATIONS

The simplest kind of differential equation of the first order is one which has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

It is an elementary example of an **"exact differential equation"** because, to find its solution, all that it is necessary to do is integrate both sides with respect to x.

In other cases of exact differential equations, the terms which are not just functions of the independent variable only, need to be recognised as the exact derivative with respect to x of some known function (possibly involving both of the variables).

The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 5,$$

subject to the boundary condition that $y = 2$ when $x = 1$.

Solution

By direct integration, the general solution is

$$y = x^3 - 3x^2 + 5x + C,$$

where C is an arbitrary constant.

From the boundary condition,

$$2 = 1 - 3 + 5 + C, \text{ so that } C = -1.$$

Thus the particular solution obeying the given boundary condition is

$$y = x^3 - 3x^2 + 5x - 1.$$

2. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3,$$

subject to the boundary condition that $y = 4$ when $x = 2$.

Solution

The left hand side of the differential equation may be recognised as the exact derivative with respect to x of the function xy.

Hence, we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3;$$

and, by direct integration, this gives

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

That is,

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Applying the boundary condition,

$$4 = 2 + \frac{C}{2},$$

which implies that $C = 4$ and the particular solution is

$$y = \frac{x^3}{4} + \frac{4}{x}.$$

3. Determine the general solution to the differential equation

$$\sin x + \sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Solution

The second and third terms on the right hand side may be recognised as the exact derivative of the function $x\sin y$; and, hence, we may write

$$\sin x + \frac{\mathrm{d}}{\mathrm{d}x}(x \sin y) = 0.$$

By direct integration, we obtain

$$-\cos x + x \sin y = C,$$

where C is an arbitrary constant.

This result counts as the general solution without further modification; but an explicit formula for y in terms of x may, in this case, be written in the form

$$y = \mathrm{Sin}^{-1}\left[\frac{C + \cos x}{x}\right].$$

15.1.3 THE METHOD OF SEPARATION OF THE VARIABLES

The method of this section relates to differential equations of the first order which may be written in the form

$$P(y)\frac{\mathrm{d}y}{\mathrm{d}x} = Q(x).$$

Integrating both sides with respect to x gives

$$\int P(y)\frac{\mathrm{d}y}{\mathrm{d}x}\ \mathrm{d}x = \int Q(x)\ \mathrm{d}x.$$

But, from the formula for integration by substitution in Units 12.3 and 12.4, this simplifies to

$$\int P(y)\ \mathrm{d}y = \int Q(x)\ \mathrm{d}x.$$

Note:
The way to remember this result is to treat $\mathrm{d}x$ and $\mathrm{d}y$, in the given differential equation, as if they were separate numbers; then rearrange the equation so that one side contains only y while the other side contains only x; that is, we **separate the variables**. The process is completed by putting an integral sign in front of each side.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

subject to the boundary condition that $y = 6$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x};$$

and, hence,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln y = \ln x + C.$$

Applying the boundary condition,

$$\ln 6 = \ln 2 + C,$$

so that

$$C = \ln 6 - \ln 2 = \ln\left(\frac{6}{2}\right) = \ln 3.$$

The particular solution is therefore

$$\ln y = \ln x + \ln 3 \quad \text{or} \quad y = 3x.$$

Note:

In a general solution where most of the terms are logarithms, the calculation can be made simpler by regarding the arbitrary constant itself as a logarithm, calling it $\ln A$, for instance, rather than C. In the above example, we would then write

$$\ln y = \ln x + \ln A \quad \text{simplifying} \quad \text{to} \quad y = Ax.$$

On applying the boundary condition, $6 = 2A$, so that $A = 3$ and the particular solution is the same as before.

2. Solve the differential equation

$$x(4-x)\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0,$$

subject to the boundary condition that $y = 7$ when $x = 2$.

Solution

The differential equation may be rearranged as

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x(4-x)}.$$

Hence,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \frac{1}{x(4-x)}\ \mathrm{d}x;$$

or, using the theory of partial fractions,

$$\int \frac{1}{y}\ \mathrm{d}y = \int \left[\frac{\frac{1}{4}}{x} + \frac{\frac{1}{4}}{4-x}\right]\ \mathrm{d}x.$$

The general solution is therefore

$$\ln y = \frac{1}{4}\ln x - \frac{1}{4}\ln(4-x) + \ln A$$

or

$$y = A\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

Applying the boundary condition, $7 = A$, so that the particular solution is

$$y = 7\left(\frac{x}{4-x}\right)^{\frac{1}{4}}.$$

15.1.4 EXERCISES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^5 + 3e^{-2x}.$$

2. Given that differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \sin x$$

is exact, determine its general solution.

3. Given that the differential equation

$$\tan x\frac{\mathrm{d}y}{\mathrm{d}x} + y\mathrm{sec}^2 x = \cos 2x$$

is exact, determine the particular solution for which $y = 1$ when $x = \frac{\pi}{4}$.

4. Use the method of separation of the variables to determine the general solution of each of the following differential equations:

(a)

$$\frac{\mathrm{d}x}{\mathrm{d}y} = (x-1)(x+2);$$

(b)

$$x(y-3)\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

5. Use the method of separation of the variables to solve the following differential equations subject to the given boundary condition:

(a)

$$(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y,$$

where $y = 2$ when $x = 1$;

(b)

$$x^3 + (y+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

where $y = 0$ when $x = 0$.

15.1.5 ANSWERS TO EXERCISES

1.

$$y = \frac{x^6}{6} - \frac{3e^{-2x}}{2} + C.$$

2.

$$y = \frac{C - \cos x}{x^2}.$$

3.

$$y = \frac{3}{2}\cot x - \cos^2 x.$$

4. (a)

$$y = \ln\left[A\left(\frac{x-1}{x+2}\right)^{\frac{1}{3}}\right];$$

(b)

$$y = \ln[Ax^4y^3].$$

5. (a)

$$y^3 = 4(1 + x^3);$$

(b)

$$4[1 - (y+1)^3] = 3x^4.$$

"JUST THE MATHS"

UNIT NUMBER

15.2

ORDINARY DIFFERENTIAL EQUATIONS 2 (First order equations (B))

by

A.J.Hobson

UNIT 15.2 - ORDINARY DIFFERENTIAL EQUATIONS 2

FIRST ORDER EQUATIONS (B)

15.2.1 HOMOGENEOUS EQUATIONS

A differential equation of the first order is said to be **"homogeneous"** if, on replacing x by λx and y by λy in all the parts of the equation except $\frac{\mathrm{d}y}{\mathrm{d}x}$, λ may be removed from the equation by cancelling a common factor of λ^n, for some integer n.

Note:
Some examples of homogeneous equations would be

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0, \quad \text{and} \quad 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0,$$

where, from the first of these, a factor of λ could be cancelled and, from the second, a factor of λ^2 could be cancelled.

15.2.2 THE STANDARD METHOD

It turns out that the substitution

$$\boxed{y = vx} \quad \left(\text{giving } \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right),$$

always converts a homogeneous differential equation into one in which the variables can be separated. The method will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2y,$$

subject to the condition that $y = 6$ when $x = 6$.

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$x\left(v + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = x + 2vx.$$

That is,

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v$$

or

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v.$$

On separating the variables,

$$\int \frac{1}{1+v}\ \mathrm{d}v = \int \frac{1}{x}\ \mathrm{d}x,$$

giving

$$\ln(1 + v) = \ln x + \ln A,$$

where A is an arbitrary constant.
An alternative form of this solution, without logarithms, is

$$Ax = 1 + v$$

and, substituting back $v = \frac{y}{x}$, the solution becomes

$$Ax = 1 + \frac{y}{x}$$

or

$$y = Ax^2 - x.$$

Finally, if $y = 6$ when $x = 1$, we have $6 = A - 1$ and, hence, $A = 7$.
The required particular solution is thus

$$y = 7x^2 - x.$$

2. Determine the general solution of the differential equation

$$(x + y)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x - y) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$(x + vx)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4x - vx) = 0.$$

That is,

$$(1 + v)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (4 - v) = 0$$

or

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1}.$$

On further rearrangement, we obtain

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 4}{v + 1} - v = \frac{-4 - v^2}{v + 1};$$

and, on separating the variables,

$$\int \frac{v + 1}{4 + v^2}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$$

or

$$\frac{1}{2}\int \left[\frac{2v}{4 + v^2} + \frac{2}{4 + v^2}\right]\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x.$$

Hence,

$$\frac{1}{2}\left[\ln(4 + v^2) + \tan^{-1}\frac{v}{2}\right] = -\ln x + C,$$

where C is an arbitrary constant.

Substituting back $v = \frac{y}{x}$, gives the general solution

$$\frac{1}{2}\left[\ln\left(4 + \frac{y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{2x}\right)\right] = -\ln x + C.$$

3. Determine the general solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0.$$

Solution

If $y = vx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$, so that the differential equation becomes

$$2vx^2\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2 + v^2x^2) = 0.$$

That is,

$$2v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (1 + v^2) = 0$$

or

$$2vx\frac{\mathrm{d}v}{\mathrm{d}x} = -(1 + 3v^2).$$

On separating the variables, we obtain

$$\int \frac{2v}{1 + 3v^2}\,\mathrm{d}x = -\int \frac{1}{x}\,\mathrm{d}x,$$

which gives

$$\frac{1}{3}\ln(1 + 3v^2) = -\ln x + \ln A,$$

where A is an arbitrary constant.
Hence,

$$(1 + 3v^2)^{\frac{1}{3}} = \frac{A}{x}$$

or, on substituting back $v = \frac{y}{x}$,

$$\left(\frac{x^2 + 3y^2}{x^2}\right)^{\frac{1}{3}} = Ax,$$

which can be written

$$x^2 + 3y^2 = Bx^5,$$

where $B = A^3$.

15.2.3 EXERCISES

Use the substitution $y = vx$ to solve the following differential equations subject to the given boundary condition:

1. $$(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y,$$

   where $y = 3$ when $x = -2$.

2. $$(x^2 - y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

   where $y = 5$ when $x = 0$.

3. $$x^3 + y^3 = 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x},$$

   where $y = 1$ when $x = 2$.

4. $$x(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^3,$$

   where $y = 2$ when $x = 1$.

5. $$x\frac{\mathrm{d}y}{\mathrm{d}x} - (y + \sqrt{x^2 - y^2}) = 0,$$

   where $y = 0$ when $x = 1$.

15.2.4 ANSWERS TO EXERCISES

1.

$$y^2 - xy - x^2 = 11.$$

2.

$$y = 5e^{-\frac{x^2}{2y^2}}.$$

3.

$$x^3 - 2y^3 = 3x.$$

4.

$$3x^2y = 2(y^2 - x^2).$$

5.

$$e^{\sin^{-1}\frac{y}{x}} = x.$$

"JUST THE MATHS"

UNIT NUMBER

15.3

ORDINARY DIFFERENTIAL EQUATIONS 3 (First order equations (C))

by

A.J.Hobson

UNIT 15.3 - ORDINARY DIFFERENTIAL EQUATIONS 3

FIRST ORDER EQUATIONS (C)

15.3.1 LINEAR EQUATIONS

For certain kinds of first order differential equation, it is possible to multiply the equation throughout by a suitable factor which converts it into an exact differential equation.

For instance, the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2$$

may be multiplied throughout by x to give

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3.$$

It may now be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3$$

and, hence, it has general solution

$$xy = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

Notes:

(i) The factor, x which has multiplied both sides of the differential equation serves as an **"integrating factor"**, but such factors cannot always be found by inspection.

(ii) In the discussion which follows, we shall develop a formula for determining integrating factors, in general, for what are known as **"linear differential equations"**.

DEFINITION

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

is said to be **"linear"**.

RESULT

Given the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x),$$

the function

$$e^{\int P(x) \, \mathrm{d}x}$$

is always an integrating factor; and, on multiplying the differential equation throughout by this factor, its left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{\int P(x) \, \mathrm{d}x}\right].$$

Proof

Suppose that the function, $R(x)$, is an integrating factor; then, in the equation

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + R(x)P(x)y = R(x)Q(x),$$

the left hand side must be the exact derivative of some function of x.

Using the formula for differentiating the product of two functions of x, we can **make** it the derivative of $R(x)y$ provided we can arrange that

$$R(x)P(x) = \frac{\mathrm{d}}{\mathrm{d}x}[R(x)].$$

But this requirement can be interpreted as a differential equation in which the variables $R(x)$ and x may be separated as follows:

$$\int \frac{1}{R(x)} \, \mathrm{d}R(x) = \int P(x) \, \mathrm{d}x.$$

Hence,

$$\ln R(x) = \int P(x) \, \mathrm{d}x.$$

That is,

$$R(x) = e^{\int P(x) \, \mathrm{d}x},$$

as required.

The solution is obtained by integrating the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times R(x)] = R(x)P(x).$$

Note:
There is no need to include an arbitrary constant, C, when $P(x)$ is integrated, since it would only serve to introduce a constant factor of e^C in the above result, which would then immediately cancel out on multiplying the differential equation by $R(x)$.

EXAMPLES

1. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2.$$

Solution

An integrating factor is

$$e^{\int \frac{1}{x} \, \mathrm{d}x} = e^{\ln x} = x.$$

On multiplying throughout by the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[y \times x] = x^3;$$

and so,

$$yx = \frac{x^4}{4} + C,$$

where C is an arbitrary constant.

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2e^{-x^2}.$$

Solution
An integrating factor is

$$e^{\int 2x \ \mathrm{d}x} = e^{x^2}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y \times e^{x^2}\right] = 2,$$

giving

$$ye^{x^2} = 2x + C,$$

where C is an arbitrary constant.

15.3.2 BERNOUILLI'S EQUATION

A similar type of differential equation to that in the previous section has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n.$$

It is called **"Bernouilli's Equation"** and may be converted to a linear differential equation by making the substitution

$$z = y^{1-n}.$$

Proof

The differential equation may be rewritten as

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x).$$

Also,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x).$$

That is,

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x),$$

which is a linear differential equation.

Note:
It is better not to regard this as a standard formula, but to apply the method of obtaining it in the case of particular examples.

EXAMPLES

1. Determine the general solution of the differential equation

$$xy - \frac{\mathrm{d}y}{\mathrm{d}x} = y^3e^{-x^2}.$$

Solution

The differential equation may be rewritten

$$-y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + x.y^{-2} = e^{-x^2}.$$

Substituting $z = y^{-2}$, we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$ and, hence,

$$\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} + xz = e^{-x^2}$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2e^{-x^2}.$$

An integrating factor for this equation is

$$e^{\int\ 2x\ \mathrm{d}x} = e^{x^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{x^2}\right) = 2,$$

giving

$$ze^{x^2} = 2x + C,$$

where C is an arbitrary constant.

Finally, replacing z by y^{-2},

$$y^2 = \frac{e^{x^2}}{2x + C}.$$

2. Determine the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = xy^2.$$

Solution

The differential equation may be rewritten

$$y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.y^{-1} = x.$$

On substituting $z = y^{-1}$ we obtain $\frac{\mathrm{d}z}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$ so that

$$-\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{1}{x}.z = x$$

or

$$\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{1}{x}.z = -x.$$

An integrating factor for this equation is

$$e^{\int \left(-\frac{1}{x}\right)\,\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(z \times \frac{1}{x}\right) = -1,$$

giving

$$\frac{z}{x} = -x + C,$$

where C is an arbitrary constant.

The general solution of the given differential equation is therefore

$$\frac{1}{xy} = -x + C \quad \text{or} \quad y = \frac{1}{Cx - x^2}.$$

15.3.3 EXERCISES

Use an integrating factor to solve the following differential equations subject to the given boundary condition:

1.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

where $y = 10$ when $x = 0$.

2.

$$3\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10,$$

where $y = 4$ when $x = 0$.

3.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = 3x,$$

where $y = 2$ when $x = -1$.

4.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{1-x} = 1 - x^2,$$

where $y = 0$ when $x = -1$.

5.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \cos x,$$

where $y = \frac{5}{2}$ when $x = \frac{\pi}{2}$.

6.

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = x,$$

where $y = 0$ when $x = 1$.

7.

$$3y - 2\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 e^{4x},$$

where $y = 1$ when $x = 0$.

8.

$$2y - x\frac{\mathrm{d}y}{\mathrm{d}x} = x(x-1)y^4,$$

where $y^3 = 14$ when $x = 1$.

15.3.4 ANSWERS TO EXERCISES

1.

$$y = 10e^{-\frac{2}{3}x}.$$

2.

$$y = 6e^{\frac{5}{3}x} - 2.$$

3.

$$yx = x^3 - 1.$$

4.

$$y = \frac{1}{2}(1-x)(1+x)^2.$$

5.

$$y = \frac{\sin x}{2} + \frac{2}{\sin x}.$$

6.

$$y = 1 + x^2 - \sqrt{2(1+x^2)}.$$

7.

$$y^2 = \frac{7e^{3x}}{e^{7x}+6}.$$

8.

$$y^3 = \frac{56x^6}{21x^6 - 24x^7 + 7}.$$

"JUST THE MATHS"

UNIT NUMBER

15.4

ORDINARY DIFFERENTIAL EQUATIONS 4 (Second order equations (A))

by

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UNIT 15.4 - ORDINARY DIFFERENTIAL EQUATIONS 4

SECOND ORDER EQUATIONS (A)

15.4.1 INTRODUCTION

In the discussion which follows, we shall consider a particular kind of second order ordinary differential equation which is called **"linear, with constant coefficients"**; it has the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

where a, b and c are the constant coefficients.

The various cases of solution which arise depend on the values of the coefficients, together with the type of function, $f(x)$, on the right hand side. These cases will now be dealt with in turn.

15.4.2 SECOND ORDER HOMOGENEOUS EQUATIONS

The term **"homogeneous"**, in the context of <u>second</u> order differential equations, is used to mean that the function, $f(x)$, on the right hand side is zero. It should not be confused with the previous use of this term in the context of <u>first</u> order differential equations.

We therefore consider equations of the general form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

Note:
A very simple case of this equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which, on integration twice, gives the general solution

$$y = Ax + B,$$

where A and B are arbitrary constants. We should therefore expect <u>two</u> arbitrary constants in the solution of <u>any</u> second order linear differential equation with constant coefficients.

The Standard General Solution

The equivalent of

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

in the discussion of <u>first</u> order differential equations would have been

$$b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0; \text{ that is, } \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{c}{b}y = 0$$

and this could have been solved using an integrating factor of

$$e^{\int \frac{c}{b}\,\mathrm{d}x} = e^{\frac{c}{b}x},$$

giving the general solution

$$y = Ae^{-\frac{c}{b}x},$$

where A is an arbitrary constant.

It seems reasonable, therefore, to make a trial solution of the form $y = Ae^{mx}$, where $A \neq 0$, in the second order case.

We shall need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ame^{mx} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Am^2e^{mx}.$$

Hence, on substituting the trial solution, we require that

$$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0;$$

and, by cancelling Ae^{mx}, this condition reduces to

$$am^2 + bm + c = 0,$$

a quadratic equation, called the **"auxiliary equation"**, having the same (constant) coefficients as the orginal differential equation.

In general, it will have two solutions, say $m = m_1$ and $m = m_2$, giving corresponding solutions $y = Ae^{m_1 x}$ and $y = Be^{m_2 x}$ of the differential equation.

However, the linearity of the differential equation implies that the sum of any two solutions is also a solution, so that

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

is another solution; and, since this contains two arbitrary constants, we shall take it to be the general solution.

Notes:

(i) It may be shown that there are no solutions other than those of the above form though special cases are considered later.

(ii) It will be possible to determine particular values of A and B if an appropriate number of boundary conditions for the differential equation are specified. These will usually be a set of given values for y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a certain value of x.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and also the particular solution for which $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -5$ when $x = 0$.

Solution

The auxiliary equation is $m^2 + 5m + 6 = 0$,

which can be factorised as

$$(m+2)(m+3)=0.$$

Its solutions are therefore $m=-2$ and $m=-3$.

Hence, the differential equation has general solution

$$y=Ae^{-2x}+Be^{-3x},$$

where A and B are arbitrary constants.

Applying the boundary conditions, we shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2Ae^{-2x}-3Be^{-3x}.$$

Hence,

$$\begin{aligned} 2 &= A+B, \\ -5 &= -2A-3B \end{aligned}$$

giving $A=1$, $B=1$ and a particular solution

$$y=e^{-2x}+e^{-3x}.$$

15.4.3 SPECIAL CASES OF THE AUXILIARY EQUATION

(a) The auxilary equation has coincident solutions

Suppose that both solutions of the auxiliary equation are the same number, m_1.

In other words, the quadratic expression am^2+bm+c is a **"perfect square"**, which means that it is actually $a(m-m_1)^2$.

Apparently, the general solution of the differential equation is

$$y=Ae^{m_1x}+Be^{m_1x},$$

which does not genuinely contain two arbitrary constants since it can be rewritten as

$$y = Ce^{m_1 x} \text{ where } C = A + B.$$

It will not, therefore, count as the general solution, though the fault seems to lie with the constants A and B rather than with m_1.

Consequently, let us now examine a new trial solution of the form

$$y = ze^{m_1 x},$$

where z denotes a function of x rather than a constant.

We shall also need

$$\frac{\mathrm{d}y}{\mathrm{d}x} = zm_1 e^{m_1 x} + e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = zm_1^2 e^{m_1 x} + 2m_1 e^{m_1 x}\frac{\mathrm{d}z}{\mathrm{d}x} + e^{m_1 x}\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}.$$

On substituting these into the differential equation, we obtain the condition that

$$e^{m_1 x}\left[a\left(zm_1^2 + 2m_1\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) + b\left(zm_1 + \frac{\mathrm{d}z}{\mathrm{d}x}\right) + cz\right] = 0$$

or

$$z(am_1^2 + bm_1 + c) + \frac{\mathrm{d}z}{\mathrm{d}x}(2am_1 + b) + a\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 0.$$

The first term on the left hand side of this condition is zero since m_1 is already a solution of the auxiliary equation; and the second term is also zero since the auxiliary equation, $am^2 + bm + c = 0$, is equivalent to $a(m - m_1)^2 = 0$; that is, $am^2 - 2am_1 m + am_1^2 = 0$. Thus $b = -2am_1$.

We conclude that $\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = 0$ with the result that $z = Ax + B$, where A and B are arbitrary constants.

The general solution of the differential equation in the case of coincident solutions to the auxiliary equation is therefore

$$y = (Ax + B)e^{m_1x}.$$

EXAMPLE

Determine the general solution of the differential equation

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

Solution

The auxilary equation is

$$4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m + 1)^2 = 0$$

and it has coincident solutions at $m = -\frac{1}{2}$.

The general solution is therefore

$$y = (Ax + B)e^{-\frac{1}{2}x}.$$

(b) The auxiliary equation has complex solutions

If the auxiliary equation has complex solutions, they will automatically appear as a pair of **"complex conjugates"**, say $m = \alpha \pm j\beta$.

Using these two solutions instead of the previous m_1 and m_2, the general solution of the differential equation will be

$$y = Pe^{(\alpha+j\beta)x} + Qe^{(\alpha-j\beta)x},$$

where P and Q are arbitrary constants.

But, by properties of complex numbers, a neater form of this result is obtainable as follows:

$$y = e^{\alpha x}[P(\cos\beta x + j\sin\beta x) + Q(\cos\beta x - j\sin\beta x)]$$

or

$$y = e^{\alpha x}[(P+Q)\cos\beta x + j(P-Q)\sin\beta x].$$

Replacing $P+Q$ and $j(P-Q)$ (which are just arbitrary quantities) by A and B, we obtain the standard general solution for the case in which the auxiliary equation has complex solutions. It is

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x].$$

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 13y = 0.$$

Solution

The auxiliary equation is

$$m^2 - 6m + 13 = 0,$$

which has solutions given by

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\times 13\times 1}}{2\times 1} = \frac{6 \pm j4}{2} = 3 \pm j2.$$

The general solution is therefore

$$y = e^{3x}[A\cos 2x + B\sin 2x],$$

where A and B are arbitrary constants.

15.4.4 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0;$$

   (b)

   $$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 6\frac{\mathrm{d}r}{\mathrm{d}\theta} + 9r = 0;$$

   (c)

   $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 4\frac{\mathrm{d}\theta}{\mathrm{d}t} + 5\theta = 0.$$

2. Solve the following differential equations, subject to the given boundary conditions:

   (a)

   $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0,$$

   where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$;

   (b)

   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = 0,$$

   where $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ when $t = 0$;

   (c)

   $$4\frac{\mathrm{d}^2z}{\mathrm{d}s^2} - 12\frac{\mathrm{d}z}{\mathrm{d}s} + 9z = 0,$$

   where $z = 1$ and $\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{5}{2}$ when $s = 0$;

(d)

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} - 2\frac{\mathrm{d}r}{\mathrm{d}\theta} + 2r = 0,$$

where $r = 5$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 7$ when $\theta = 0$.

15.4.5 ANSWERS TO EXERCISES

1. (a)

$$y = Ae^{-3x} + Be^{-4x};$$

(b)

$$r = (A\theta + B)e^{-3\theta};$$

(c)

$$\theta = e^{-2t}[A\cos 2t + B\sin 2t].$$

2. (a)

$$y = 3e^{x} - e^{2x};$$

(b)

$$x = 2e^{t} + e^{3t};$$

(c)

$$z = (s+1)e^{\frac{3}{2}s};$$

(d)

$$r = e^{\theta}[5\cos\theta + 2\sin\theta].$$

"JUST THE MATHS"

UNIT NUMBER

15.5

ORDINARY DIFFERENTIAL EQUATIONS 5
(Second order equations (B))

by

A.J.Hobson

UNIT 15.5 - ORDINARY DIFFERENTIAL EQUATIONS 5

SECOND ORDER EQUATIONS (B)

15.5.1 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

The following discussion will examine the solution of the second order linear differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

in which a, b and c are constants, but $f(x)$ is not identically equal to zero.

The Particular Integral and Complementary Function

(i) Suppose that $y = u(x)$ is any particular solution of the differential equation; that is, it contains no arbitrary constants. In the present context, we shall refer to such particular solutions as **"particular integrals"** and systematic methods of finding them will be discussed later.

It follows that

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu = f(x).$$

(ii) Suppose also that we make the substitution $y = u(x) + v(x)$ in the original differential equation to give

$$a\frac{\mathrm{d}^2(u+v)}{\mathrm{d}x^2} + b\frac{\mathrm{d}(u+v)}{\mathrm{d}x} + c(u+v) = f(x).$$

That is,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + b\frac{\mathrm{d}u}{\mathrm{d}x} + cu + a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = f(x);$$

and, hence,

$$a\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + b\frac{\mathrm{d}v}{\mathrm{d}x} + cv = 0.$$

This means that the function $v(x)$ is the general solution of the homogeneous differential equation whose auxiliary equation is

$$am^2 + bm + c = 0.$$

In future, $v(x)$ will be called the **"complementary function"** in the general solution of the original (non-homogeneous) differential equation. It complements the particular integral to provide the general solution.

Summary

$$\boxed{\textbf{General solution = particular integral + complementary function.}}$$

15.5.2 DETERMINATION OF SIMPLE PARTICULAR INTEGRALS

(a) Particular integrals, when $f(x)$ is a constant, k.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = k,$$

it is easy to see that a particular integral will be $y = \frac{k}{c}$, since its first and second derivatives are both zero, while $cy = k$.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 20.$$

Solution

(i) By inspection, we may observe that a particular integral is $y = 2$.

(ii) The auxiliary equation is

$$m^2 + 7m + 10 = 0 \quad \text{or} \quad (m + 2)(m + 5) = 0,$$

having solutions $m = -2$ and $m = -5$.

(iii) The complementary function is

$$Ae^{-2x} + Be^{-5x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 2 + Ae^{-2x} + Be^{-5x}.$$

(b) Particular integrals, when $f(x)$ is of the form $px + q$.

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = px + q,$$

it is possible to determine a particular integral by assuming one which has the same form as the right hand side; that is, in this case, another expression consisting of a multiple of x and constant term. The method is, again, illustrated by an example.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 84x - 5.$$

Solution

(i) First, we assume a particular integral of the form

$$y = \alpha x + \beta,$$

which implies that $\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Substituting into the differential equation, we require that

$$-11\alpha + 28(\alpha x + \beta) \equiv 84x - 5.$$

Hence, $28\alpha = 84$ and $-11\alpha + 28\beta = -5$, giving $\alpha = 3$ and $\beta = 1$.

Thus, the particular integral is

$$y = 3x + 1.$$

(ii) The auxiliary equation is

$$m^2 - 11m + 28 = 0 \quad \text{or} \quad (m-4)(m-7) = 0,$$

having solutions $m = 4$ and $m = 7$.

(iii) The complementary function is

$$Ae^{4x} + Be^{7x},$$

where A and B are arbitrary constants.

(iv) The general solution is

$$y = 3x + 1 + Ae^{4x} + Be^{7x}.$$

Note:
In examples of the above types, the complementary function must not be prefixed by "$y =$", since the given differential equation, as a whole, is not normally satisfied by the complementary function alone.

15.5.3 EXERCISES

1. Determine the general solutions of the following differential equations:

   (a)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6;$$

   (b)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 16y = 7;$$

   (c)
$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - y = x + 1;$$

   (d)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 18x + 28.$$

2. Solve, completely, the following differential equations, subject to the given boundary condidtions:

   (a)
$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 100,$$

   where $y = -26$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 5$ when $x = 0$;

   (b)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 12x + 16,$$

   where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$;

   (c)
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 10x + 14,$$

   where $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.5.4 ANSWERS TO EXERCISES

1. (a)

$$y = -3 + Ae^x + Be^{2x};$$

   (b)

$$y = \frac{7}{16} + A\cos 4x + B\sin 4x;$$

   (c)

$$y = 1 - x + Ae^x + Be^{-\frac{1}{3}x};$$

   (d)

$$y = 2x + 5 + (Ax + B)e^{3x}.$$

2. (a)

$$y = -25 + e^{4x} - 2e^{\frac{1}{2}x};$$

   (b)

$$y = 3x + 1 - (x + 1)e^{-2x};$$

   (c)

$$y = x + 2 + e^{3x}(\cos x - 2\sin x).$$

"JUST THE MATHS"

UNIT NUMBER

15.6

ORDINARY DIFFERENTIAL EQUATIONS 6 (Second order equations (C))

by

A.J.Hobson

UNIT 15.6 - ORDINARY DIFFERENTIAL EQUATIONS 6

SECOND ORDER EQUATIONS (C)

15.6.1 RECAP

For the differential equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x),$$

it was seen, in Unit 15.5, that

(a) when $f(x) \equiv k$, a given **constant**, a particular integral is $y = \frac{k}{c}$;

(b) when $f(x) \equiv px + q$, a **linear** function in which p and q are given constants, it is possible to obtain a particular integral by assuming that y also has the form of a linear function; that is, we make a **"trial solution"**, $y = \alpha x + \beta$.

15.6.2 FURTHER TYPES OF PARTICULAR INTEGRAL

We now examine particular integrals for other cases of $f(x)$, the method being illustrated by examples. Also, for reasons relating to certain problematic cases discussed in Unit 15.7, we shall determine the complementary function **before** determining the particular integral.

1. $f(x) \equiv px^2 + qx + r$, a **quadratic** function in which p, q and r are given constants; $p \neq 0$.

$$\text{Trial solution :} \quad y = \alpha x^2 + \beta x + \gamma.$$

Note:
This is the trial solution even if q or r (or both) are zero.

EXAMPLE

Determine the general solution of the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 4x^2 + 10x - 23.$$

Solution

The auxilary equation is

$$2m^2 - 7m - 4 = 0 \ \text{ or } \ (2m+1)(m-4) = 0,$$

having solutions $m = 4$ and $m = -\frac{1}{2}$.
Thus, the complementary function is

$$Ae^{4x} + Be^{-\frac{1}{2}x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form $y = \alpha x^2 + \beta x + \gamma$, giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x + \beta$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha$.
We thus require that

$$4\alpha - 14\alpha x - 7\beta - 4\alpha x^2 - 4\beta x - 4\gamma \equiv 4x^2 + 10x - 23.$$

That is,

$$-4\alpha x^2 - (14\alpha + 4\beta)x + 4\alpha - 7\beta - 4\gamma \equiv 4x^2 + 10x - 23.$$

Comparing corresponding coefficents on both sides, this means that

$$-4\alpha = 4, \quad -(14\alpha + 4\beta) = 10 \ \text{ and } \ 4\alpha - 7\beta - 4\gamma = -23,$$

which give $\alpha = -1$, $\beta = 1$ and $\gamma = 3$.
Hence, the particular integral is

$$y = 3 + x - x^2.$$

Finally, the general solution is

$$y = 3 + x - x^2 + Ae^{4x} + Be^{-\frac{1}{2}x}.$$

2. $f(x) \equiv p\sin kx + q\cos kx$, a **trigonometric** function in which p, q and k are given constants.

$$\text{Trial solution :} \quad y = \alpha\sin kx + \beta\cos kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 8\cos 3x - 19\sin 3x.$$

Solution

The auxiliary equation is

$$m^2 - 2m + 2 = 0,$$

which has complex number solutions given by

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j.$$

Hence, the complementary function is

$$e^x(A\cos x + B\sin x),$$

where A and B are arbitrary constants.

To determine a particular integral, we may make a trial solution of the form

$$y = \alpha\sin 3x + \beta\cos 3x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha\cos 3x - 3\beta\sin 3x$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\alpha\sin 3x - 9\beta\cos 3x$.

We thus require that

$$-9\alpha\sin 3x - 9\beta\cos 3x - 6\alpha\cos 3x + 6\beta\sin 3x + 2\alpha\sin 3x + 2\beta\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

That is,

$$(-9\alpha + 6\beta + 2\alpha)\sin 3x + (-9\beta - 6\alpha + 2\beta)\cos 3x \equiv 8\cos 3x - 19\sin 3x.$$

Comparing corresponding coefficients on both sides, we have

$$\begin{aligned} -7\alpha + 6\beta &= -19, \\ -6\alpha - 7\beta &= 8. \end{aligned}$$

These equations are satisfied by $\alpha = 1$ and $\beta = -2$, so that the particular integral is

$$y = \sin 3x - 2\cos 3x.$$

Finally, the general solution is

$$y = \sin 3x - 2\cos 3x + e^x(A\cos x + B\sin x).$$

3. $f(x) \equiv pe^{kx}$, an **exponential** function in which p and k are given constants.

$$\text{Trial solution :} \quad y = \alpha e^{kx}.$$

EXAMPLE

Determine the general solution of the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y = 50e^{3x}.$$

Solution

The auxiliary equation is

$$9m^2 + 6m + 1 = 0 \quad \text{or} \quad (3m+1)^2 = 0,$$

which has coincident solutions at $m = -\frac{1}{3}$.
The complementary function is therefore

$$(Ax + B)e^{-\frac{1}{3}x}.$$

To find a particular integral, we may make a trial solution of the form

$$y = \alpha e^{3x},$$

which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\alpha e^{3x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 9\alpha e^{3x}$.
Hence, on substituting into the differential equation, it is necessary that

$$81\alpha e^{3x} + 18\alpha e^{3x} + \alpha e^{3x} = 50e^{3x}.$$

That is, $100\alpha = 50$, from which we deduce that $\alpha = \frac{1}{2}$ and a particular integral is

$$y = \frac{1}{2}e^{3x}.$$

Finally, the general solution is

$$y = \frac{1}{2}e^{3x} + (Ax + B)e^{-\frac{1}{3}x}.$$

4. $f(x) \equiv p \sinh kx + q \cosh kx$, a **hyperbolic** function in which p, q and k are given constants.

$$\text{Trial solution : } \quad y = \alpha \sinh kx + \beta \cosh kx.$$

Note:
This is the trial solution even if p or q is zero.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 93 \cosh 5x - 75 \sinh 5x.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \quad \text{or} \quad (m-2)(m-3) = 0,$$

which has solutions $m = 2$ and $m = 3$ so that the complementary function is

$$Ae^{2x} + Be^{3x},$$

where A and B are arbitrary constants.
To determine a particular integral, we may make a trial solution of the form

$$y = \alpha \sinh 5x + \beta \cosh 5x,$$

giving $\frac{\mathrm{d}y}{\mathrm{d}x} = 5\alpha \cosh 5x + 5\beta \sinh 5x$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 25\alpha \sinh 5x + 25\beta \cosh 5x$.
Substituting into the differential equation, the left-hand-side becomes

$$25\alpha \sinh 5x + 25\beta \cosh 5x - 25\alpha \cosh 5x - 25\beta \sinh 5x + 6\alpha \sinh 5x + 6\beta \cosh 5x.$$

This simplifies to

$$(31\alpha - 25\beta) \sinh 5x + (31\beta - 25\alpha) \cosh 5x,$$

so that we require

$$\begin{aligned} 31\alpha - 25\beta &= -75, \\ -25\alpha + 31\beta &= 93, \end{aligned}$$

and these are satisfied by $\alpha = 0$ and $\beta = 3$.
The particular integral is thus

$$y = 3\cosh 5x$$

and, hence, the general solution is

$$y = 3\cosh 5x + Ae^{2x} + Be^{3x}.$$

5. Combinations of Different Types of Function

In cases where $f(x)$ is the sum of two or more of the various types of function discussed previously, then the particular integrals for each type (determined separately) may be added together to give an overall particular integral.

15.6.3 EXERCISES

1. Determine the general solution for each of the following differential equations:

(a)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 4x^2 + 2x - 4;$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\cos 2x - \sin 2x;$$

(c)

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 12\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 27e^{-x};$$

(d)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} - 10y = \cosh 3x - \sinh 3x.$$

2. Solve completely the following differential equations subject to the given boundary conditions:

(a)

$$\frac{d^2y}{dx^2} - y = 10 - 5x^2 - x + 16e^{-3x},$$

where $y = 13$ and $\frac{dy}{dx} = -2$ when $x = 0$;

(b)

$$4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 3y = 9x + 6\cos x - 19\sin x,$$

where $y = -2$ and $\frac{dy}{dx} = 0$ when $x = 0$.

15.6.4 ANSWERS TO EXERCISES

1. (a)

$$y = x^2 - 2x + 1 + Ae^{-x} + Be^{-4x};$$

(b)

$$y = \sin 2x + e^{-2x}(A\cos x + B\sin x);$$

(c)

$$y = 3e^{-x} + (Ax + B)e^{-\frac{3}{2}x};$$

(d)

$$y = \frac{1}{8}(\cosh 3x - \sinh 3x) + Ae{-}2x + Be^{5x}.$$

2. (a)

$$y = 5x^2 + x + 2e^{-3x} + 3e^{x} - 2e^{-x};$$

(b)

$$y = 3x - 8 + 2\cos x + \sin x + 2e^{-\frac{1}{2}x} + 2e^{-\frac{3}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.7

ORDINARY DIFFERENTIAL EQUATIONS 7 (Second order equations (D))

by

A.J.Hobson

UNIT 15.7 - ORDINARY DIFFERENTIAL EQUATIONS 7

SECOND ORDER EQUATIONS (D)

15.7.1 PROBLEMATIC CASES OF PARTICULAR INTEGRALS

Difficulties can arise if all or part of any trial solution would already be included in the complementary function. We illustrate with some examples:

EXAMPLES

1. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{2x}.$$

Solution

The auxiliary equation is $m^2 - 3m + 2 = 0$, with solutions $m = 1$ and $m = 2$ and hence the complementary function is $Ae^x + Be^{2x}$, where A and B are arbitrary constants.

A trial solution of $y = \alpha e^{2x}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha e^{2x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\alpha e^{2x}$$

and, on substituting these into the differential equation, it is necessary that

$$4\alpha e^{2x} - 6\alpha e^{2x} + 2\alpha e^{2x} \equiv e^{2x}.$$

That is, $0 \equiv e^{2x}$ which is impossible.

However, if $y = \alpha e^{2x}$ has proved to be unsatisfactory, let us investigate, as an alternative, $y = F(x)e^{2x}$ (where $F(x)$ is a function of x instead of a constant).

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2F(x)e^{2x} + F'(x)e^{2x}$$

and, hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4F(x)e^{2x} + 2F'(x)e^{2x} + F''(x)e^{2x} + 2F'(x)e^{2x}.$$

On substituting these into the differential equation, it is necessary that

$$(4F(x) + 2F'(x) + F''(x) + 2F'(x) - 6F(x) - 3F'(x) + 2F(x))\, e^{2x} \equiv e^{2x}.$$

That is,

$$F''(x) + F'(x) = 1,$$

which is satisfied by the function $F(x) \equiv x$ and thus a suitable particular integral is

$$y = xe^{2x}.$$

Note:
It may be shown, in other cases too that, if the standard trial solution is already contained in the complementary function, then it is is necessary to multiply it by x in order to obtain a suitable particular integral.

2. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \sin x.$$

Solution
The auxilary equation is $m^2 + 1 = 0$, with solutions $m = \pm j$ and, hence, the complementary function is $A\sin x + B\cos x$, where A and B are arbitrary constants.
A trial solution of $y = \alpha \sin x + \beta \cos x$ gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\alpha \sin x - \beta \cos x;$$

and, on substituting into the differential equation, it is necessary that $0 \equiv \sin x$, which is impossible.
Here, we may try $y = x(\alpha \sin x + \beta \cos x)$, giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha \sin x + \beta \cos x + x(\alpha \cos x - \beta \sin x) = (\alpha - \beta x)\sin x + (\beta + \alpha x)\cos x$$

and, therefore,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\alpha - \beta x)\cos x - \beta \sin x - (\beta + \alpha x)\sin x + \alpha \cos x = (2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x.$$

Substituting into the differential equation, we thus require that

$$(2\alpha - \beta x)\cos x - (2\beta + \alpha x)\sin x + x(\alpha \sin x + \beta \cos x) \equiv \sin x,$$

which simplifies to

$$2\alpha \cos x - 2\beta \sin x \equiv \sin x.$$

Thus $2\alpha = 0$ and $-2\beta = 1$.

An appropriate particular integral is now

$$y = -\frac{1}{2}x\cos x.$$

3. Determine the complementary function and a particular integral for the differential equation

$$9\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + y == 50e^{-\frac{1}{3}x}.$$

Solution

The auxiliary equation is $9m^2 + 6m + 1 = 0$, or $(3m+1)^2 = 0$, which has coincident solutions $m = -\frac{1}{3}$ and so the complementary function is

$$(Ax + B)e^{-\frac{1}{3}x}.$$

In this example, both $e^{-\frac{1}{3}x}$ **and** $xe^{-\frac{1}{3}x}$ are contained in the complementary function.

Thus, in the trial solution, it is necessary to multiply by a **further** x, giving

$$y = \alpha x^2 e^{-\frac{1}{3}x}.$$

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\alpha x e^{-\frac{1}{3}x} - \frac{1}{3}x^2 e^{\frac{1}{3}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} - \frac{2}{3}\alpha x e^{-\frac{1}{3}x} + \frac{1}{9}\alpha x^2 e^{-\frac{1}{3}x}.$$

Substituting these into the differential equation, it is necessary that

$$\left(18\alpha - 12\alpha x + \alpha x^2 + 12\alpha x - 2\alpha x^2 + \alpha x^2\right) e^{-\frac{1}{3}x} = 50e^{-\frac{1}{3}x}$$

and, hence, $18\alpha = 50$ or $\alpha = \frac{25}{9}$.
An appropriate particular integral is

$$y = \frac{25}{9}x^2 e^{-\frac{1}{3}x}.$$

4. Determine the complementary function and a particular integral for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = \sinh 2x.$$

Solution

The auxiliary equation is $m^2 - 5m + 6 = 0$ or $(m-2)(m-3) = 0$ which has solutions $m = 2$ and $m = 3$ and, hence, the complementary function is

$$Ae^{2x} + Be^{3x}.$$

However, since $\sinh 2x \equiv \frac{1}{2}(e^{2x} - e^{-2x})$, **part** of it is contained in the complementary function and we must find a particular integral for each part separately.
(a) For $\frac{1}{2}e^{2x}$, we may try

$$y = x\alpha e^{2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \alpha e^{2x} + 2x\alpha e^{2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\alpha e^{2x} + 2\alpha e^{2x} + 4x\alpha e^{2x}.$$

Substituting these into the differential equation, it is necessary that

$$\left(4\alpha + 4x\alpha - 5\alpha - 10x\alpha + 6x\alpha\right) e^{2x} \equiv \frac{1}{2}e^{2x},$$

which gives $\alpha = -\frac{1}{2}$.

(b) For $-\frac{1}{2}e^{-2x}$, we may try

$$y = \beta e^{-2x},$$

giving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\beta e^{-2x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4\beta e^{-2x}.$$

Substituting these into the differential equation, it is necessary that

$$(4\beta + 10\beta + 6\beta)e^{-2x} \equiv -\frac{1}{2}e^{-2x},$$

which gives $\beta = -\frac{1}{40}$.
The overall particular integral is thus

$$y = -\frac{1}{2}xe^{2x} - \frac{1}{40}e^{-2x}.$$

15.7.2 EXERCISES

Solve completely the following differential equations subject to the given boundary conditions:

1.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-x},$$
where $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2}$ when $x = 0$.

2.
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 9y = 2\sin 3x,$$
where $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8}{3}$ when $x = 0$.

3.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 10\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 8e^{3x} + 25x^2 - 20x + 27,$$

where $y = 5$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 13$ when $x = 0$.

4.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \cosh x,$$

where $y = \frac{7}{12}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$.

5.

$$4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 24e^{-\frac{1}{2}x}$$

where $y = 6$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$.

15.7.3 ANSWERS TO EXERCISES

1.

$$y = \frac{1}{2}xe^{-x} + Ae^{-x} + Be^{-3x}.$$

2.

$$y = -\frac{1}{3}x\cos 3x + 2\cos 3x + \sin 3x.$$

3.

$$y = 2e^{3x} + x^2 + 1 + (2 - 3x)e^{5x}.$$

4.

$$y = \frac{1}{12}\left(e^{-x} - 6xe^{x} - e^{x} + 7e^{2x}\right).$$

5.

$$y = 3x^2e^{-\frac{1}{2}x} + (5x + 6)e^{-\frac{1}{2}x}.$$

"JUST THE MATHS"

UNIT NUMBER

15.8

ORDINARY DIFFERENTIAL EQUATIONS 8 (Simultaneous equations (A))

by

A.J.Hobson

UNIT 15.8 - ORDINARY DIFFERENTIAL EQUATIONS 8

SIMULTANEOUS EQUATIONS (A)

15.8.1 THE SUBSTITUTION METHOD

The methods discussed in previous Units for the solution of <u>second</u> order ordinary linear differential equations with constant coefficients may now be used for cases of two <u>first</u> order differential equations which must be satisfied simultaneously. The technique will be illustrated by the following examples:

EXAMPLES

1. Determine the general solutions for y and z in the case when

$$
\begin{aligned}
5\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + 4y - z &= e^{-x}, - - - - - - - - - - - (1)\\
\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 3z &= 5e^{-x}. - - - - - - - - - - (2)
\end{aligned}
$$

Solution

First, we eliminate one of the dependent variables from the two equations; in this case, we eliminate z.

From equation (2),

$$z = \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right)$$

and, on substituting this into equation (1), we obtain

$$5\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{3}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 5e^{-x}\right) + 4y - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 8y - 5e^{-x}\right) = e^{-x}.$$

That is, $$-\frac{2}{3}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{2}{3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{3}y = \frac{8}{3}e^{-x}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = -4e^{-x}.$$

The auxiliary equation is

$$m^2 + m - 2 = 0 \quad \text{or} \quad (m-1)(m+2) = 0,$$

giving a complementary function of $Ae^x + Be^{-2x}$, where A and B are arbitrary constants. A particular integral will be of the form ke^{-x}, where $k - k - 2k = -4$ and hence $k = 2$. Thus,

$$y = 2e^{-x} + Ae^x + Be^{-2x}.$$

Finally, from the formula for z in terms of y,

$$z = \frac{1}{3}\left(-2e^{-x} + Ae^x - 2Be^{-2x} + 16e^{-x} + 8Ae^x + 8Be^{-2x} - 5e^{-x}\right).$$

That is,

$$z = 3e^{-x} + 3Ae^{-x} + 2Be^{-2x}.$$

Note:
The above example would have been a little more difficult if the second differential equation had contained a term in $\frac{\mathrm{d}z}{\mathrm{d}x}$. But, if this were the case, we could eliminate $\frac{\mathrm{d}z}{\mathrm{d}x}$ between the two equations in order to obtain a statement with the same form as Equation (2).

2. Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} + 2y &= e^x, ---------(1) \\ \frac{\mathrm{d}y}{\mathrm{d}x} - 2z &= 1 + x, --------(2) \end{aligned}$$

given that $y = 1$ and $z = 2$ when $x = 0$.

Solution:

From equation (2), we have

$$z = \frac{1}{2}\left[\frac{\mathrm{d}y}{\mathrm{d}x} - 1 - x\right].$$

Substituting into the first differential equation gives

$$\frac{1}{2}\left[\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 1\right] + 2y = e^x$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 2e^x + 1.$$

The auxiliary equation is therefore $m^2+4=0$, having solutions $m=\pm j2$, which means that the complementary function is

$$A\cos 2x + B\sin 2x,$$

where A and B are arbitrary constants.
The particular integral will be of the form $y = pe^x + q$,
where

$$pe^x + 4pe^x + 4q = 2e^x + 1.$$

We require, then, that $5p = 2$ and $4q = 1$; and so the general solution for y is

$$y = A\cos 2x + B\sin 2x + \frac{2}{5}e^x + \frac{1}{4}.$$

Using the earlier formula for z, we obtain

$$z = \frac{1}{2}\left[-2A\sin 2x + 2B\cos 2x + \frac{2}{5}e^x - 1 - x\right] = B\cos 2x - A\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

Applying the boundary conditions,

$$1 = A + \frac{2}{5} + \frac{1}{4} \quad \text{giving} \quad A = \frac{7}{20}$$

and

$$2 = B + \frac{1}{5} - \frac{1}{2} \quad \text{giving} \quad B = \frac{23}{10}.$$

The required solutions are therefore

$$y = \frac{7}{20}\cos 2x + \frac{23}{10}\sin 2x + \frac{2}{5}e^x + \frac{1}{4}$$

and

$$z = \frac{23}{10}\cos 2x - \frac{7}{20}\sin 2x + \frac{1}{5}e^x - \frac{1}{2} - \frac{x}{2}.$$

15.8.2 EXERCISES

Solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + 2z &= e^{-x}, \\
\frac{\mathrm{d}z}{\mathrm{d}x} + 3z &= y,
\end{aligned}
$$

given that $y = 1$ and $z = 0$ when $x = 0$.

2.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} - z &= \sin x, \\
\frac{\mathrm{d}z}{\mathrm{d}x} + y &= \cos x,
\end{aligned}
$$

given that $y = 3$ and $z = 4$ when $x = 0$.

3.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} + 2y - 3z &= 1, \\
\frac{\mathrm{d}z}{\mathrm{d}x} - y &= e^{-2x},
\end{aligned}
$$

given that $y = 0$ and $z = 0$ when $x = 0$.

4.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2z, \\
\frac{\mathrm{d}z}{\mathrm{d}x} &= 8y,
\end{aligned}
$$

given that $y = 1$ and $z = 0$ when $x = 0$.

5.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}z}{\mathrm{d}x} + 6z &= 0, \\ 5\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 6y &= 0, \end{aligned}$$

given that $y = 3$ and $z = 0$ when $x = 0$.

Hint: First eliminate the $\frac{\mathrm{d}z}{\mathrm{d}x}$ terms to obtain a formula for z in terms of y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

6.

$$\begin{aligned} 10\frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{\mathrm{d}z}{\mathrm{d}x} + 6y + 5z &= 0, \\ 2\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}z}{\mathrm{d}x} + 2y + z &= 2e^{-x}, \end{aligned}$$

given that $y = 2$ and $z = -1$ when $x = 0$.

Hint: First, eliminate the $\frac{\mathrm{d}y}{\mathrm{d}x}$ and z terms in one step, to obtain a formula for y in terms of $\frac{\mathrm{d}z}{\mathrm{d}x}$ and x.

15.8.3 ANSWERS TO EXERCISES

1.

$$y = (2x+1)e^{-x} \quad \text{and} \quad z = xe^{-x}.$$

2.

$$y = (x+4)\sin x + 3\cos x \quad \text{and} \quad z = (x+4)\cos x - 3\sin x.$$

3.

$$y = \frac{1}{2}e^{x} + \frac{1}{2}e^{-3x} - e^{-2x} \quad \text{and} \quad z = \frac{1}{2}e^{x} - \frac{1}{6}e^{-3x} - \frac{1}{3}.$$

4.

$$y = \frac{1}{2}e^{4x} - \frac{1}{2}e^{-4x} \equiv \sinh 4x \quad \text{and} \quad z = e^{4x} + e^{-4x} \equiv 2\cosh 4x.$$

5.

$$y = 2e^{-x} + e^{-2x} \quad \text{and} \quad z = e^{-x} - e^{-2x}.$$

6.

$$y = \sin x + 2e^{-x} \quad \text{and} \quad z = e^{-x} - 2\cos x.$$

"JUST THE MATHS"

UNIT NUMBER

15.9

ORDINARY DIFFERENTIAL EQUATIONS 9 (Simultaneous equations (B))

by

A.J.Hobson

UNIT 15.9 - ORDINARY DIFFERENTIAL EQUATIONS 9

SIMULTANEOUS EQUATIONS (B)

15.9.1 INTRODUCTION

For students who have studied the principles of eigenvalues and eigenvectors (see Unit 9.6), a second method of solving two simultaneous linear differential equations is to interpret them as a single equation using matrix notation. The discussion will be limited to the simpler kinds of example, and we shall find it convenient to use t, x_1 and x_2 rather than x, y and z.

15.9.2 MATRIX METHODS FOR HOMOGENEOUS SYSTEMS

To introduce the technique, we begin by considering two simultaneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2.\end{aligned}$$

which are of the "homogeneous" type, since no functions of t, other than x_1 and x_2, appear on the right hand sides.

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(ii) Secondly, in a similar way to the method appropriate to a single differential equation, we make a trial solution of the form

$$\mathrm{X} = \mathrm{K}e^{\lambda t},$$

where

$$\mathrm{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

is a constant matrix of order 2×1.

This requires that

$$\lambda \mathrm{K} e^{\lambda t} = \mathrm{MK} e^{\lambda t} \quad \text{or} \quad \lambda \mathrm{K} = \mathrm{MK},$$

which we may recognise as the condition that λ is an eigenvalue of the matrix M, and K is an eigenvector of M.

The solutions for λ are obtained from the "characteristic equation"

$$|\mathrm{M} - \lambda \mathrm{I}| = 0.$$

In other words,

$$\begin{vmatrix} a - \lambda & b \\ c & b - \lambda \end{vmatrix} = 0,$$

leading to a quadratic equation having real and distinct solutions ($\lambda = \lambda_1$ and $\lambda = \lambda_2$), real and coincident solutions (λ only) or conjugate complex solutions ($\lambda = l \pm jm$).

(iii) The possibilities for the matrix K are obtained by solving the homogeneous linear equations

$$\begin{aligned} (a - \lambda_1 k_1 + bk_2 &= 0, \\ ck_1 + (d - \lambda_1)k_2 &= 0, \end{aligned}$$

giving $k_1 : k_2 = 1 : \alpha$ (say)

and

$$
\begin{aligned}
(a-\lambda_2)k_1 + bk_2 &= 0,\\
ck_1 + (d-\lambda_2)k_2 &= 0,
\end{aligned}
$$

giving $k_1 : k_2 = 1 : \beta$ (say).

Finally, it may be shown that, according to the types of solution to the auxiliary equation, the solution of the differential equation will take one of the following three forms, in which A and B are arbitrary constants:

(a)

$$
A\begin{bmatrix}1\\ \alpha\end{bmatrix}e^{\lambda_1 t} + B\begin{bmatrix}1\\ \beta\end{bmatrix}e^{\lambda_2 t};
$$

(b)

$$
\left\{(At+B)\begin{bmatrix}1\\ \alpha\end{bmatrix} + \frac{A}{b}\begin{bmatrix}0\\ 1\end{bmatrix}\right\}e^{\lambda t};
$$

or

(c)

$$
e^{lt}\left\{\begin{bmatrix}A\\ pA+qB\end{bmatrix}\cos mt + \begin{bmatrix}B\\ pB-qA\end{bmatrix}\sin mt\right\},
$$

where, in (c), $1 : \alpha = 1 : p + jq$ and $1 : \beta = 1 : p - jq$.

EXAMPLES

1. Determine the general solution of the simultaneous differential equations

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1 + 5x_2,\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -x_1 + 2x_2.
\end{aligned}
$$

Solution

The characteristic equation is

$$\begin{vmatrix} -4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 + 2\lambda - 3 = 0 \ \text{ or } \ (\lambda - 1)(\lambda + 3) = 0.$$

When $\lambda = 1$, we need to solve the homogeneous equations

$$\begin{aligned} -5k_1 + 5k_2 &= 0, \\ -k_1 + k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : 1$.
When $\lambda = -3$, we need to solve the homogeneous equations

$$\begin{aligned} -k_1 + 5k_2 &= 0, \\ -k_1 + 5k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{1}{5}$.
The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} e^{-3t}$$

or, alternatively,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + B \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{-3t},$$

where A and B are arbitrary constants.

2. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - x_2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 + 3x_2.
\end{aligned}$$

Solution

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 4 = 0 \quad \text{or} \quad (\lambda - 2)^2 = 0.$$

When $\lambda = 2$, we need to solve the homogeneous equations

$$\begin{aligned}
-k_1 - k_2 &= 0, \\
k_1 + k_2 &= 0,
\end{aligned}$$

both of which give $k_1 : k_2 = 1 : -1$.

The general solution is therefore

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ (At + B) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} e^{2t},$$

where A and B are arbitrary constants.

3. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 - 5x_2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 + 3x_2.
\end{aligned}$$

Solution
The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0.$$

That is,

$$\lambda^2 - 4\lambda + 13 = 0,$$

which gives $\lambda = 2 \pm j3$.
When $\lambda = 2 + j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1-j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1-j3)k_2 = 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1-j3}{5}$.
When $\lambda = 2 - j3$, we need to solve the homogeneous equations

$$\begin{aligned} (-1+j3)k_1 - 5k_2 &= 0, \\ 2k_1 + (1+j3)k_2 &= 0, \end{aligned}$$

both of which give $k_1 : k_2 = 1 : \frac{-1+j3}{5}$.
The general solution is therefore

$$\frac{e^{2t}}{5}\left\{\begin{bmatrix} A \\ -A+3B \end{bmatrix}\cos 3t + \begin{bmatrix} B \\ -B-3A \end{bmatrix}\sin 3t\right\},$$

where A and B are arbitrary constants.
Note:
From any set of simultaneous differential equations of the form

$$\begin{aligned} a\frac{\mathrm{d}x_1}{\mathrm{d}t} + b\frac{\mathrm{d}x_2}{\mathrm{d}t} + cx_1 + dx_2 = 0, \\ a'\frac{\mathrm{d}x_1}{\mathrm{d}t} + b'\frac{\mathrm{d}x_2}{\mathrm{d}t} + b'x_1 + c'x_2 = 0, \end{aligned}$$

it is possible to eliminate $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ in turn, in order to obtain two equivalent equations of the form discussed in the above examples.

15.9.3 EXERCISES

1. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 + 3x_2,\\ \frac{dx_2}{dt} &= 3x_1 + x_2.\end{aligned}$$

2. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= 3x_1 + 2x_2,\\ \frac{dx_2}{dt} &= 4x_1 + x_2,\end{aligned}$$

given that $x_1 = 3$ and $x_2 = -3$ when $t = 0$.

3. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= -x_1 + 9x_2,\\ \frac{dx_2}{dt} &= 11x_1 + x_2,\end{aligned}$$

given that $x_1 = 20$ and $x_2 = 20$ when $t = 0$.

4. Solve completely, the simultaneous differential equations

$$\begin{aligned}\frac{dx_1}{dt} - \frac{dx_2}{dt} + 2x_1 - 2x_2 &= 0,\\ \frac{dx_1}{dt} + 2\frac{dx_2}{dt} - 7x_1 - 5x_2 &= 0,\end{aligned}$$

given that $x_1 = 2$ and $x_2 = 0$ when $t = 0$.

5. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2.\end{aligned}$$

6. Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2.\end{aligned}$$

15.9.4 ANSWERS TO EXERCISES

1.

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

2.

$$\begin{bmatrix}1\\1\end{bmatrix}e^{5t} + 2\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}.$$

3.

$$2\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t} + \begin{bmatrix}18\\22\end{bmatrix}e^{10t}.$$

4.

$$\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}.$$

5.

$$\left\{(At+B)\begin{bmatrix}1\\-1\end{bmatrix} + \frac{A}{2}\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}.$$

6.

$$e^{7t}\left\{\begin{bmatrix}A\\-A+2B\end{bmatrix}\cos 2t + \begin{bmatrix}B\\-B-2A\end{bmatrix}\sin 2t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

15.10

ORDINARY DIFFERENTIAL EQUATIONS 10 (Simultaneous equations (C))

by

A.J.Hobson

UNIT 15.10 - ORDINARY DIFFERENTIAL EQUATIONS 10

SIMULTANEOUS EQUATIONS (C)

15.10.1 MATRIX METHODS FOR NON-HOMOGENEOUS SYSTEMS

In Units 15.5, 15.6 and 15.7 , it was seen that, for a single linear differential equation with constant coefficients, the general solution is made up of a particular integral and a complementary function (the latter being the general solution of the corresponding homogeneous differential equation).

In the work which follows, a similar principle is applied to a pair of simultaneous non-homogeneous differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= ax_1 + bx_2 + f(t),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= cx_1 + dx_2 + g(t).\end{aligned}$$

The method will be illustrated by the following example, in which $f(t) \equiv 0$:

EXAMPLE

Determine the general solution of the simultaneous differential equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, - - - - - - - - - - - - -(1)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -4x_1 - 5x_2 + g(t), - - - - -(2)\end{aligned}$$

where $g(t)$ is (a) t, (b) e^{2t} (c) $\sin t$, (d) e^{-t}.

Solutions

(i) First, we write the differential equations in matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t),$$

which may be interpreted as

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t) \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathrm{M} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \quad \text{and} \quad \mathrm{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(ii) Secondly, we consider the corresponding "homogeneous" system

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX},$$

for which the characteristic equation is

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & -5-\lambda \end{vmatrix} = 0,$$

and gives

$$\lambda(5+\lambda)+4=0 \quad \text{or} \quad \lambda^2+5\lambda+4=0 \quad \text{or} \quad (\lambda+1)(\lambda+4)=0.$$

(iii) The eigenvectors of M are obtained from the homogeneous equations

$$\begin{aligned} -\lambda k_1 + k_2 &= 0, \\ -4k_1 - (5+\lambda)k_2 &= 0. \end{aligned}$$

Hence, in the case when $\lambda = -1$, we solve

$$\begin{aligned} k_1 + k_2 &= 0, \\ -4k_1 - 4k_2 &= 0, \end{aligned}$$

and these are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -1$.

Also, when $\lambda = -4$, we solve

$$\begin{aligned} 4k_1 + k_2 &= 0, \\ -4k_1 - k_2 &= 0 \end{aligned}$$

which are satisfied by any two numbers in the ratio $k_1 : k_2 = 1 : -4$.

The complementary function may now be written in the form

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t},$$

where A and B are arbitrary constants.

(iv) In order to obtain a particular integral for the equation

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}g(t),$$

we note the second term on the right hand side and investigate a trial solution of a similar form. The three cases in this example are as follows:

(a) $g(t) \equiv t$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P} + \mathrm{Q}t,$$

where P and Q are constant matrices of order 2×1.

We require that

$$\mathrm{Q} = \mathrm{M}(\mathrm{P} + \mathrm{Q}t) + \mathrm{N}t,$$

whereupon, equating the matrix coefficients of t and the constant matrices,

$$\mathrm{MQ} + \mathrm{N} = \mathbf{0} \quad \text{and} \quad \mathrm{Q} = \mathrm{MP},$$

giving

$$\mathrm{Q} = -\mathrm{M}^{-1}\mathrm{N} \quad \text{and} \quad \mathrm{P} = \mathrm{M}^{-1}\mathrm{Q}.$$

Thus, using

$$\mathrm{M}^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix},$$

we obtain

$$\mathrm{Q} = -\frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$$

and

$$\mathrm{P} = \frac{1}{4}\begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \begin{bmatrix} -0.3125 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} t.$$

(b) $g(t) \equiv e^{2t}$

$$\text{Trial solution} \quad \mathrm{X} = \mathrm{P}e^{2t}$$

We require that

$$2\mathrm{P}e^{2t} = \mathrm{MP}e^{2t} + \mathrm{N}e^{2t}.$$

That is,

$$2\mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The matrix, P, may now be determined from the formula

$$(2\mathrm{I} - \mathrm{M})\mathrm{P} = \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix}.\mathrm{P} = \mathrm{N}.$$

Hence,

$$\mathrm{P} = \frac{1}{18}\begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}.\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}.$$

The general solution, in this case, is

$$\mathrm{X} = A\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + B\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-4t} + \frac{1}{18}\begin{bmatrix} 7 \\ -4 \end{bmatrix}e^{2t}.$$

(c) $g(t) \equiv \sin t$

$$\text{Trial solution } \ \mathrm{X} = \mathrm{P}\sin t + \mathrm{Q}\cos t.$$

We require that

$$\mathrm{P}\cos t - \mathrm{Q}\sin t = \mathrm{M}(\mathrm{P}\sin t + \mathrm{Q}\cos t) + \mathrm{N}\sin t.$$

Equating the matrix coefficients of $\cos t$ and $\sin t$,

$$\mathrm{P} = \mathrm{MQ} \ \text{ and } \ -\mathrm{Q} = \mathrm{MP} + \mathrm{N},$$

which means that

$$-\mathrm{Q} = \mathrm{M}^2\mathrm{Q} + \mathrm{N} \ \text{ or } \ (\mathrm{M}^2 + \mathrm{I})\mathrm{Q} = -\mathrm{N}.$$

Thus,

$$Q = -(M^2 + I)^{-1}N,$$

where

$$M^2 + I = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 20 & 22 \end{bmatrix}$$

and, hence,

$$Q = -\frac{1}{34} \begin{bmatrix} 22 & 5 \\ -20 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix}.$$

Also,

$$P = MQ = \frac{1}{34} \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

The general solution, in this case, is

$$X = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + B \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-4t} + \frac{1}{34} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin t + \frac{1}{34} \begin{bmatrix} -5 \\ 3 \end{bmatrix} \cos t.$$

(d) $g(t) \equiv e^{-t}$

In this case, the function, $g(t)$, is already included in the complementary function and it becomes necessary to assume a particular integral of the form

$$X = (P + Qt)e^{-t},$$

where P and Q are constant matrices of order 2×1.

We require that

$$Qe^{-t} - (P + Qt)e^{-t} = M(P + Qt)e^{-t} + Ne^{-t},$$

whereupon, equating the matrix coefficients of te^{-t} and e^{-t}, we obtain

$$-\mathrm{Q} = \mathrm{MQ} \ \text{ and } \ \mathrm{Q} - \mathrm{P} = \mathrm{MP} + \mathrm{N}.$$

The first of these conditions shows that Q is an eigenvector of the matrix M corresponding the eigenvalue -1 and so, from earlier work,

$$\mathrm{Q} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any constant k.

Also,

$$(\mathrm{M} + \mathrm{I})\mathrm{P} = \mathrm{Q} - \mathrm{N};$$

or, in more detail,

$$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} p_1 + p_2 &= k, \\ -4p_1 - 4p_2 &= -k - 1. \end{aligned}$$

Using $p_1 + p_2 = k$ and $p_1 + p_2 = \frac{k+1}{4}$, we deduce that $k = \frac{1}{3}$ and that the matrix P is given by

$$\mathrm{P} = \begin{bmatrix} l \\ \frac{1}{3} - l \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + l\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for any number, l.

Taking $l = 0$ for simplicity, a particular integral is therefore

$$\mathrm{X} = \frac{1}{3}\left\{\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}t\right\}e^{-t}.$$

and the general solution is

$$\mathrm{X} = A\begin{bmatrix}1\\-1\end{bmatrix}e^{-t} + B\begin{bmatrix}1\\-4\end{bmatrix}e^{-4t} + \frac{1}{3}\left\{\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}t\right\}e^{-t}.$$

Note:
In examples for which neither $f(t)$ nor $g(t)$ is identically equal to zero, the particular integral may be found by adding together the separate forms of particular integral for $f(t)$ and $g(t)$ and writing the system of differential equations in the form

$$\frac{\mathrm{dX}}{\mathrm{dt}} = \mathrm{MX} + \mathrm{N}_1 f(t) + \mathrm{N}_2 g(t),$$

where

$$\mathrm{N}_1 = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } \mathrm{N}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

For instance, if $f(t) \equiv t$ and $g(t) \equiv e^{2t}$, the particular integral would take the form

$$\mathrm{X} = \mathrm{P} + \mathrm{Q}t + \mathrm{R}e^{2t},$$

where P, Q and R are matrices of order 2×1.

15.10.2 EXERCISES

1. Determine the general solutions of the following systems of simultaneous differential equations:

   (a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + 3x_2 + 5t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 3x_1 + x_2 + e^{3t}.\end{aligned}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 3x_1 + 2x_2 + t^2,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 4x_1 + x_2 + e^{-2t}.\end{aligned}$$

2. Determine the complete solutions of the following systems of differential equations, subject to the conditions given:

(a)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + 9x_2 + 3,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 11x_1 + x_2 + e^{10t},\end{aligned}$$

given that $x_1 = \frac{1}{225}$ and $x_2 = -\frac{1}{100}$ when $t = 0$.

(b)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 5x_1 + 2x_2 + 2t^2 + t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -2x_1 + x_2,\end{aligned}$$

given that $x_1 = \frac{32}{27}$ and $x_2 = -\frac{12}{27}$ when $t = 0$.

(c)

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 8x_1 + x_2 + \sin t,\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -5x_1 + 6x_2 + \cos t,\end{aligned}$$

given that $x_1 = 0$ and $x_2 = 0$ when $t = 0$.

15.10.3 ANSWERS TO EXERCISES

1. (a)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{4t}+B\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}+\frac{1}{32}\begin{bmatrix}-25\\15\end{bmatrix}+\frac{1}{8}\begin{bmatrix}5\\-15\end{bmatrix}t-\frac{1}{5}\begin{bmatrix}3\\2\end{bmatrix}e^{3t};$$

(b)

$$A\begin{bmatrix}1\\1\end{bmatrix}e^{5t}+B\begin{bmatrix}1\\-2\end{bmatrix}e^{-t}+\frac{2}{125}\begin{bmatrix}41\\-84\end{bmatrix}+\frac{2}{25}\begin{bmatrix}-9\\16\end{bmatrix}t+\frac{1}{5}\begin{bmatrix}1\\-4\end{bmatrix}t^2+\frac{1}{7}\begin{bmatrix}2\\-5\end{bmatrix}e^{-2t}.$$

2. (a)

$$-\frac{7}{45}\begin{bmatrix}1\\-1\end{bmatrix}e^{-10t}+\frac{13}{900}\begin{bmatrix}9\\11\end{bmatrix}e^{10t}+\frac{3}{100}\begin{bmatrix}1\\-11\end{bmatrix}+\frac{1}{180}\begin{bmatrix}0\\1\end{bmatrix}e^{10t}+\frac{1}{20}\begin{bmatrix}9\\11\end{bmatrix}te^{10t};$$

(b)

$$\left\{(2t+1)\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right\}e^{3t}+\frac{1}{27}\begin{bmatrix}5\\-12\end{bmatrix}+\frac{1}{27}\begin{bmatrix}1\\-22\end{bmatrix}t-\frac{2}{9}\begin{bmatrix}1\\2\end{bmatrix}t^2;$$

(c)

$$\frac{1}{145}\left\{e^{7t}\left(\begin{bmatrix}-1\\25\end{bmatrix}\cos 2t+\begin{bmatrix}-12\\-10\end{bmatrix}\sin 2t\right)+\begin{bmatrix}-17\\-10\end{bmatrix}\sin t+\begin{bmatrix}1\\-25\end{bmatrix}\cos t\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

A.J.Hobson

UNIT 16.1 - LAPLACE TRANSFORMS 1 - DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** to be discussed in the following notes will be for the purpose of solving certain kinds of **"differential equation"**; that is, an equation which involves a derivative or derivatives.

The particular differential equation problems to be encountered will be limited to the two types listed below:

(a) Given the **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of x when $t = 0$ (that is, $x(0)$), determine a formula for x in terms of t, which does not include any derivatives.

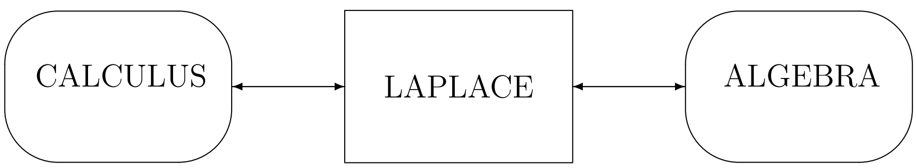
(b) Given the **"second order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of x and $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 0$ (that is, $x(0)$ and $x'(0)$), determine a formula for x in terms of t which does not include any derivatives.

Roughly speaking, the method of Laplace Transforms is used to convert a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.



The background to the development of Laplace Transforms would be best explained using certain other techniques of solving differential equations which may not have been part of earlier work. This background will therefore be omitted here.

DEFINITION
The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t)\ \mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes
(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$, since the result of the definite integral in the definition will be an expression involving s.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations; (see the note to the second standard result below).

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

The following is a list of standard results on which other Laplace Transforms will be based:

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n\ \mathrm{d}t = I_n \ \text{ say}.$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1}\ \mathrm{d}t = \frac{n}{s}.I_{n-1},$$

using the fact that e^{-st} tends to zero much faster than any other function of t can tend to infinity. That is, a decaying exponential will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st}\ \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:
This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^\infty e^{-st}e^{-at} \text{ d}t = \int_0^\infty e^{-(s+a)t} \text{ d}t$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:
A slightly different form of this result, less commonly used in applications to science and engineering, is

$$L[e^{bt}] = \frac{1}{s-b};$$

but, to obtain this result by integration, we would need to assume that $s > b$ to ensure that $e^{-(s-b)t}$ is genuinely a **decaying** exponential.

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^\infty e^{-st}\cos at \text{ d}t = \left[\frac{e^{-st}\sin at}{a}\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at \text{ d}t$$

using Integration by Parts, once.

Using Integration by Parts a second time,

$$F(s) = 0 + \frac{s}{a}\left\{\left[-\frac{e^{-st}\cos at}{a}\right]_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at \text{ d}t\right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2}.F(s).$$

That is,

$$F(s) = \frac{s}{s^2+a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2+a^2}.$$

4. $f(t) \equiv \sin at$.

The method is similar to that for $\cos at$, and we obtain

$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

The following list of results is of use in finding the Laplace Transform of a function which is made up of **basic** functions, such as those encountered in the previous section.

1. LINEARITY

If A and B are constants, then

$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

Proof:

This follows easily from the linearity of an integral.

EXAMPLE

Determine the Laplace Transform of the function,

$$2t^5 + 7\cos 4t - 1.$$

Solution

$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2 + 4^2} - \frac{1}{s} = \frac{240}{s^6} + \frac{7s}{s^2 + 16} - \frac{1}{s}.$$

2. THE TRANSFORM OF A DERIVATIVE

The two results which follow are of special use when solving first and second order differential equations. We shall begin by discussing them in relation to an arbitrary function, $f(t)$; then we shall restate them in the form which will be needed for solving differential equations.

(a)

$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\ \mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\ \mathrm{d}t$$

using integration by parts.

Thus,

$$L[f'(t)] = -f(0) + sL[f(t)],$$

as required.

(b)

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0).$$

Proof:

Treating $f''(t)$ as the first derivative of $f'(t)$, we have

$$L[f''(t)] = sL[f'(t)] - f'(0),$$

which gives the required result on substituting from (a) the expression for $L[f'(t)]$.

Alternative Forms (Using $L[x(t)] = X(s)$)**:**

(i)

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

(ii)

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:

$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\ \mathrm{d}t,$$

which can be regarded as the effect of replacing s by $s+a$ in $L[f(t)]$. In other words, $F(s+a)$.

Notes:

(i) Sometimes, this result is stated in the form

$$L[e^{bt}f(t)] = F(s-b)$$

but, in science and engineering, the exponential is more likely to be a **decaying** exponential.

(ii) There is, in fact, a Second Shifting Theorem, encountered in more advanced courses; but we do not include it in this Unit (see Unit 16.5).

EXAMPLE

Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.

Solution

First of all, we note that

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

Replacing s by $(s+2)$ in this result, the First Shifting Theorem gives

$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t = \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -\ L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right] = -\ \frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

THE USE OF A TABLE OF LAPLACE TRANSFORMS AND RULES

For the purposes of these Units, the following **brief** table may be used to determine the Laplace Transforms of functions of t without having to use integration:

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \quad \text{or} \quad s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t f(T)g(t-T)\ \mathrm{d}T\right] = F(s)G(s).$$

16.1.5 EXERCISES

1. Use a table the table of Laplace Transforms to find $L[f(t)]$ in the following cases:

(a)

$$3t^2 + 4t - 1;$$

(b)

$$t^3 + 3t^2 + 3t + 1 \quad (\equiv (t+1)^3);$$

(c)

$$2e^{5t} - 3e^t + e^{-7t};$$

(d)

$$2\sin 3t - 3\cos 2t;$$

(e)

$$t\sin 6t;$$

(f)

$$t(e^t + e^{-2t});$$

(g)

$$\frac{1}{2}(1 - \cos 2t) \quad (\equiv \sin^2 t).$$

2. Using the First Shifting Theorem, obtain the Laplace Transforms of the following functions of t:

(a)

$$e^{-3t}\cos 5t;$$

(b)

$$t^2 e^{2t};$$

(c)

$$e^{-2t}\left(2t^3+3t-2\right);$$

(d)

$$\cosh 2t.\sin t;$$

(e)

$$e^{-at}f'(t),$$

where $L[f(t)] = F(s)$.

3. (a) If

$$x = t^3e^{-t},$$

determine the Laplace Transform of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ without differentiating x more than once with respect to t.

(b) If

$$\frac{\mathrm{d}x}{\mathrm{d}t}+x=e^t,$$

where $x(0) = 0$, show that

$$X(s)=\frac{1}{s^2-1}.$$

4. Verify the Initial and Final Value Theorems for the function

$$f(t) = te^{-3t}.$$

16.1.6 ANSWERS TO EXERCISES

1. (a)

$$\frac{6}{s^3}+\frac{4}{s^2}-\frac{1}{s};$$

(b)

$$\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s};$$

(c)

$$\frac{2}{s-5}-\frac{3}{s-1}+\frac{1}{s+7};$$

(d)

$$\frac{6}{s^2+9}-\frac{3s}{s^2+4};$$

(e)

$$\frac{12s}{(s^2+36)^2};$$

(f)

$$\frac{1}{(s-1)^2}+\frac{1}{(s+2)^2};$$

(g)

$$\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right].$$

2. (a)

$$\frac{s+3}{(s+3)^2+25};$$

(b)

$$\frac{2}{(s-2)^3};$$

(c)

$$\frac{12}{(s+2)^4}+\frac{3}{(s+2)^2}-\frac{2}{s+2};$$

(d)

$$\frac{1}{2}\left[\frac{1}{(s-2)^2+1}+\frac{1}{(s+2)^2+1}\right];$$

(e)

$$(s+a)F(s+a)-f(0).$$

3. (a)

$$\frac{6s^2}{(s+1)^4};$$

(b) On the left hand side, use the formula for $L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]$.

4.

$$\lim_{t\to 0} f(t)=0 \quad\text{and}\quad \lim_{t\to\infty} f(t)=0.$$

"JUST THE MATHS"

UNIT NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

A.J.Hobson

UNIT 16.2 - LAPLACE TRANSFORMS 2
INVERSE LAPLACE TRANSFORMS

In order to solve differential equations, we now examine how to determine a function of the variable, t, whose Laplace Transform is already known.

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORMS

A function of t, whose Laplace Transform is the given expression, $F(s)$, is called the **"Inverse Laplace Transform"** of $f(t)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:
(i) Since two functions which coincide for $t > 0$ will have the same Laplace Transform, we can determine the Inverse Laplace Transform of $F(s)$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear** since

$$L^{-1}\left[AF(s) + BG(s)\right]$$

is a function of t whose Laplace Transform is

$$AF(s) + BG(s);$$

and, by the linearity of Laplace Transforms, discussed in Unit 16.1, such a function is

$$AL^{-1}[F(s)] + BL^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

The type of differential equation to be encountered in simple practical problems usually lead to Laplace Transforms which are **"rational functions of s"**. We shall restrict the discussion to such cases, as illustrated in the following examples, where the table of standard Laplace Transforms is used whenever possible. The partial fractions are discussed in detail, but other, shorter, methods may be used if known (for example, the "Cover-up Rule" and "Keily's Method"; see Unit 1.9)

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s+3 \equiv A(s+3) + Bs$$

Note:
Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions. Otherwise, entire partial fractions exercises would have to be carried out by equating coefficients of appropriate powers of s on both sides.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B;$$

so that

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2+9}.$$

Solution

$$f(t) = \frac{1}{3}\sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2+5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}}\sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2+4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2+4) + (Bs + C)(s+1).$$

Substituting $s = -1$, we obtain

$$5 = 5A \text{ which implies that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24} t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t} \sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**, giving

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

In order to use the First Shifting Theorem, we must try to include $s+2$ in the numerator; so we write

$$F(s) = \frac{(s+2) - 2}{(s+2)^2 + 9} = \frac{s+2}{(s+2)^2 + 3^2} - \frac{2}{3}.\frac{3}{(s+2)^2 + 3^2}.$$

Hence,

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t = \frac{1}{3}e^{-2t}\left[3\cos 3t - 2\sin 3t\right] \quad t > 0.$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2 + 4s + 8)}.$$

Solution

Applying the principles of partial fractions,

$$\frac{8(s+1)}{s(s^2 + 4s + 8)} \equiv \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2 + 4s + 8) + (Bs + C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s + 4}{s^2 + 4s + 8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square to give

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

which, on rearrangement, becomes

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

Thus, from the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$, and partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s + 10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s = -2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s = 6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

However, if we did not factorise the denominator, a different form of solution could be obtained as follows:

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\text{cosh}4t + 3\text{sinh}4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform of this function could certainly be obtained by using partial fractions, but we note here how it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T.e^{-2(t-T)}\ \mathrm{d}T = \int_0^t e^{(3T-2t)}\ \mathrm{d}T = \left[\frac{e^{3T-2t}}{3}\right]_0^t.$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$

16.2.3 EXERCISES

Determine the Inverse Laplace Transforms of the following rational functions of s:

1. (a)

$$\frac{1}{(s-1)^2};$$

(b)

$$\frac{1}{(s+1)^2+4};$$

(c)

$$\frac{s+2}{(s+2)^2+9};$$

(d)

$$\frac{s-2}{(s-3)^3};$$

(e)

$$\frac{1}{(s^2+4)^2};$$

(f)

$$\frac{s+1}{s^2+2s+5};$$

(g)

$$\frac{s-3}{s^2-4s+5};$$

(h)

$$\frac{s-3}{(s-1)^2(s-2)};$$

(i)

$$\frac{5}{(s+1)(s^2-2s+2)};$$

(j)

$$\frac{2s-9}{(s-3)(s+2)};$$

(k)

$$\frac{3}{s(s^2+9)};$$

(l)

$$\frac{2s-1}{(s-1)(s^2+2s+2)}.$$

2. Use the Convolution Theorem to obtain the Inverse Laplace Transform of

$$\frac{s}{(s^2+1)^2}.$$

16.2.4 ANSWERS TO EXERCISES

1. (a)

$$te^t \quad t>0;$$

(b)

$$\frac{1}{2}e^{-t}\sin 2t \quad t>0;$$

(c)

$$e^{-2t}\cos 3t \quad t>0;$$

(d)

$$e^{3t}\left[t+\frac{1}{2}t^2\right] \quad t>0;$$

(e)

$$\frac{1}{16}[\sin 2t - 2t\cos 2t] \quad t>0;$$

(f)

$$e^{-t}\cos 2t \quad t>0;$$

(g)

$$e^{2t}[\cos t - \sin t] \quad t>0;$$

(h)

$$2te^{t}+e^{t}-e^{2t} \quad t>0;$$

(i)

$$e^{-t}+e^{t}[2\sin t - \cos t] \quad t>0;$$

(j)

$$\frac{1}{5}[13e^{-2t}-3e^{3t}] \quad t>0;$$

(k)

$$\frac{1}{3}[1-\cos 3t] \quad t>0;$$

(l)

$$\frac{1}{5}[e^{t}-e^{-t}\cos t+8e^{-t}\sin t] \quad t>0.$$

2.

$$\frac{1}{2}t\sin t \quad t>0.$$

"JUST THE MATHS"

UNIT NUMBER

16.3

LAPLACE TRANSFORMS 3
(Differential equations)

by

A.J.Hobson

UNIT 16.3 - LAPLACE TRANSFORMS 3 - DIFFERENTIAL EQUATIONS

16.3.1 EXAMPLES OF SOLVING DIFFERENTIAL EQUATIONS

In the work which follows, the problems considered will usually take the form of a linear differential equation of the second order with constant coefficients.

That is,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

However, the method will apply equally well to the corresponding first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t).$$

The technique will be illustrated by examples.

EXAMPLES

1. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0,$$

given that $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 3] + 4[sX(s) - 3] + 13X(s) = 0.$$

Hence,

$$(s^2 + 4s + 13)X(s) = 3s + 12,$$

giving

$$X(s) \equiv \frac{3s + 12}{s^2 + 4s + 13}.$$

The denominator does not factorise, therefore we complete the square to obtain

$$X(s) \equiv \frac{3s + 12}{(s + 2)^2 + 9} \equiv \frac{3(s + 2) + 6}{(s + 2)^2 + 9} \equiv 3.\frac{s + 2}{(s + 2)^2 + 9} + 2.\frac{3}{(s + 2)^2 + 9}.$$

Thus,

$$x(t) = 3e^{-2t}\cos 3t + 2e^{-2t}\sin 3t \quad t > 0$$

or

$$x(t) = e^{-2t}[3\cos 3t + 2\sin 3t] \quad t > 0.$$

2. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = 50\sin t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] - 4 + 6[sX(s) - 1] + 9X(s) = \frac{50}{s^2 + 1},$$

giving

$$(s^2 + 6s + 9)X(s) = \frac{50}{s^2 + 1} + s + 10.$$

Hint: Do <u>not</u> combine the terms on the right into a single fraction - it won't help !

Thus,

$$X(s) \equiv \frac{50}{(s^2 + 6s + 9)(s^2 + 1)} + \frac{s + 10}{s^2 + 6s + 9}$$

or

$$X(s) \equiv \frac{50}{(s + 3)^2(s^2 + 1)} + \frac{s + 10}{(s + 3)^2}.$$

Using the principles of partial fractions in the first term on the right,

$$\frac{50}{(s + 3)^2(s^2 + 1)} \equiv \frac{A}{(s + 3)^2} + \frac{B}{s + 3} + \frac{Cs + D}{s^2 + 1}.$$

Hence,

$$50 \equiv A(s^2 + 1) + B(s + 3)(s^2 + 1) + (Cs + D)(s + 3)^2.$$

Substituting $s = -3$,

$$50 = 10A \text{ giving } A = 5.$$

Equating coefficients of s^3 on both sides,

$$0 = B + C. \quad (1)$$

Equating the coefficients of s on both sides (we shall not need the s^2 coefficients in this example),

$$0 = B + 9C + 6D. \qquad (2)$$

Equating the constant terms on both sides,

$$50 = A + 3B + 9D = 5 + 3B + 9D. \qquad (3)$$

Putting $C = -B$ into (2), we obtain

$$-8B + 6D = 0, \qquad (4)$$

and we already have

$$3B + 9D = 45. \qquad (3)$$

These last two solve easily to give $B = 3$ and $D = 4$ so that $C = -3$.

We conclude that

$$\frac{50}{(s+3)^2(s^2+1)} \equiv \frac{5}{(s+3)^2} + \frac{3}{s+3} + \frac{-3s+4}{s^2+1}.$$

In addition to this, we also have

$$\frac{s+10}{(s+3)^2} \equiv \frac{s+3}{(s+3)^2} + \frac{7}{(s+3)^2} \equiv \frac{1}{s+3} + \frac{7}{(s+3)^2}.$$

The total for $X(s)$ is therefore given by

$$X(s) \equiv \frac{12}{(s+3)^2} + \frac{4}{s+3} - 3.\frac{s}{s^2+1} + 4.\frac{1}{s^2+1}.$$

Finally,

$$x(t) = 12te^{-3t} + 4e^{-3t} - 3\cos t + 4\sin t \quad t > 0.$$

3. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 3x = 4e^t,$$

given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s[sX(s) - 1] + 2 + 4[sX(s) - 1] - 3X(s) = \frac{4}{s-1}.$$

This gives

$$(s^2 + 4s - 3)X(s) = \frac{4}{s-1} + s + 2.$$

Therefore,

$$X(s) \equiv \frac{4}{(s-1)(s^2+4s-3)} + \frac{s+2}{s^2+4s-3}.$$

Applying the principles of partial fractions,

$$\frac{4}{(s-1)(s^2+4s-3)} \equiv \frac{A}{s-1} + \frac{Bs+C}{s^2+4s-3}.$$

Hence,

$$4 \equiv A(s^2+4s-3) + (Bs+C)(s-1).$$

Substituting $s = 1$, we obtain

$$4 = 2A; \quad \text{that is,} \quad A = 2.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B, \quad \text{so that} \quad B = -2.$$

Equating constant terms on both sides,

$$4 = -3A - C, \quad \text{so that} \quad C = -10.$$

Thus, in total,

$$X(s) \equiv \frac{2}{s-1} + \frac{-s-8}{s^2+4s-3} \equiv \frac{2}{s-1} + \frac{-s-8}{(s+2)^2-7}$$

or

$$X(s) \equiv \frac{2}{s-1} - \frac{s+2}{(s+2)^2-7} - \frac{6}{(s+2)^2-7}.$$

Finally,

$$x(t) = 2e^t - e^{-2t}\text{cosh}t\sqrt{7} - \frac{6}{\sqrt{7}}e^{-2t}\text{sinh}t\sqrt{7} \quad t > 0.$$

16.3.2 THE GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

On some occasions, we may either be given no boundary conditions at all; or else the boundary conditions given do not tell us the values of $x(0)$ and $x'(0)$.

In such cases, we simply let $x(0) = A$ and $x'(0) = B$ to obtain a solution in terms of A and B called the **"general solution"**.

If any non-standard boundary conditions are provided, we then substitute them into the general solution to obtain particular values of A and B.

EXAMPLE

Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 0$$

and, hence, determine the particular solution in the case when $x(\frac{\pi}{2}) = -3$ and $x'(\frac{\pi}{2}) = 10$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - A) - B + 4X(s) = 0.$$

That is,

$$(s^2 + 4)X(s) = As + B.$$

Hence,

$$X(s) \equiv \frac{As + B}{s^2 + 4} \equiv A.\frac{s}{s^2 + 4} + B.\frac{1}{s^2 + 4}.$$

This gives

$$x(t) = A\cos 2t + \frac{B}{2}\sin 2t \quad t > 0;$$

but, since A and B are **arbitrary** constants, this may be written in the simpler form

$$x(t) = A\cos 2t + B\sin 2t \quad t > 0,$$

in which $\frac{B}{2}$ has been rewritten as B.

To apply the boundary conditions, we require also the formula for $x'(t)$, namely

$$x'(t) = -\ 2A\sin 2t + 2B\cos 2t.$$

Hence, $-\ 3 = -\ A$ and $10 = -\ 2B$ giving $A = 3$ and $B =$ - 5.

Therefore, the particular solution is

$$x(t) = 3\cos 2t - 5\sin 2t \quad t > 0.$$

16.3.3 EXERCISES

1. Solve the following differential equations subject to the conditions given:

   (a)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 0,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (b)
   $$4\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$
   given that $x(0) = 4$ and $x'(0) = 1$;

   (c)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} - 8x = 2t,$$
   given that $x(0) = 3$ and $x'(0) = 1$;

   (d)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 4x = 2e^{2t},$$
   given that $x(0) = 1$ and $x'(0) = 10.5$;

   (e)
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 3\cos^2 t,$$
   given that $x(0) = 1$ and $x'(0) = 2$.
   Hint: $\cos 2t \equiv 2\cos^2 t - 1$.

2. Determine the particular solution of the differential equation
   $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}x}{\mathrm{d}t} = e^t(t-3)$$
   in the case when $x(0) = 2$ and $x(3) = -1$.

   Hint:
   Since $x(0)$ is given, just let $x'(0) = B$ to obtain a solution in terms of B; then substitute the second boundary condition at the end.

16.3.4 ANSWERS TO EXERCISES

1. (a)

$$X(s) = \frac{3s-5}{s^2-2s+5},$$

giving

$$x(t) = e^t(3\cos 2t - \sin 2t) \quad t > 0;$$

(b)

$$X(s) = \frac{4}{s+\frac{1}{2}} + \frac{3}{(s+\frac{1}{2})^2},$$

giving

$$x(t) = 4e^{-\frac{1}{2}t} + 3te^{-\frac{1}{2}t} = e^{-\frac{1}{2}t}[4+3t] \quad t > 0;$$

(c)

$$X(s) = \frac{27}{12}.\frac{1}{s-2} + \frac{39}{48}.\frac{1}{s+4} - \frac{1}{4}.\frac{1}{s^2} - \frac{1}{16}.\frac{1}{s},$$

giving

$$x(t) = \frac{27}{12}e^{2t} + \frac{39}{48}e^{-4t} - \frac{1}{4}t - \frac{1}{16} \quad t > 0;$$

(d)

$$X(s) = \frac{\frac{1}{2}}{(s-2)^2} + \frac{3}{s-2} - \frac{2}{s+2},$$

giving

$$x(t) = \frac{1}{2}te^{2t} + 3e^{2t} - 2e^{-2t} \quad t > 0;$$

(e)

$$X(s) = \frac{3}{2}.\frac{s}{(s^2+4)^2} + \frac{3}{8}.\frac{1}{s} + \frac{5}{8}.\frac{s}{s^2+4} + \frac{2}{s^2+4},$$

giving

$$x(t) = \frac{3}{8}t\sin 2t + \frac{3}{8} + \frac{5}{8}\cos 2t + \sin 2t \quad t > 0.$$

2.

$$x(t) = 3e^t - te^t - 1 \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.4

LAPLACE TRANSFORMS 4
(Simultaneous differential equations)

by

A.J.Hobson

UNIT 16.4 - LAPLACE TRANSFORMS 4
SIMULTANEOUS DIFFERENTIAL EQUATIONS

16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In this Unit, we shall consider a pair of differential equations involving an independent variable, t, such as a time variable, and two dependent variables, x and y, such as electric currents or linear displacements.

The general format is as follows:

$$\begin{aligned} a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y &= f_1(t), \\ a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y &= f_2(t). \end{aligned}$$

To solve these equations simultaneously, we take the Laplace Transform of each equation obtaining two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t, \end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

Solution

Taking the Laplace Transforms of the differential equations,

$$sY(s) - 2 + 2X(s) = \frac{1}{s-1},$$

$$sX(s) - 1 - 2Y(s) \quad = \quad \frac{1}{s} + \frac{1}{s^2}.$$

That is,

$$\begin{aligned} 2X(s) + sY(s) &= \frac{1}{s-1} + 2, \qquad (1) \\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. \qquad (2) \end{aligned}$$

Using $(1) \times 2 + (2) \times s$, we obtain

$$(4 + s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Applying the methods of partial fractions, this gives

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating x from equations (1) and (2) in order to calculate y, and this is often necessary; but, since

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t$$

in the current example,

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$

16.4.2 EXERCISES

Use Laplace Transforms to solve the following pairs of simultaneous differential equations, subject to the given boundary conditions:

1.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2y &= e^{-t},\\ \frac{\mathrm{d}y}{\mathrm{d}t} + 3y &= x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

2.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} - y &= \sin t,\\ \frac{\mathrm{d}y}{\mathrm{d}t} + x &= \cos t,\end{aligned}$$

given that $x = 3$ and $y = 4$ when $t = 0$.

3.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 3y &= 1,\\ \frac{\mathrm{d}y}{\mathrm{d}t} - x + 2y &= e^{-2t},\end{aligned}$$

given that $x = 0$ and $y = 0$ when $t = 0$.

4.
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 8x,\end{aligned}$$

given that $x = 1$ and $y = 0$ when $t = 0$.

5.

$$\begin{aligned}
10\frac{\mathrm{d}x}{\mathrm{d}t} - 3\frac{\mathrm{d}y}{\mathrm{d}t} + 6x + 5y &= 0, \\
2\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} + 2x + y &= 2e^{-t},
\end{aligned}$$

given that $x = 2$ and $y = -1$ when $t = 0$.

6.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y &= 0, \\
5\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 6x &= 0,
\end{aligned}$$

given that $x = 3$ and $y = 0$ when $t = 0$.

7.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2y, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 2z, \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= 2x,
\end{aligned}$$

given that $x = 1$, $y = 0$ and $z = -1$ when $t = 0$.

16.4.3 ANSWERS TO EXERCISES

1.

$$x = (2t+1)e^{-t} \quad \text{and} \quad y = te^{-t}.$$

2.

$$x = (t+4)\sin t + 3\cos t \quad \text{and} \quad y = (t+4)\cos t - 3\sin t.$$

3.

$$x = 2 - e^{-2t}[1 + \sqrt{3}\sinh t\sqrt{3} + \cosh t\sqrt{3}]$$

and

$$y = 1 - e^{-2t}\left[\cosh t\sqrt{3} + \frac{1}{\sqrt{3}}\sinh t\sqrt{3}\right].$$

4.

$$x = \sinh 4t \quad \text{and} \quad y = 2\cosh 4t.$$

5.

$$x = 4\cos t - 2e^{-t} \quad \text{and} \quad y = e^{-t} - 2\cos t.$$

6.

$$x = 2e^{-t} + e^{-2t} \quad \text{amd} \quad y = e^{-t} - e^{-2t}.$$

7.

$$x = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} + \cos t\sqrt{3}\right],$$

$$y = \frac{-2}{\sqrt{3}}e^{-t}\sin t\sqrt{3}$$

and

$$z = e^{-t}\left[\frac{1}{\sqrt{3}}\sin t\sqrt{3} - \cos t\sqrt{3}\right].$$

"JUST THE MATHS"

UNIT NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

A.J.Hobson

UNIT 16.5 - LAPLACE TRANSFORMS 5

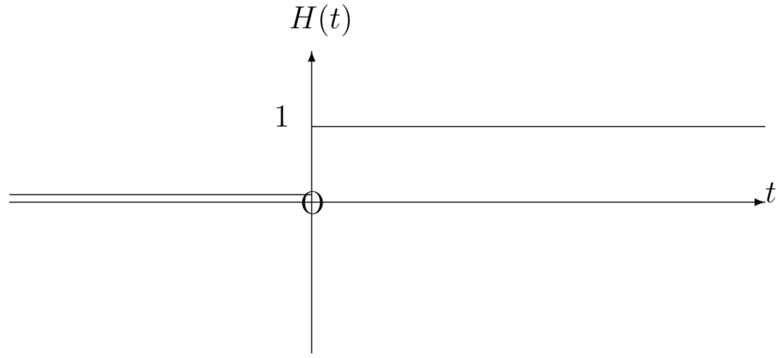
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The Heaviside Step Function, $H(t)$, is defined by the statements

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note:
$H(t)$ is undefined when $t = 0$.

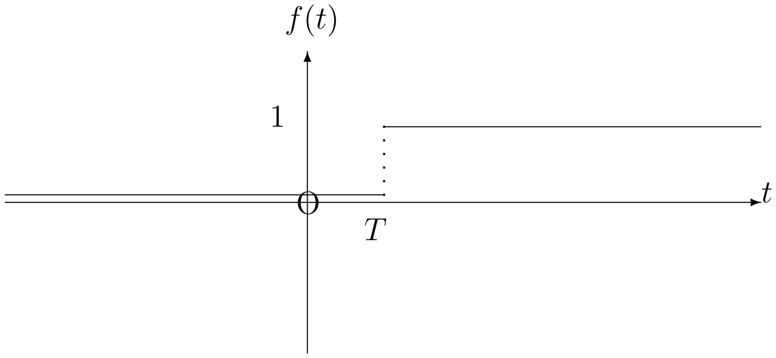


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



Clearly, $f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$. Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function $H(t-T)$ is of importance in constructing what are known as **"pulse functions"** (see later).

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t - T)] = \int_0^\infty e^{-st} H(t - T) \text{ d}t$$

$$= \int_0^T e^{-st}.0 \text{ d}t + \int_T^\infty e^{-st}.1 \text{ d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$, we have

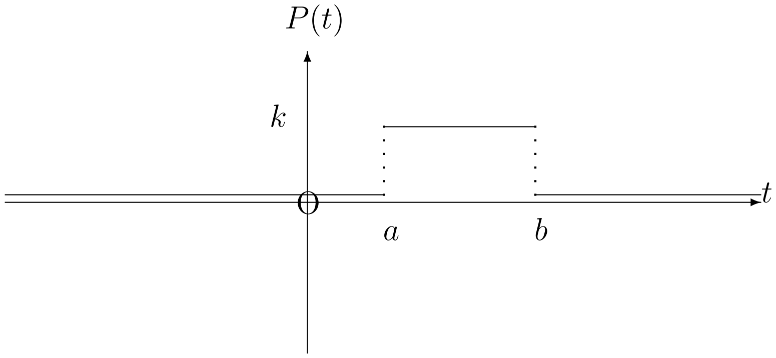
$$L[H(t)] = \frac{1}{s},$$

which can be expected since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration, $b - a$, and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



We can show that, in terms of Heaviside functions, the above pulse may be represented by

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$. Hence, the above right-hand side = 0.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$. Hence, the above right-hand side = 0.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$. Hence, the above right-hand side = k.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration, $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right] = k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse, described above, is defined as the area of the rectangle with base, $b - a$, and height, k. That is,

$$\text{strength} = k(b - a).$$

(ii) In general, the expression,

$$[H(t - a) - H(t - b)]f(t),$$

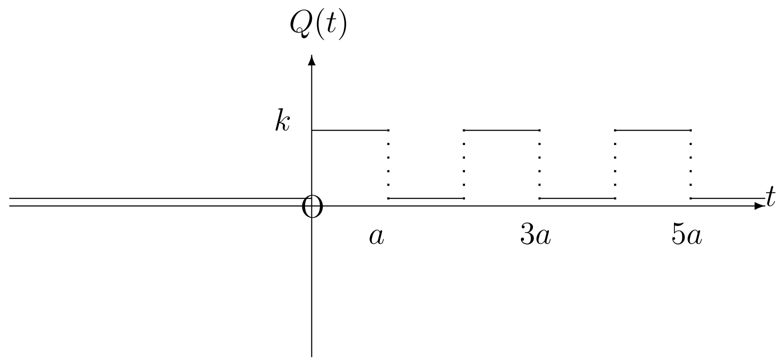
may be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) Similarly, the expression,

$$H(t - a)f(t),$$

may be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, the train of rectangular pulses, $Q(t)$, in the following diagram:



may be represented by the function

$$Q(t) \equiv k\left\{[H(t) - H(t - a)] + [H(t - 2a) - H(t - 3a)] + [H(t - 4a) - H(t - 5a)] + \ldots\ldots\ldots\right\}.$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$

$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$

$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution $u = t - T$, we obtain

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$

$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1. \end{cases}$$

and, hence, determine its Laplace Transform.

Solution

For values of $t > 0$, we may write

$$f(t) = (t-1)^2H(t-1).$$

Therefore, using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s}L[t^2] = e^{-s}.\frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression

$$\frac{e^{-7s}}{s^2+4s+5}.$$

Solution

First, we find the inverse Laplace Transform of the expression,

$$\frac{1}{s^2+4s+5} \equiv \frac{1}{(s+2)^2+1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t}\sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t-7)e^{-2(t-7)}\sin(t-7), \quad t > 0.$$

16.5.5 EXERCISES

1. (a) For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} e^{-t} & \text{for } 0 < t < 3; \\ 0 & \text{for } t > 3. \end{cases}$$

   (b) Determine the Laplace Transform of the function, $f(t)$, in part (a).

2. For values of $t > 0$, express, in terms of Heaviside functions, the function,

$$f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a; \\ f_2(t) & \text{for } t > a. \end{cases}$$

3. For values of $t > 0$, express the following functions in terms of Heaviside functions:

   (a)

$$f(t) = \begin{cases} t^2 & \text{for } 0 < t < 2; \\ 4t & \text{for } t > 2. \end{cases}$$

   (b)

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi; \\ \sin 2t & \text{for } \pi < t < 2\pi; \\ \sin 3t & \text{for } t > 2\pi. \end{cases}$$

4. Use the second shifting theorem to determine the Laplace Transform of the function,

$$f(t) \equiv t^3 H(t-1).$$

Hint:
Write $t^3 \equiv [(t-1)+1]^3$.

5. Determine the inverse Laplace Transforms of the following:

(a)

$$\frac{e^{-2s}}{s^2};$$

(b)

$$\frac{8e^{-3s}}{s^2+4};$$

(c)

$$\frac{se^{-2s}}{s^2+3s+2};$$

(d)

$$\frac{e^{-3s}}{s^2-2s+5}.$$

6. Solve the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = H(t-2),$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

16.5.6 ANSWERS TO EXERCISES

1. (a)

$$e^{-t}[H(t) - H(t-3)];$$

(b)

$$L[f(t)] = \frac{1-e^{-3(s+1)}}{s+1}.$$

2.

$$f(t) \equiv f_1(t)[H(t) - H(t-a)] + f_2(t)H(t-a)].$$

3. (a)

$$f(t) \equiv t^2[H(t) - H(t-2)] + 4tH(t-2);$$

(b)

$$f(t) \equiv \sin t[H(t) - H(t-\pi)] + \sin 2t[H(t-\pi) - H(t-2\pi)] + \sin 3t[H(t-2\pi).$$

4.

$$L[f(t)] = \left[\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right] e^{-s}.$$

5. (a)

$$H(t-2)(t-2);$$

(b)

$$4H(t-3)\sin 2(t-3);$$

(c)

$$H(t-2)[2e^{-2(t-2)} - e^{-(t-2)};$$

(d)

$$\frac{1}{2}H(t-3)e^{(t-3)}\sin 2(t-3).$$

6.

$$x = \frac{1}{2}\sin 2t + \frac{1}{4}H(t-2)[1 - \cos 2(t-2)].$$

"JUST THE MATHS"

UNIT NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

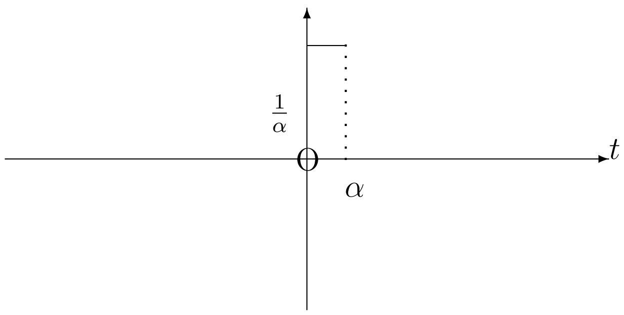
UNIT 16.6 - LAPLACE TRANSFORMS 6
THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**. In particular, a **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude, $\frac{1}{\alpha}$. The strength of the pulse is then 1.



From Unit 16.5, this pulse is given by

$$\frac{H(t) - H(t-\alpha)}{\alpha}.$$

If we now allow α to tend to zero, we obtain a unit impulse located at $t = 0$. This leads to the following definition:

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$ is defined to be an impulse of unit strength located at $t = 0$. It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t-\alpha)}{\alpha}.$$

Notes:

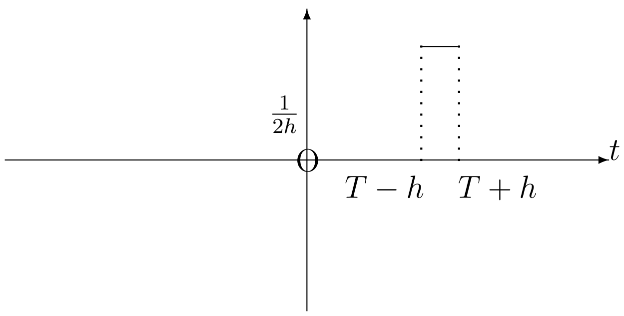
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t - T)$.

(ii) An alternative definition of the function $\delta(t - T)$ is as follows:

$$\delta(t - T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T. \end{cases}$$

and

$$\lim_{h \to 0} \int_{T-h}^{T+h} \delta(t - T) \ \mathrm{d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t - T) \ \mathrm{d}t = f(T) \ \text{ if } \ a < T < b.$$

Proof:

Since $\delta(t - T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h \to 0} \int_{T-h}^{T+h} f(t)\delta(t - T) \ \mathrm{d}t.$$

But, in the <u>small</u> interval from $T - h$ to $T + h$, $f(t)$ is approximately constant and equal to $f(T)$. Hence, the left-hand side may be written

$$f(T) \left[\lim_{h \to 0} \int_{T-h}^{T+h} \delta(T - T) \ \mathrm{d}t \right],$$

which reduces to $f(T)$, using note (ii) in the definition of the Dirac unit impulse function.

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular,

$$L[\delta(t)] = 1.$$

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T)\ \mathrm{d}t.$$

But, from the Theorem discussed above, with $f(t) = e^{-st}$, we have

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

given that $x = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$3sX(s) + 4X(s) = 1.$$

That is,

$$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

Hence,

$$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution

Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t = a$. The required result follows from the Theorem discussed earlier, with $T = a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of an ordinary differential equation having the form

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function** $f(t)$**"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

It is also customary to refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$; that is, we shall assume zero initial conditions.

Impulse response function and transfer function

Consider, for the moment, the differential equation having the form,

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

Here, we refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2 + bs + c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c},$$

which is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1$$

and, hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System response for any input

Assuming zero initial conditions, the Laplace Transform of the differential equation

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t)$$

is given by

$$(as^2 + bs + c)X(s) = F(s),$$

which means that

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$ which may possibly be found using partial fractions but may, if necessary, be found by using the Convolution Theorem referred to in Unit 16.1

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s);$$

in other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be $u(t) = \frac{10e^{-t}}{3}$.

Determine the response, $x(t)$, of the system to an input of $f(t) \equiv \sin 3t$.

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \quad \text{and} \quad F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9},$$

using partial fractions.

Thus

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \sin 3T.\frac{10e^{-(t-T)}}{3}\,\mathrm{d}T;$$

but the integration here can be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \frac{10}{3}e^{-t}.e^{(1+j3)T}\,\mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m \left(\frac{10}{3} \left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3} \right] \right)$$

$$= I_m \left(\frac{10}{3} \left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10} \right] \right)$$

$$= \frac{10}{3} \left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10} \right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
Clearly, in this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

In the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots; that is, it gives rise to an impulse response, $u(t)$, involving negative powers of e and, hence, tending to zero as t tends to infinity.

Suppose also that $f(t)$ takes one of the forms, $\cos\omega t$ or $\sin\omega t$, which may be regarded, respectively, as the real and imaginary parts of the function, $e^{j\omega t}$.

It turns out that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity, together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = e^{j7t},$$

where $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - 2) - 1 + 3(sX(s) - 2) + 2X(s) = \frac{1}{s - j7}.$$

That is,

$$(s^2 + 3s + 2)X(s) = 2s + 7 + \frac{1}{s - j7},$$

giving

$$X(s) = \frac{2s + 7}{s^2 + 3s + 2} + \frac{1}{(s - j7)(s^2 + 3s + 2)} = \frac{2s + 7}{(s + 2)(s + 1)} + \frac{1}{(s - j7)(s + 2)(s + 1)}.$$

Using the "cover-up" rule for partial fractions, we obtain

$$X(s) = \frac{5}{s + 1} - \frac{3}{s + 2} + \frac{1}{(-1 - j7)(s + 1)} + \frac{1}{(2 + j7)(s + 2)} + \frac{U(j7)}{(s - j7)},$$

where

$$U(s) \equiv \frac{1}{s^2 + 3s + 2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t} + \frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity, so that the final term represents the steady state response; we need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$

16.6.5 EXERCISES

1. Evaluate
$$\int_0^\infty e^{-4t}\delta'(t-2)\ \mathrm{d}t.$$

2. In the following cases, solve the differential equation
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = f(t),$$
where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$:

   (a)
$$f(t) \equiv \delta(t);$$

   (b)
$$f(t) \equiv \delta(t-2).$$

3. Determine the transfer function and impulse response function for the differential equation,
$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t),$$
assuming zero initial conditions.

4. The impulse response function of a system is known to be $u(t) = e^{3t}$. Determine the response, $x(t)$, of the system to an input of $f(t) \equiv 6\cos 3t$.
5. Determine the steady-state response to the system

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + x = f(t)$$

in the cases when

(a)

$$f(t) \equiv e^{j2t};$$

(b)

$$f(t) \equiv 3\cos 2t.$$

16.6.6 ANSWERS TO EXERCISES

1.

$$4e^{-8}.$$

2. (a)

$$x = \sin 2t \quad t > 0;$$

(b)

$$x = \sin t + H(t-2)\sin(t-2) \quad t \neq 2.$$

3.

$$U(s) = \frac{1}{2s^2 - 3s + 1} \quad \text{and} \quad u(t) = [e^t - e^{\frac{1}{2}t}].$$

4.

$$\frac{1}{13}\left[18e^{3t} - 18\cos 2t + 12\sin 2t\right] \quad t > 0.$$

5. (a)

$$\frac{(1-j6)e^{j2t}}{37} \quad t > 0;$$

(b)

$$\frac{1}{37}(\cos 2t + 6\sin 2t) \quad t > 0.$$

"JUST THE MATHS"

UNIT NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

by

A.J.Hobson

One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) Let us consider that our main problem is to solve a second order linear differential equation with constant coefficients, the general form of which is

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

(ii) Assuming that the solution of an equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

has already been included in previous knowledge, we examine a typical worked example as follows:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{2t},$$

given that $x = 0$ when $t = 0$.

Solution

A method called the **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation.

The integrating factor in the current example is e^{3t} since the coefficient of x is 3.

We obtain, therefore,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t} + 3x\right] = e^{5t}.$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] = e^{5t}.$$

On integrating both sides with respect to t,

$$xe^{3t} = \frac{e^{5t}}{5} + C$$

or

$$x = \frac{e^{2t}}{5} + Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$ and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) As a lead up to what follows, we shall now examine a different way of setting out the above working in which we do not leave the substitution of the boundary condition until the very end.

We multiply both sides of the differential equation by e^{3t} as before, but we then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right] \mathrm{d}t = \int_0^t e^{5t} \mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5}.$$

since $x = 0$ when $t = 0$.

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before.

(iv) Now let us consider whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

Supposing that there might be an integrating factor for this equation, we shall take it to be e^{st} where s, at present, is unknown, but assumed to be positive.

Multiplying throughout by e^{st} and integrating from 0 to t, as in the previous example,

$$\int_0^t e^{st}\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x\right]\ \mathrm{d}t = \int_0^t e^{(s+9)t}\ \mathrm{d}t = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Now, using integration by parts, with the boundary condition,

$$\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = e^{st}x - s\int_0^t e^{st}x\ \mathrm{d}t$$

and

$$\int_0^t e^{st}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\ \mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - s\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - se^{st}x + s^2\int_0^t e^{st}x\ \mathrm{d}t.$$

On substituting these results into the differential equation, we may collect together (on the left hand side) terms which involve $\int_0^t e^{st}x\ \mathrm{d}t$ and e^{st} as follows:

$$(s^2 + 10s + 21)\int_0^t e^{st}x\ \mathrm{d}t + e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right] = \frac{e^{(s+9)t}}{s+9} - \frac{1}{s+9}.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation; that is, $(s^2 - 10s + 21)$.

(b) Using e^{-st} with $s > 0$, if we had integrated from 0 to ∞ instead of 0 to t, the second term on the left hand side above would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞ to obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \text{ d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Of course, this works only if $s > 9$, but we can easily assume that it is so. Hence,

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{(s-9)(s^2-10s+21)} = \frac{1}{(s-9)(s-3)(s-7)}.$$

Applying the principles of partial fractions, we obtain

$$\int_0^\infty e^{-st}x \text{ d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) But, finally, it can be shown by an independent method of solution that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

and we may conclude that the solution of the differential equation is closely linked to the integral

$$\int_0^\infty e^{-st}x \text{ d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.

"JUST THE MATHS"

UNIT NUMBER

16.8

Z-TRANSFORMS 1
(Definition and rules)

by

A.J.Hobson

UNIT 16.8 - Z TRANSFORMS 1 - DEFINITION AND RULES

16.8.1 INTRODUCTION - Linear Difference Equations

Closely linked with the concept of a linear <u>differential</u> equation with constant coefficients is that of a **"linear <u>difference</u> equation with constant coefficients"**.

Two particular types of difference equation to be discussed in the present section may be defined as follows:

DEFINITION 1

A first-order linear difference equation with constant coefficients has the general form,

$$a_1 u_{n+1} + a_0 u_n = f(n),$$

where a_0, a_1 are constants, n is a positive integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots .$

DEFINITION 2

A second-order linear difference equation with constant coefficients has the general form,

$$a_2 u_{n+2} + a_1 u_{n+1} + a_0 u_n = f(n),$$

where a_0, a_1, a_2 are constants, n is an integer, $f(n)$ is a given function of n (possibly zero) and u_n is the general term of an infinite sequence of numbers, $\{u_n\} \equiv u_0,\ u_1,\ u_2\ , u_3,\ \ldots .$

Notes:

(i) We shall assume that the sequences under discussion are such that $u_n = 0$ whenever $n < 0$.

(ii) Difference equations are usually associated with given "boundary conditions", such as the value of u_0 for a first-order equation or the values of u_0 and u_1 for a second-order equation.

ILLUSTRATION

Certain **simple** difference equations may be solved by very elementary methods.

For example, suppose that we wish to solve the difference equation,

$$u_{n+1} - (n+1)u_n = 0,$$

subject to the boundary condition that $u_0 = 1$.

We may rewrite the difference equation as

$$u_{n+1} = (n+1)u_n$$

and, by using this formula repeatedly, we obtain

$$u_1 = u_0 = 1, \quad u_2 = 2u_1 = 2, \quad u_3 = 3u_2 = 3 \times 2, \quad u_4 = 4u_3 = 4 \times 3 \times 2, \quad . \ . \ . \ .$$

In general, for this illustration, $u_n = n!$.

However, not all difference equations can be solved as easily as this and we shall now discuss the Z-Transform method of solving more advanced types.

16.8.2 STANDARD DEFINITION AND RESULTS

THE DEFINITION OF A Z-TRANSFORM (WITH EXAMPLES)

The Z-Transform of the sequence of numbers, $\{u_n\} \equiv u_0, \ u_1, \ u_2, \ u_3, \ . \ . \ . \ .$, is defined by the formula,

$$Z\{u_n\} = \sum_{r=0}^{\infty} u_r z^{-r},$$

provided that the series converges (allowing for z to be a complex number if necessary).

EXAMPLES

1. Determine the Z-Transform of the sequence,

   $$\{u_n\} \equiv \{a^n\},$$

   where a is a non-zero constant.

Solution

$$Z\{a^n\} = \sum_{r=0}^{\infty} a^r z^{-r}.$$

That is,

$$Z\{a^n\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ .\ .\ .\ .\ = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a},$$

by properties of infinite geometric series.
Thus,

$$Z\{a^n\} = \frac{z}{z - a}.$$

2. Determine the Z-Transform of the sequence,

$$\{u_n\} = \{n\}.$$

Solution

$$Z\{n\} = \sum_{r=0}^{\infty} rz^{-r}.$$

That is,

$$Z\{n\} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ .\ .\ .\ .\ ,$$

which may be rearranged as

$$Z\{n\} = \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ .\ .\ .\ .\right) + \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ .\ .\ .\ .\right) + \left(\frac{1}{z^3} + \frac{1}{z^4} + \ .\ .\ .\ .\right),$$

giving

$$Z\{n\} = \frac{\frac{1}{z}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^2}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^3}}{1 - \frac{1}{z}} + \ .\ .\ .\ . = \frac{1}{1 - \frac{1}{z}}\left[\frac{\frac{1}{z}}{1 - \frac{1}{z}}\right],$$

by properties of infinite geometric series.
Thus,

$$Z\{n\} = \frac{z}{(1 - z)^2} = \frac{z}{(z - 1)^2}.$$

Note:
Other Z-Transforms may be obtained, in the same way as in the above examples, from the definition.

We list, here, for reference, a short table of standard Z-Transforms, including those already proven:

A SHORT TABLE OF Z-TRANSFORMS

$\{u_n\}$	$Z\{u_n\}$	Region of Existence
$\{1\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{a^n\}$ (a constant)	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\left\{e^{-nT}\right\}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\lvert z\rvert > e^{-T}$
$\sin nT$ (T constant)	$\frac{z\sin T}{z^2-2z\cos T+1}$	$\lvert z\rvert > 1$
$\cos nT$ (T constant)	$\frac{z(z-\cos T)}{z^2-2z\cos T+1}$	$\lvert z\rvert > 1$
1 for $n=0$ 0 for $n>0$ (Unit pulse sequence)	1	All z
0 for $n=0$ $\{a^{n-1}\}$ for $n>0$	$\frac{1}{z-a}$	$\lvert z\rvert > \lvert a\rvert$

16.8.3 PROPERTIES OF Z-TRANSFORMS

(a) Linearity

If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, while A and B are constants, then

$$Z\{Au_n + Bv_n\} \equiv A.Z\{u_n\} + B.Z\{v_n\}.$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} (Au_r + Bv_r)z^{-r} \equiv A\sum_{r=0}^{\infty} u_r z^{-r} + B\sum_{r=0}^{\infty} v_r z^{-r},$$

which, in turn, is equivalent to the right-hand side.

EXAMPLE

$$Z\{5.2^n - 3n\} = \frac{5z}{z-2} - \frac{3z}{(z-1)^2}.$$

(b) The First Shifting Theorem

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.Z\{u_n\},$$

where $\{u_{n-1}\}$ denotes the sequence whose first term, corresponding to $n = 0$, is taken as zero and whose subsequent terms, corresponding to $n = 1, \ 2, \ 3, \ 4, \ \ldots.$, are the terms $u_0, \ u_1, \ u_3, \ u_4, \ \ldots$ of the original sequence.

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r-1} z^{-r} \equiv \frac{u_0}{z} + \frac{u_1}{z^2} + \frac{u_2}{z^3} + \frac{u_3}{z^4} + \ \ldots.,$$

since it is assumed that $u_n = 0$ whenever $n < 0$.

Thus,

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \ .\ .\ .\ .\right],$$

which is equivalent to the right-hand side.

Note:
A more general form of the first shifting theorem states that

$$Z\{u_{n-k}\} \equiv \frac{1}{z^k}.Z\{u_n\},$$

where $\{u_{n-k}\}$ denotes the sequence whose first k terms, corresponding to $n = 0,\ 1,\ 2,\ .\ .\ .\ .\ , k-1$, are taken as zero and whose subsequent terms, corresponding to $n = k,\ k+1,\ k+2,\ .\ .\ .\ .$ are the terms $u_0,\ u_1,\ u_2,\ .\ .\ .\ .$ of the original sequence.

ILLUSTRATION

Given that $\{u_n\} \equiv \{4^n\}$, we may say that

$$Z\{u_{n-2}\} \equiv \frac{1}{z^2}.Z\{u_n\} \equiv \frac{1}{z^2}.\frac{z}{z-4} \equiv \frac{1}{z(z-4)}.$$

Note:
In this illustration, the sequence, $\{u_{n-2}\}$ has terms $0,\ 0,\ 1,\ 4,\ 4^2,\ 4^3,\ .\ .\ .$ and, by applying the definition of a Z-Transform directly, we would obtain

$$Z\{u_{n-2}\} = \frac{1}{z^2} + \frac{4}{z^3} + \frac{4^2}{z^4} + \frac{4^3}{z^5}\ .\ .\ .\ .,$$

which gives

$$Z\{u_n\} \equiv \frac{1}{z^2}.\frac{1}{1-\frac{4}{z}} \equiv \frac{1}{z(z-4)},$$

by properties of infinite geometric series.

(c) The Second Shifting Theorem

$$Z\{u_{n+1}\} \equiv z.Z\{u_n\} - z.u_0$$

Proof:

The left-hand side of the above identity is equivalent to

$$\sum_{r=0}^{\infty} u_{r+1}z^{-r} \equiv u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \frac{u_4}{z^4} + \ \ldots .$$

This may be rearranged as

$$z.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \ \ldots .\right] - z.u_0$$

which, in turn, is equivalent to the right-hand side.

Note:
This **"recursive relationship"** may be applied repeatedly. For example, we may deduce that

$$Z\{u_{n+2}\} \equiv z.Z\{u_{n+1}\} - z.u_1 \equiv z^2.Z\{u_n\} - z^2.u_0 - z.u_1$$

16.8.4 EXERCISES

1. Determine, from first principles, the Z-Transforms of the following sequences, $\{u_n\}$:

   (a)

   $$\{u_n\} \equiv \{e^{-n}\};$$

   (b)

   $$\{u_n\} \equiv \{\cos \pi n\}.$$

2. Determine the Z-Transform of the following sequences:

   (a)
$$\{u_n\} \equiv \{7.(3)^n - 4.(-1)^n\}\,;$$

   (b)
$$\{u_n\} \equiv \left\{6n + 2e^{-5n}\right\};$$

   (c)
$$\{u_n\} \equiv \{13 + \sin 2n - \cos 2n\}\,.$$

3. Determine the Z-Transform of $\{u_{n-1}\}$ and $\{u_{n-2}\}$ for the sequences in question 1.

4. Determine the Z-Transform of $\{u_{n+1}\}$ and $\{u_{n+2}\}$ for the sequences in question 1.

16.8.5 ANSWERS TO EXERCISES

1. (a)
$$\frac{ez}{ez-1};$$

   (b)
$$\frac{z}{z+1}.$$

2. (a)
$$\frac{7z}{z-3} - \frac{4z}{z+1};$$

   (b)
$$\frac{6z}{(z-1)^2} + \frac{2z}{z-e^{-5}};$$

   (c)
$$\frac{13z}{z-1} + \frac{z(\sin 2 + \cos 2 - z)}{z^2 - 2z\cos 2 + 1}.$$

3. (a)

$$Z\{u_{n-1}\} \equiv \frac{e}{ez-1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{e}{z(ez-1)}\ (n>1);$$

(b)

$$Z\{u_{n-1}\} \equiv \frac{1}{z+1}\ (n>0), \quad Z\{u_{n-2}\} \equiv \frac{1}{z(z+1)}\ (n>1).$$

Note:
$u_{-2} = 0$ and $u_{-1} = 0$.

4. (a)

$$Z\{u_{n+1}\} \equiv \frac{z}{ez-1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{e(ez-1)};$$

(b)

$$Z\{u_{n+1}\} \equiv -\frac{z}{z+1}, \quad Z\{u_{n+2}\} \equiv \frac{z}{z+1}.$$

"JUST THE MATHS"

UNIT NUMBER

16.9

Z-TRANSFORMS 2
(Inverse Z-Transforms)

by

A.J.Hobson

UNIT 16.9 - Z TRANSFORMS 2

INVERSE Z - TRANSFORMS

16.9.1 THE USE OF PARTIAL FRACTIONS

When solving linear difference equations by means of Z-Transforms, it is necessary to be able to determine a sequence, $\{u_n\}$, of numbers, whose Z-Transform is a known function, $F(z)$, of z. Such a sequence is called the **"inverse Z-Transform of $F(z)$"** and may be denoted by $\mathrm{Z}^{-1}[F(z)]$.

For simple difference equations, the function $F(z)$ turns out to be a rational function of z, and the method of partial fractions may be used to determine the corresponding inverse Z-Transform.

EXAMPLES

1. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{10z(z+5)}{(z-1)(z-2)(z+3)}.$$

Solution

Bearing in mind that

$$Z\{a^n\} = \frac{z}{z-a},$$

for any non-zero constant, a, we shall write

$$F(z) \equiv z.\left[\frac{10(z+5)}{(z-1)(z-2)(z+3)}\right],$$

which gives

$$F(z) \equiv z.\left[\frac{-15}{z-1} + \frac{14}{z-2} + \frac{1}{z+3}\right]$$

or

$$F(z) \equiv \frac{z}{z+3} + 14\frac{z}{z-2} - 15\frac{z}{z-1}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \{(-3)^n + 14(2)^n - 15\}.$$

2. Determine the Inverse Z-Transform of the function

$$F(z) \equiv \frac{1}{z-a}.$$

Solution

In this example, there is no factor, z, in the function $F(z)$ and we shall see that it is necessary to make use of the first shifting theorem.

First, we may write

$$F(z) \equiv \frac{1}{z}\left[\frac{z}{z-a}\right]$$

and, since the inverse Z-Transform of the expression inside the brackets is a^n, the first shifting theorem tells us that

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ a^{n-1} & \text{when } n > 0. \end{cases}$$

Note:

This may now be taken as a standard result.

3. Determine the inverse Z-Transform of the function

$$F(z) \equiv \frac{4(2z+1)}{(z+1)(z-3)}.$$

Solution

Expressing $F(z)$ in partial fractions, we obtain

$$F(z) \equiv \frac{1}{z+1} + \frac{7}{(z-3)}.$$

Hence,

$$\mathrm{Z}^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ (-1)^{n-1} + 7.(3)^{n-1} & \text{when } n > 0. \end{cases}$$

16.9.2 EXERCISES

1. Determine the inverse Z-Transforms of each of the following functions, $F(z)$:

   (a)
   $$F(z) \equiv \frac{z}{z-1};$$

   (b)
   $$F(z) \equiv \frac{z}{z+1};$$

   (c)
   $$F(z) \equiv \frac{2z}{2z-1};$$

   (d)
   $$F(z) \equiv \frac{z}{3z+1};$$

   (e)
   $$F(z) \equiv \frac{z}{(z-1)(z+2)};$$

   (f)
   $$F(z) \equiv \frac{z}{(2z+1)(z-3)};$$

   (g)
   $$F(z) \equiv \frac{z^2}{(2z+1)(z-1)}.$$

2. Determine the inverse Z-Transform of each of the following functions, $F(z)$, and list the first five terms of the sequence obtained:

   (a)
   $$F(z) \equiv \frac{1}{z-1};$$

   (b)
   $$F(z) \equiv \frac{z+2}{z+1};$$

(c)

$$F(z) \equiv \frac{z-3}{(z-1)(z-2)};$$

(d)

$$F(z) \equiv \frac{2z^2 - 7z + 7}{(z-1)^2(z-2)}.$$

16.9.3 ANSWERS TO EXERCISES

1. (a)

$$\mathrm{Z}^{-1}[F(z)] = \{1\}$$

(b)

$$\mathrm{Z}^{-1}[F(z)] = \{(-1)^n\}$$

(c)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\left(\frac{1}{2}\right)^n\right\};$$

(d)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}\left(-\frac{1}{3}\right)^n\right\};$$

(e)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3}[1 - (-2)^n]\right\};$$

(f)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{7}\left[(3)^n - \left(-\frac{1}{2}\right)^n\right]\right\};$$

(g)

$$\mathrm{Z}^{-1}[F(z)] = \left\{\frac{1}{3} + \frac{1}{6}\left(-\frac{1}{2}\right)^n\right\}.$$

2. (a)

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 1 & \text{when } n > 0; \end{cases}$$

The first five terms are 0,1,1,1,1

(b)

$$Z^{-1}[F(z)] = \begin{cases} 1 & \text{when } n = 0; \\ (-1)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 1,1,−1,1,−1

(c)

$$Z^{-1}[F(z)] = \begin{cases} 0 & \text{when } n = 0; \\ 2 - (2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,1,0,−2,−6

(d)

$$Z^{-1}[F(z] = \begin{cases} 0 & \text{when } n = 0; \\ 3 - 2n + (2)^{n-1} & \text{when } n > 0. \end{cases}$$

The first five terms are 0,2,1,1,3

"JUST THE MATHS"

UNIT NUMBER

16.10

Z-TRANSFORMS 3
(Solution of linear difference equations)

by

A.J.Hobson

UNIT 16.10 - Z TRANSFORMS 3

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear difference equations may be solved by constructing the Z-Transform of both sides of the equation. The method will be illustrated with linear difference equations of the first and second orders (with constant coefficients).

16.10.1 FIRST ORDER LINEAR DIFFERENCE QUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that $u_0 = 2/5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}$$

$$\equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z - \frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z - \frac{1}{3}}.$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and $u_0 = 5$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.5$$

Taking the Z-Transform of the difference equation, we obtain

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation

$$u_{n+2} = u_{n+1} + u_n,$$

given that $u_0 = 0$ and $u_1 = 1$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0 \equiv z.Z\{u_n\}$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\},$$

so that, on rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1},$$

which may be written

$$Z\{u_n\} = \frac{z}{(z-\alpha)(z-\beta)},$$

where, from the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta}\left[\frac{z}{z-\alpha} - \frac{z}{z-\beta}\right].$$

Taking the inverse Z-Transform of this function of z gives the solution

$$\{u_n\} \equiv \left\{\frac{1}{\alpha - \beta}\left[(\alpha)^n - (\beta)^n\right]\right\}.$$

2. Solve the linear difference equation

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that $u_0 = 6$ and $u_1 = 2$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2},$$

which, on rearrangement, gives

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2};$$

and, hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions, we obtain

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

The solution of the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation

$$u_{n+2} + 2u_n = 0$$

given that $u_0 = 1$ and $u_1 = \sqrt{2}$.

Solution

First of all, using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation, we obtain

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0,$$

which, on rearrangement, gives

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{z^2 + 2} \equiv z.\frac{z + \sqrt{2}}{(z + j\sqrt{2})(z - j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z\left[\frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z - j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z + j\sqrt{2})}\right] \equiv z.\left[\frac{(1-j)}{2(z - j\sqrt{2})} + \frac{(1+j)}{2(z + j\sqrt{2})}\right],$$

so that

$$\{u_n\} \equiv \left\{\frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[(1-j)(j)^n + (1+j)(-j)^n\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[\sqrt{2}e^{-j\frac{\pi}{4}}.e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}}.e^{-j\frac{n\pi}{2}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}\left[e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}.2\cos\frac{(2n-1)\pi}{4}\right\}$$

$$\equiv \left\{(\sqrt{2})^{n+1}\cos\frac{(2n-1)\pi}{4}\right\}.$$

16.10.3 EXERCISES

1. Solve the following first-order linear difference equations:

   (a)
   $$3u_{n+1} + 2u_n = (-1)^n,$$
   given that $u_0 = 0$;

   (b)
   $$u_{n+1} - 5u_n = 3(2)^n,$$
   given that $u_0 = 1$;

   (c)
   $$u_{n+1} + u_n = n,$$
   given that $u_0 = 1$;

   (d)
   $$u_{n+1} + 2u_n = f(n),$$
   where
   $$f(n) \equiv \begin{cases} 3 & \text{when } n = 0; \\ 0 & \text{when } n > 0; \end{cases}$$
   and $u_0 = 2$;

   (e)
   $$u_{n+1} - 3u_n = \sin\frac{n\pi}{2} + \frac{1}{2}\cos\frac{n\pi}{2},$$
   given that $u_0 = 0$.

2. Solve the following second-order linear difference equations:

   (a)
   $$u_{n+2} - 2u_{n+1} + u_n = 0,$$
   given that $u_0 = 0$ and $u_1 = 1$;

   (b)
   $$u_{n+2} - 4u_n = n,$$
   given that $u_0 = 0$ and $u_1 = 1$;

(c)

$$u_{n+2} - 8u_{n+1} - 9u_n = 24,$$

given that $u_0 = 2$ and $u_1 = 0$;

(d)

$$6u_{n+2} + 5u_{n+1} - u_n = 20,$$

given that $u_0 = 3$ and $u_1 = 8$;

(e)

$$u_{n+2} + 2u_{n+1} - 15u_n = 32\cos n\pi,$$

given that $u_0 = 0$ and $u_1 = 0$;

(f)

$$u_{n+2} - 3u_{n+1} + 3u_n = 5,$$

given that $u_0 = 5$ and $u_1 = 8$.

16.10.4 ANSWERS TO EXERCISES

1. (a)

$$\{u_n\} \equiv \left\{\left(-\frac{2}{3}\right)^n - (-1)^n\right\};$$

(b)

$$\{u_n\} \equiv \{2(5)^n - (2)^n\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}n - \frac{1}{4} + \frac{5}{4}(-1)^n\right\};$$

(d)

$$\{u_n\} \equiv 2(-2)^n + 3(-2)^{n-1} \quad \text{when} \quad n > 0;$$

(e)

$$\{u_n\} \equiv \left\{\frac{1}{4}\left[(3^n - \sqrt{2}\cos\frac{(2n-1)\pi}{4}\right]\right\}.$$

2. (a)

$$\{u_n\} \equiv \{n\};$$

(b)

$$\{u_n\} \equiv \left\{\frac{1}{2}(2)^n - \frac{1}{3}n - \frac{5}{18}(-2)^n - \frac{2}{9}\right\};$$

(c)

$$\{u_n\} \equiv \left\{\frac{1}{2}(9)^n + 3(-1)^n - \frac{3}{2}\right\};$$

(d)

$$\{u_n\} \equiv \left\{2 + (6)^{1-n} - 5(-1)^n\right\};$$

(e)

$$\{u_n\} \equiv \left\{2(-1)^{n+1} + (3)^n + (-5)^n\right\};$$

(f)

$$\{u_n\} \equiv \left\{5 + (2\sqrt{3})^{n+1} \cos\frac{(n-3)\pi}{6}\right\}.$$

"JUST THE MATHS"

UNIT NUMBER

17.1

NUMERICAL MATHEMATICS 1
(Approximate solution of equations)

by

A.J.Hobson

UNIT 17.1 - NUMERICAL MATHEMATICS 1

THE APPROXIMATE SOLUTION OF ALGEBRAIC EQUATIONS

17.1.1 INTRODUCTION

In the work which follows, we shall consider the solution of the equation

$$f(x) = 0,$$

where $f(x)$ is a given function of x.

It is assumed that examples of such equations will have been encountered earlier at an elementary level; as, for instance, with quadratic equations where there is simple formula for obtaining solutions.

However, the equation

$$f(x) = 0$$

cannot, in general, be solved algebraically to give **exact** solutions and we have to be satisfied, at most, with **approximate** solutions. Nevertheless, it is often possible to find approximate solutions which are correct to any specified degree of accuracy; and this is satisfactory for the applications of mathematics to science and engineering.

It is certainly possible to consider **graphical** methods of solving the equation

$$f(x) = 0,$$

where we try to plot a graph of the equation

$$y = f(x),$$

then determine where the graph crosses the x-axis. But this method can be laborious and inaccurate and will not be discussed, here, as a viable method.

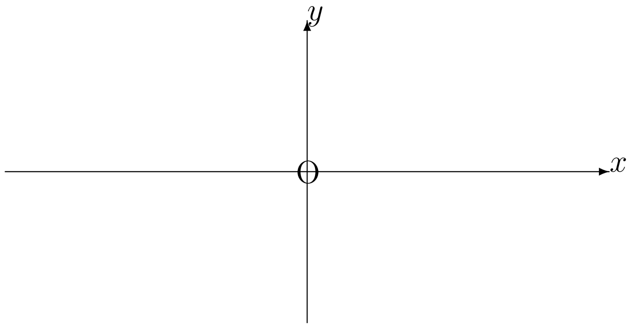
Three so called **"iterative"** methods will be included, below, where repeated use of the method is able to improve the accuracy of an approximate solution, already obtained.

17.1.2 THE BISECTION METHOD

Suppose a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. We may obtain these by trial and error or by sketching, roughly, the graph of the equation

$$y = f(x),$$

in order to estimate convenient values a and b between which the graph crosses the x-axis; whole numbers will usually suffice.



If we let $c = (a + b)/2$, there are three possibilities;

(i) $f(c) = 0$, in which case we have solved the equation;

(ii) $f(c) < 0$, in which case there is a solution between c and b enabling us repeat the procedure with these two numbers;

(iii) $f(c) > 0$, in which case there is a solution between c and a enabling us to repeat the procedure with these two numbers.

Each time we apply the method, we bisect the interval between the two numbers being used so that, eventually, the two numbers used will be very close together. The method stops when two consecutive values of the mid-point agree with each other to the required number of decimal places or significant figures.

Convenient labels for the numbers used at each stage (or iteration) are

$$a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, ..., a_n, b_n, c_n, ...$$

EXAMPLE

Determine, correct to three decimal places, the positive solution of the equation

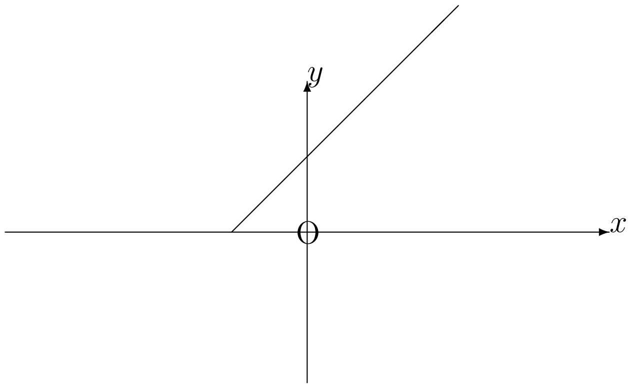
$$e^x = x + 2.$$

Solution

We could first observe, from a rough sketch of the graphs of

$$y = e^x \quad \text{and} \quad y = x + 2,$$

that the graphs intersect each other at a positive value of x. This confirms that there is indeed a positive solution to our equation.



But now let

$$f(x) = e^x - x - 2$$

and look for two numbers between which $f(x)$ changes sign from positive to negative. By trial and error, suitable numbers are 1 and 2, since

$$f(1) = e - 3 < 0 \quad \text{and} \quad f(2) = e^2 - 5 > 0.$$

The rest of the solution may be set out in the form of a table as follows:

n	a_n	b_n	c_n	$f(c_n)$
0	1.00000	2.00000	1.50000	0.98169
1	1.00000	1.50000	1.25000	0.24034
2	1.00000	1.25000	1.12500	- 0.04478
3	1.12500	1.25000	1.18750	0.09137
4	1.12500	1.18750	1.15625	0.02174
5	1.12500	1.15625	1.14062	- 0.01191
6	1.14063	1.15625	1.14844	0.00483
7	1.14063	1.14844	1.14454	- 0.00354
8	1.14454	1.14844	1.14649	0.00064
9	1.14454	1.14649	1.14552	- 0.00144

As a general rule, it is appropriate to work to two more places of decimals than that of the required accuracy; and so, in this case, we work to five.

We can stop at stage 9, since c_8 and c_9 are the same value when rounded to three places of decimals. The required solution is therefore $\boxed{x = 1.146}$

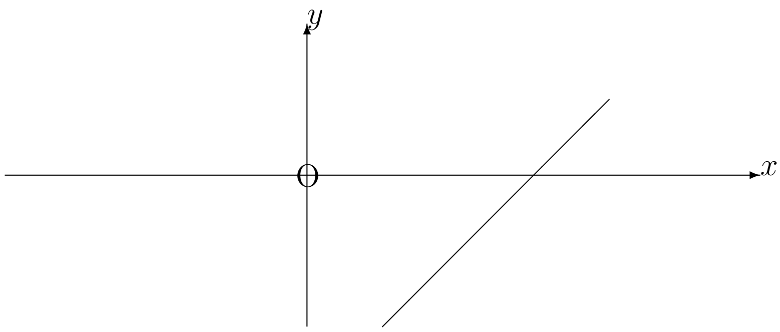
17.1.3 THE RULE OF FALSE POSITION

This method is commonly known by its Latin name, **"Regula Falsi"**, and tries to compensate a little for the shortcomings of the Bisection Method.

Instead of taking c as the average of a and b, we consider that the two points, $(a, f(a))$ and $(b, f(b))$, on the graph of the equation,

$$y = f(x),$$

are joined by a straight line; and the point at which this straight line crosses the x-axis is taken as c.



From elementary co-ordinate geometry, the equation of the straight line is given by

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}.$$

Hence, when $y = 0$, we obtain

$$x = a - \frac{(b - a)f(a)}{f(b) - f(a)}.$$

That is,

$$x = \frac{a[f(b) - f(a)] - (b - a)f(a)}{f(b) - f(a)}.$$

Hence,

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

In setting out the tabular form of a Regula Falsi solution, the c_n column uses the general formula

$$c_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}.$$

EXAMPLE

For the equation

$$f(x) \equiv x^3 + 2x - 1 = 0$$

use the Regula Falsi method with $a_0 = 0$ and $b_0 = 1$ to determine the first approximation, c_0, to the solution between $x = 0$ and $x = 1$.

Solution

We have $f(0) = -1$ and $f(1) = 2$, so that there is certainly a solution between $x = 0$ and $x = 1$.

From the general formula,

$$c_0 = \frac{0 \times 2 - 1 \times (-1)}{2 - (-1)} = \frac{1}{3}$$

and, if we were to continue with the method, we would observe that $f(1/3) < 0$ so that $a_1 = 1/3$ and $b_1 = 1$.

Note:
The Bisection Method would have given $c_0 = \frac{1}{2}$.

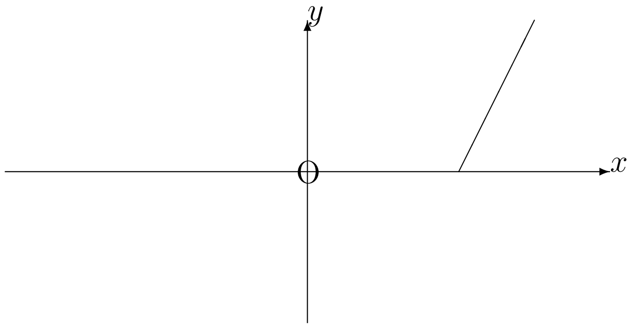
17.1.4 THE NEWTON-RAPHSON METHOD

This method is based on the guessing of an approximate solution, $x = x_0$, to the equation $f(x) = 0$.

We then draw the tangent to the curve whose equation is

$$y = f(x)$$

at the point $x_0, f(x_0)$ to find out where this tangent crosses the x-axis. The point obtained is normally a better approximation x_1 to the solution.



In the diagram,

$$f'(x_0) = \frac{\text{AB}}{\text{AC}} = \frac{f(x_0)}{h}.$$

Hence,

$$h = \frac{f(x_0)}{f'(x_0)},$$

so that a better approximation to the exact solution at point D is given by

$$x_1 = x_0 - h.$$

Repeating the process, gives rise to the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Notes:

(i) To guess the starting approximation, x_0, it is normally sufficient to use a similar technique to that in the Bisection Method; that is, we find a pair of whole numbers, a and b, such that $f(a) < 0$ and $f(b) > 0$; then we take $x_0 = (a + b)/2$. In some exercises, however, an alternative starting approximation may be suggested in order to speed up the rate of convergence to the final solution.

(ii) There are situations where the Newton-Raphson Method fails to give a better approximation; as, for example, when the tangent to the curve has a very small gradient, and consequently meets the x-axis at a relatively great distance from the previous approximation. In this Unit, we shall consider only examples in which the successive approximations converge rapidly to the required solution.

EXAMPLE

Use the Newton-Raphson method to calculate $\sqrt{5}$, correct to three places of decimals.

Solution

We are required to solve the equation

$$f(x) \equiv x^2 - 5 = 0.$$

By trial and error, we find that a solution exists between $x = 2$ and $x = 3$ since $f(2) = -1 < 0$ and $f(3) = 4 > 0$. Hence, we use $x_0 = 2.5$

Furthermore,

$$f'(x) = 2x,$$

so that

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

Thus,

$$x_1 = 2.5 - \frac{1.25}{5} = 2.250,$$

$$x_2 = 2.250 - \frac{0.0625}{4.5} \simeq 2.236,$$

$$x_3 = 2.236 - \frac{-0.000304}{4.472} \simeq 2.236$$

At each stage, we round off the result to the required number of decimal places and use the rounded figure in the next iteration.

The last two iterations give the same result to three places of decimals and this is therefore the required result.

17.1.5 EXERCISES

1. Determine the smallest positive solution to the following equations (i) by the Bisection Method and (ii) by the Regula Falsi Method, giving your answers correct to four significant figures:

   (a)
$$x - 2\sin^2 x = 0;$$
   (b)
$$e^x - \cos(x^2) - 1 = 0.$$

2. Use the Newton-Raphson Method to determine the smallest positive solution to each of the following equations, correct to five decimal places:

   (a)
$$x^4 = 5;$$
   (b)
$$x^3 + x^2 - 4x + 1 = 0;$$
   (c)
$$x - 2 = \ln x.$$

17.1.6 ANSWERS TO EXERCISES

1. (a)
$$x \simeq 1.849;$$
   (b)
$$x \simeq 0.6486$$

2. (a)
$$x \simeq 1.49535;$$
   (b)
$$x \simeq 0.27389;$$
   (c)
$$x \simeq 3.14619$$

"JUST THE MATHS"

UNIT NUMBER

17.2

NUMERICAL MATHEMATICS 2
(Approximate integration (A))

by

A.J.Hobson

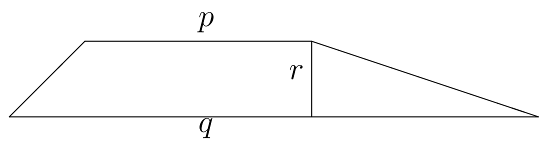
UNIT 17.2 - NUMERICAL MATHEMATICS 2

APPROXIMATE INTEGRATION (A)

17.2.1 THE TRAPEZOIDAL RULE

The rule which is explained below is based on the formula for the area of a trapezium. If the parallel sides of a trapezium are of length p and q while the perpendicular distance between them is r, then the area A is given by

$$A = \frac{r(p+q)}{2}.$$



Let us assume first that the curve whose equation is

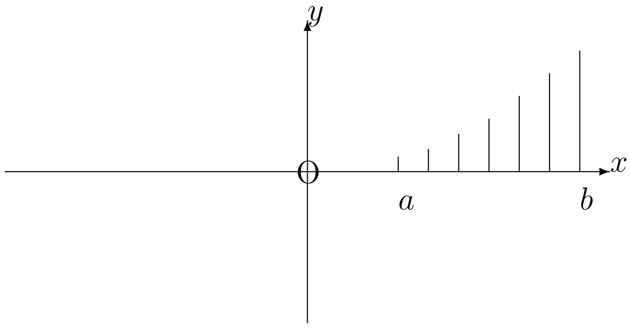
$$y = f(x)$$

lies wholly above the x-axis between $x = a$ and $x = b$. It has already been established, in Unit 13.1, that the definite integral

$$\int_a^b f(x)\ \mathrm{d}x$$

can be regarded as the area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

However, suppose we divided this area into several narrow strips of equal width, h, by marking the values $x_1, x_2, x_3, ..., x_n$ along the x-axis (where $x_1 = a$ and $x_n = b$) and drawing in the corresponding lines of length $y_1, y_2, y_3, ..., y_n$ parallel to the y-axis.



Each narrow strip of width h may be considered approximately as a trapezium whose parallel sides are of lengths y_i and y_{i+1}, where $i = 1, 2, 3, ..., n-1$.

Thus, the area under the curve, and hence the value of the definite integral, approximates to

$$\frac{h}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + ... + (y_{n-1} + y_n)].$$

That is,

$$\int_a^b f(x)\ \mathrm{d}x \simeq \frac{h}{2}[y_1 + y_n + 2(y_2 + y_3 + y_4 + ... + y_{n-1})];$$

or, what amounts to the same thing,

$$\int_a^b f(x)\ \mathrm{d}x = \frac{h}{2}[\text{First} + \text{Last} + 2 \times \text{ The Rest}].$$

Note:
Care must be taken at the beginning to ascertain whether or not the curve $y = f(x)$ crosses the x-axis between $x = a$ and $x = b$. If it does, then allowance must be made for the fact that areas below the x-axis are negative and should be calculated separately from those above the x-axis.

EXAMPLE

Use the trapezoidal rule with five divisions of the x-axis in order to evaluate, approximately, the definite integral:

$$\int_0^1 e^{x^2}\ \mathrm{d}x.$$

Solution

First we make up a table of values as follows:

x	0	0.2	0.4	0.6	0.8	1.0
e^{x^2}	1	1.041	1.174	1.433	1.896	2.718

Then, using $h = 0.2$, we have

$$\int_0^1 e^{x^2}\ \mathrm{d}x \simeq \frac{0.2}{2}[1 + 2.718 + 2(1.041 + 1.174 + 1.433 + 1.896)] \simeq 1.481$$

17.2.2 EXERCISES

Use the trapezoidal rule with six divisions of the x-axis to determine an approximation for each of the following, working to three decimal places throughout:

1. $$\int_1^7 x \ln x\ \mathrm{d}x.$$

2. $$\int_{-2}^1 \frac{1}{5 - x^2}\ \mathrm{d}x.$$

3. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\ \mathrm{d}x.$$

4. $$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2 + 1}\ \mathrm{d}x.$$

5. $$\int_0^{\frac{\pi}{2}} \ln(1 + \sin x)\ \mathrm{d}x.$$

17.2.3 ANSWERS TO EXERCISES

1. 35.836 2. 0.931 3. 0.348 4. 1.468 5. 0.737

"JUST THE MATHS"

UNIT NUMBER

17.3

NUMERICAL MATHEMATICS 3
(Approximate integration (B))

by

A.J.Hobson

UNIT 17.3 - NUMERICAL MATHEMATICS 3

APPROXIMATE INTEGRATION (B)

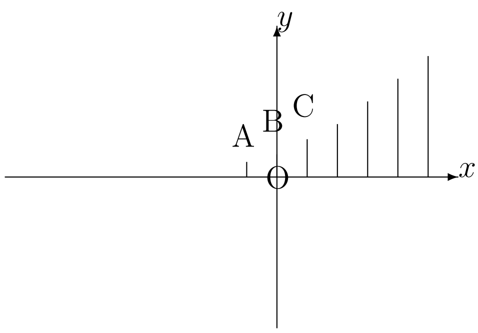
17.3.1 SIMPSON'S RULE

A better approximation to

$$\int_a^b f(x)\mathrm{d}x$$

than that provided by the Trapezoidal rule (Unit 17.2) may be obtained by usingan **even** number of narrow strips of width, h, and considering them in pairs.

To begin with, we examine a **special** case in which the first strip lies to the left of the y-axis as in the following diagram:



The arc of the curve passing through the points A$(-h, y_1)$, B$(0, y_2)$ and C(h, y_3) may be regarded as an arc of a parabola whose equation is

$$y = Lx^2 + Mx + N,$$

provided that the coefficients L, M and N satisfy the equations

$$
\begin{aligned}
y_1 &= Lh^2 - Mh + N, \\
y_2 &= N, \\
y_3 &= Lh^2 + Mh + N.
\end{aligned}
$$

Also, the area of the first pair of strips is given by

$$\text{Area} = \int_{-h}^{h} (Lx^2 + Mx + N) \, \mathrm{d}x$$

$$= \left[L\frac{x^3}{3} + M\frac{x^2}{2} + Nx \right]_{-h}^{h}$$

$$= \frac{2Lh^3}{3} + 2Nh$$

$$= \frac{h}{3}[2Lh^2 + 6N],$$

which, from the simultaneous equations earlier, gives

$$\text{Area} = \frac{h}{3}[y_1 + y_3 + 4y_2].$$

But the area of **every** pair of strips will be dependent only on the three corresponding y co-ordinates, together with the value of h.

Hence, the area of the next pair of strips will be

$$\frac{h}{3}[y_3 + y_5 + 4y_4],$$

and the area of the pair after that will be

$$\frac{h}{3}[y_5 + y_7 + 4y_6].$$

Thus, the total area is given by

$$\text{Area} = \frac{h}{3}[y_1 + y_n + 4(y_2 + y_4 + y_6 + \ldots) + 2(y_3 + y_5 + y_7 + \ldots),$$

usually interpreted as

$$\text{Area} = \frac{h}{3}[\text{First+Last+4}\times\text{The even numbered } y \text{ co} - \text{ords.+2}\times\text{The remaining } y \text{ co} - \text{ords.}]$$

or

$$\text{Area} = \frac{h}{3}[\text{F} + \text{L} + 4\text{E} + 2\text{R}]$$

This result is known as SIMPSON'S RULE.

Notes:

(i) Since the area of the pairs of strips depends only on the three corresponding y co-ordinates, together with the value of h, the Simpson's rule formula provides an approximate value of the definite integral

$$\int_a^b f(x)\ \mathrm{d}x$$

whatever the values of a and b are, as long as the curve does not cross the x-axis between $x = a$ and $x = b$.

(ii) If the curve **does** cross the x-axis between $x = a$ and $x = b$, it is necessary to consider separately the positive parts of the area above the x-axis and the negative parts below the x-axis.

(iii) The approximate evaluation, by Simpson's rule, of a definite integral should be set out in **tabular form**, as illustrated in the examples overleaf.

EXAMPLES

1. Working to a maximum of three places of decimals throughout, use Simpson's rule with ten divisions to evaluate, approximately, the definite integral

$$\int_0^1 e^{x^2}\ \mathrm{d}x.$$

Solution

x_i	$y_i = e^{x_i^2}$	F & L	E	R
0	1	1		
0.1	1.010		1.010	
0.2	1.041			1.041
0.3	1.094		1.094	
0.4	1.174			1.174
0.5	1.284		1.284	
0.6	1.433			1.433
0.7	1.632		1.632	
0.8	1.896			1.896
0.9	2.248		2.248	
1.0	2.718	2.718		
	F + L $\rightarrow$	3.718	7.268	5.544
	4E $\rightarrow$	29.072	$\times 4$	$\times 2$
	2R $\rightarrow$	11.088	29.072	11.088
	(F + L) + 4E + 2R $\rightarrow$	43.878	//////	//////

Hence,

$$\int_0^1 e^{x^2}\ \mathrm{d}x \simeq \frac{0.1}{3} \times 43.878 \simeq 1.463$$

2. Working to a maximum of three places of decimals throughout, use Simpson's rule with eight divisions between $x = -1$ and $x = 1$ and four divisions between $x = 1$ and $x = 2$ in order to evaluate, approximately, the area between the curve whose equation is

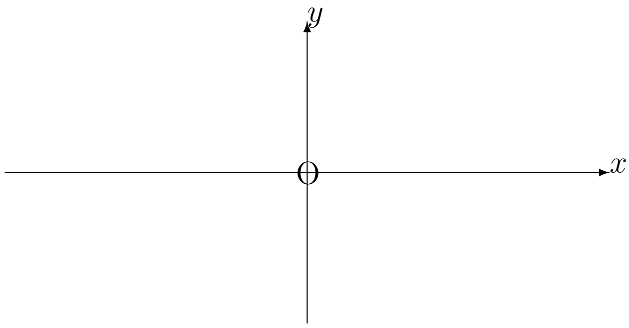
$$y = (x^2 - 1)e^{-x}$$

and the x-axis from $x = -1$ to $x = 2$.

Solution

We note that the curve crosses the x-axis when $x = -1$ and when $x = 1$, the y co-ordinates being negative in the interval between these two values of x and positive outside this interval.

Hence, we need to evaluate the negative area between $x = -1$ and $x = 1$ and the positive area between $x = 1$ and $x = 2$; then we add their numerical values together to find the total area.



(a) The Negative Area

x_i	$y_i = (x^2 - 1)e^{-x}$	F & L	E	R
-1	0	0		
-0.75	-0.926		-0.926	
-0.5	-1.237			-1.237
-0.25	-1.204		-1.204	
0	-1			-1
0.25	-0.730		-0.730	
0.50	-0.455			-0.455
0.75	-0.207		-0.207	
1	0	0		
F + L $\rightarrow$		0	-2.860	-2.692
4E $\rightarrow$		-11.440	$\times 4$	$\times 2$
2R $\rightarrow$		-5.384	-11.440	-5.384
(F + L) + 4E + 2R $\rightarrow$		-16.824	//////	//////

(b) The Positive Area

x_i	$y_i = (x^2-1)e^{-x}$	F & L	E	R
1	0	0		
1.25	0.161		0.161	
1.5	0.279			0.279
1.75	0.358		0.358	
2	0.406	0.406		
	F + L $\rightarrow$	0.406	0.519	0.279
	4E $\rightarrow$	2.076	$\times 4$	$\times 2$
	2R $\rightarrow$	0.558	2.076	0.558
	(F + L) + 4E + 2R $\rightarrow$	3.040	//////	//////

The total area is thus

$$\frac{0.25}{3} \times (16.824 + 3.040) \simeq 1.655$$

17.3.2 EXERCISES

Use Simpson's rule with six divisions of the x-axis to find an approximation for each of the following, working to a maximum of three decimal places throughout:

1.
$$\int_1^7 x \ln x \; \mathrm{d}x.$$

2.
$$\int_{-2}^1 \frac{1}{5-x^2} \; \mathrm{d}x.$$

3.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \; \mathrm{d}x.$$

4.
$$\int_0^{\frac{\pi}{2}} \sin\sqrt{x^2+1} \; \mathrm{d}x.$$

5.
$$\int_0^{\frac{\pi}{2}} \ln(1+\sin x) \; \mathrm{d}x.$$

17.3.3 ANSWERS TO EXERCISES

1. 35.678 2. 0.882 3. 0.347 4. 1.469 5. 0.743

"JUST THE MATHS"

UNIT NUMBER

17.4

NUMERICAL MATHEMATICS 4
(Further Gaussian elimination)

by

A.J.Hobson

UNIT 17.4 - NUMERICAL MATHEMATICS 4
FURTHER GAUSSIAN ELIMINATION

The **elementary** method of Gaussian Elimination, for simultaneous linear equations, was discussed in Unit 9.4. We introduce, here, a more **general** method, suitable for use with sets of equations having **decimal** coefficients.

17.4.1 GAUSSIAN ELIMINATION BY "PARTIAL PIVOTING" WITH A CHECK COLUMN

Let us first consider an example in which the coefficients are **integers**.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 2x + y + z &= 3, \\ x - 2y - z &= 2, \\ 3x - y + z &= 8. \end{aligned}$$

Solution

We may set out the solution, in the form of a **table** (rather than a **matrix**) indicating each of the **"pivot elements"** in a box as follows:

	x	y	z	constant	Σ
	$\boxed{2}$	1	1	3	7
$\frac{1}{2}$	1	-2	-1	2	0
$\frac{3}{2}$	3	-1	1	8	11
		$\boxed{\frac{-5}{2}}$	$\frac{-3}{2}$	$\frac{1}{2}$	$\frac{-7}{2}$
1		$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{1}{2}$
			1	3	4

INSTRUCTIONS

(i) Divide the coefficients of x in lines 2 and 3 by the coefficient of x in line 1 and write the respective results at the side of lines 2 and 3; (that is, $\frac{1}{2}$ and $\frac{3}{2}$ in this case).

(ii) Eliminate x by subtracting $\frac{1}{2}$ times line 1 from line 2 and $\frac{3}{2}$ times line 1 from line 3.

(iii) Repeat the process starting with lines 4 and 5.

(iv) line 6 implies that $z = 3$ and by substitution back into earlier lines, we obtain the values $y = -2$ and $x = 1$.

OBSERVATIONS

Difficulties could arise if the pivot element were very small compared with the other quantities in the same column, since the errors involved in dividing by small numbers are likely to be large.

A better choice of pivot element would be the one with the **largest** numerical value in its column.

We shall consider an example, now, in which this choice of pivot is made. The working will be carried out using fractional quantities; though, in practice, decimals would normally be used instead.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 5, \\ 2x + y - z &= 1, \\ x + 3y - z &= 4. \end{aligned}$$

Solution

	x	y	z	constant	Σ
$\frac{1}{2}$	1	-1	2	5	7
	$\boxed{2}$	1	-1	1	3
$\frac{1}{2}$	1	3	-1	4	7

On eliminating x, we obtain the new table:

	y	z	constant	Σ
$\frac{-3}{5}$	$\frac{-3}{2}$	$\frac{5}{2}$	$\frac{9}{2}$	$\frac{11}{2}$
	$\boxed{\frac{5}{2}}$	$\frac{-1}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Eliminating y takes us to the final table as follows:

z	constant	Σ
$\frac{11}{5}$	$\frac{33}{5}$	$\frac{44}{5}$

We conclude that
$11z = 33$ and, hence, $\boxed{z = 3}$.

Substituting into the second table (either line will do), we have
$5y - 3 = 7$ and, hence, $\boxed{y = 2}$.

Substituting into the original table (any line will do), we have
$x - 2 + 6 = 5$, so that $\boxed{x = 1}$.

Notes:

(i) In questions which involve decimal quantities stated to n decimal places, the calculations should be carried out to $n + 2$ decimal places to allow for rounding up.

(ii) A final check on accuracy in the above example is obtained by adding the original three equations together and verifying that the solution obtained also satisfies the further equation

$$4x + 3y = 10.$$

(iii) It is not essential to set out the solution in the form of separate tables (at each step) with their own headings. A continuation of the first table is acceptable.

17.4.2 EXERCISES

1. Use Gaussian Elimination by Partial Pivoting with a check column to solve the following sets of simultaneous linear equations:

   (a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5, \\ 3x_1 - x_2 + 2x_3 &= 8, \\ 4x_1 - 6x_2 - 4x_3 &= -2. \end{aligned}$$

   (b)

$$\begin{aligned} 5i_1 - i_2 + 2i_3 &= 3, \\ 2i_1 + 4i_2 + i_3 &= 8, \\ i_1 + 3i_2 - 3i_3 &= 2; \end{aligned}$$

(c)

$$
\begin{aligned}
i_1 + 2i_2 + 3i_3 &= -4, \\
2i_1 + 6i_2 - 3i_3 &= 33, \\
4i_1 - 2i_2 + i_3 &= 3;
\end{aligned}
$$

(d)

$$
\begin{aligned}
7i_1 - 4i_2 \qquad &= 12, \\
-4i_1 + 12i_2 - 6i_3 &= 0, \\
-6i_2 + 14i_3 &= 0;
\end{aligned}
$$

2. Use Gaussian Elimination with Partial Pivoting and a check column to solve the following sets of simultaneous linear equations:

(a)

$$
\begin{aligned}
1.202x_1 - 4.371x_2 + 0.651x_3 &= 19.447, \\
-3.141x_1 + 2.243x_2 - 1.626x_3 &= -13.702, \\
0.268x_1 - 0.876x_2 + 1.341x_3 &= 6.849;
\end{aligned}
$$

(b)

$$
\begin{aligned}
-2.381x_1 + 1.652x_2 - 1.243x_3 &= 12.337, \\
2.151x_1 - 3.427x_2 + 3.519x_3 &= 9.212, \\
1.882x_1 + 2.734x_2 - 1.114x_3 &= 5.735;
\end{aligned}
$$

17.4.3 ANSWERS TO EXERCISES

1. (a) $x_1 = -1, \quad x_2 = -3, \quad x_3 = 4;$
   (b) $i_1 = 0.5, \quad i_2 = 1.5, \quad i_3 = 1.0;$
   (c) $i_1 = 3.0, \quad i_2 = 2.5, \quad i_3 = -4.0;$
   (d) $i_1 = 2.26, \quad i_2 = 0.96, \quad i_3 = 0.41$
2. (a) $x_1 = 0.229, \quad x_2 = -4.024, \quad x_3 = 2.433;$
   (b) $x_1 = -5.753, \quad x_2 = 14.187, \quad x_3 = 19.951$

"JUST THE MATHS"

UNIT NUMBER

17.5

NUMERICAL MATHEMATICS 5
(Iterative methods)
for solving
(simultaneous linear equations)

by

A.J.Hobson

UNIT 17.5 - NUMERICAL MATHEMATICS 5

ITERATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATIONS

17.5.1 INTRODUCTION

An iterative method is one which is used repeatedly until the results obtained acquire a pre-assigned degree of accuracy. For example, if results are required to be accurate to five places of decimals, the number of **"iterations"** (that is, stages of the method) is continued until two consecutive iterations give the same result when rounded off to that number of decimal places. It is usually enough for the calculations themselves to be carried out to **two extra** places of decimals.

A similar interpretation holds for accuracy which requires a certain number of **significant figures**.

In the work which follows, we shall discuss two standard methods of solving a set of simultaneous linear equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3, \end{aligned}$$

when the system is **"diagonally dominant"**, which, in this case, means that

$$\begin{aligned} |a_1| &> |b_1| + |c_1|, \\ |b_2| &> |a_2| + |c_2|, \\ |c_3| &> |a_3| + |b_3|. \end{aligned}$$

The methods would be adaptable to a different number of simultaneous equations.

17.5.2 THE GAUSS-JACOBI ITERATION

This method begins by making x the subject of the first equation, y the subject of the second equation and z the subject of the third equation.
An initial approximation such as $x_0 = 1, y_0 = 1, z_0 = 1$ is substituted on the new right-hand sides to give values $x = x_1, y = y_1$ and $z = z_1$ on the new left-hand sides.

A continuation of the process leads to the following general scheme for the results of the $(n+1)$-th iteration:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_n \qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_n - b_3 y_n \qquad\right).
\end{aligned}
$$

This scheme will now be illustrated by numerical examples:

EXAMPLES

1. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$
\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12,
\end{aligned}
$$

obtaining x, y and z correct to the nearest whole number.

Solution

We have

$$
\begin{aligned}
x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\
y_{n+1} &= 3.75 - 0.25x_n \qquad - 0.5z_n, \\
z_{n+1} &= 2.4 - 0.2x_n + 0.4y_n.
\end{aligned}
$$

Using

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

we obtain

$$x_1 = 0.8, y_1 = 3.0, z_1 = 2.6,$$
$$x_2 = 0.72, y_2 = 2.25, z_2 = 3.44,$$
$$x_3 = 1.038, y_3 = 1.85, z_3 = 3.156$$

The results of the last two iterations both give

$$x = 1, y = 2, z = 3,$$

when rounded to the nearest whole number.

In fact, these whole numbers are clearly seen to be the **exact** solutions.

2. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned} x + 7y - z &= 3, \\ 5x + y + z &= 9, \\ -3x + 2y + 7z &= 17, \end{aligned}$$

obtaining x, y and z correct to the nearest whole number.

Solution

This set of equations is not diagonally dominant; but they can be rewritten as

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17, \end{aligned}$$

which **is** a diagonally dominant set. We could also interchange the first two of the original equations.

We have now

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad - 0.2y_n - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_n - 0.29y_n. \end{aligned}$$

Using

$$y_0 = 1, x_0 = 1, z_0 = 1,$$

we obtain

$$y_1 = 0.43, x_1 = 1.4, z_1 = 2.57,$$
$$y_2 = 0.59, x_2 = 1.2, z_2 = 2.91,$$
$$y_3 = 0.67, x_3 = 1.1, z_3 = 2.78$$

This is now enough to conclude that $x = 1, y = 1, z = 3$ to the nearest whole number though, this time, they are not the exact solutions.

17.5.3 THE GAUSS-SEIDEL ITERATION

This method differs from the Gauss-Jacobi Iteration in that successive approximations are used within each step **as soon as they become available**.

It turns out that the rate of convergence of this method is usually faster than that of the Gauss-Jacobi method.

The scheme of the calculations is according to the following pattern:

$$\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad\qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_{n+1} \qquad\qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_{n+1} - b_3 y_{n+1} \qquad\qquad\right).
\end{aligned}$$

EXAMPLES

1. Use the Gauss-Seidel method to solve the simultaneous linear equations

$$\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12.
\end{aligned}$$

Solution

This time, we write:

$$
\begin{aligned}
x_{n+1} &= 0.8 \qquad\qquad - 0.2y_n + 0.2z_n, \\
y_{n+1} &= 3.75 - 0.25x_{n+1} \qquad\quad - 0.5z_n, \\
z_{n+1} &= 2.4 - 0.2x_{n+1} + 0.4y_{n+1},
\end{aligned}
$$

and the sequence of successive results is as follows:

$$x_0 = 1, y_0 = 1, z_0 = 1,$$

$$x_1 = 0.8, y_1 = 3.05, z_1 = 3.46,$$

$$x_2 = 0.88, y_2 = 1.80, z_2 = 2.94,$$

$$x_3 = 1.03, y_3 = 2.02, z_3 = 3.00$$

In this particular example, the rate of convergence is about the same as for the Gauss-Jacobi method, giving $x = 1, y = 2, z = 3$ to the nearest whole number; but we would normally expect the Gauss-Seidel method to converge at a faster rate.

2. Use the Gauss Seidel method to solve the simultaneous linear equations:

$$
\begin{aligned}
7y + x - z &= 3, \\
y + 5x + z &= 9, \\
2y - 3x + 7z &= 17.
\end{aligned}
$$

Solution

These equations give rise to the following iterative scheme:

$$
\begin{aligned}
y_{n+1} &= 0.43 - 0.14x_n \qquad\quad + 0.14z_n, \\
x_{n+1} &= 1.8 \qquad\qquad - 0.2y_{n+1} - 0.2z_n, \\
z_{n+1} &= 2.43 + 0.43x_{n+1} - 0.29y_{n+1},
\end{aligned}
$$

The sequence of successive results is:

$$y_0 = 1, x_0 = 1, z_0 = 1,$$

$$y_1 = 0.43, x_1 = 1.51, z_1 = 2.96,$$

$$y_2 = 0.63, x_2 = 1.08, z_2 = 2.71,$$

$$y_3 = 0.66, x_3 = 1.13, z_3 = 2.73$$

Once more, to the nearest whole number, the solutions are $x = 1, y = 1, z = 3$.

17.5.4 EXERCISES

1. Setting $x_0 = y_0 = z_0 = 1$ and working to three places of decimals, complete four iterations of

   (a) The Gauss-Jacobi method

   and

   (b) The Gauss-Seidel method

   for the system of simultaneous linear equations

$$\begin{aligned} 7x - y + z &= 7.3, \\ 2x - 8y - z &= -6.4, \\ x + 2y + 9z &= 13.6 \end{aligned}$$

   To how many decimal places are your results accurate ?

2. Rearrange the following equations to form a diagonally dominant system and perform the first four iterations of the Gauss-Seidel method, setting $x_0 = 1$, $y_0 = 1$ and $z_0 = 1$ and working to two places of decimals:

$$\begin{aligned} x + 5y - z &= 8, \\ -9x + 3y + 2z &= 3, \\ x + 2y + 7z &= 26. \end{aligned}$$

   Estimate the accuracy of your results and suggest the exact solutions (checking that they are valid).

3. Use an appropriate number of iterations of the Gauss-Seidel method to solve accurately, to three places of decimals, the simultaneous linear equations

$$
\begin{aligned}
7x + y + z &= 5, \\
-2x + 9y + 3z &= 4, \\
x + 4y + 8z &= 3.
\end{aligned}
$$

17.5.5 ANSWERS TO EXERCISES

1. (a) $x_4 \simeq 1.014$, $y_4 \simeq 0.850$, $z_4 \simeq 1.199$, which are accurate to one decimal place.
   (b) $x_4 \simeq 0.999$, $y_4 \simeq 0.900$, $z_4 \simeq 1.200$, which are accurate to two decimal places.
2. On interchanging the first two equations, $x_4 \simeq 0.98$, $y_4 \simeq 2.00$, $z_4 \simeq 2.99$, which are accurate to the nearest whole number. The exact solutions are $x = 1$, $y = 2$, $z = 3$.
3. $x \simeq 0.630$, $y \simeq 0.582$, $z \simeq 0.004$.

"JUST THE MATHS"

UNIT NUMBER

17.6

NUMERICAL MATHEMATICS 6
(Numerical solution)
of
(ordinary differential equations (A))

by

A.J.Hobson

UNIT 17.6 - NUMERICAL MATHEMATICS 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (A)

17.6.1 EULER'S UNMODIFIED METHOD

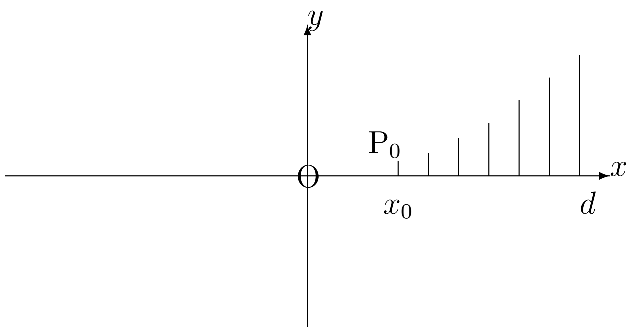
Every first order ordinary differential equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y);$$

and, if it is given that $y = y_0$ when $x = x_0$, then the solution for y in terms of x represents some curve through the point $\mathrm{P}_0(x_0, y_0)$.

Suppose that we wish to find the solution for y at $x = d$, where $d > x_0$.

We sub-divide the interval from $x = x_0$ to $x = d$ into n equal parts of width, δx.



Letting $x_1, x_2, x_3, ...$ be the points of subdivision, we have

$$x_1 = x_0 + \delta x,$$

$$x_2 = x_0 + 2\delta x,$$

$$x_3 = x_0 + 3\delta x,$$

$$...,$$

$$...,$$

$$d = x_n = x_0 + n\delta x.$$

If $y_1, y_2, y_3, ...$ are the y co-ordinates of $x_1, x_2, x_3, ...$, we are required to find y_n.

From elementary calculus, the increase in y, when x increases by δx, is given approximately by $\frac{\mathrm{d}y}{\mathrm{d}x}\delta x$; and since, in our case, $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, we have

$$y_1 = y_0 + f(x_0, y_0)\delta x,$$

$$y_2 = y_1 + f(x_1, y_1)\delta x,$$

$$y_3 = y_2 + f(x_2, y_2)\delta x,$$

$$...,$$

$$...,$$

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\delta x,$$

each stage using the previously calculated y value.

Note:
The method will be the same if $d < x_0$, except that δx will be negative.

In general, each intermediate value of y is given by the formula

$$y_{i+1} = y_i + f(x_i, y_i)\delta x.$$

EXAMPLE

Use Euler's method with 5 sub-intervals to continue, to $x = 0.5$, the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

Solution

i	x_i	y_i	$f(x_i, y_i)$	$y_{i+1} = y_i + f(x_i, y_i)\delta x$
0	0	1	0	1
1	0.1	1	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.061106	0.4244424	1.1035524
5	0.5	1.1035524	-	-

Accuracy
The differential equation in the above example is simple to solve by an elementary method,

such as separation of the variables. It is therefore useful to compare the exact result so obtained with the approximation which comes from Eulers' method.

$$\int \frac{\mathrm{d}y}{y} = \int x \mathrm{d}x.$$

Therefore

$$\ln y = \frac{x^2}{2} + C;$$

that is,

$$y = Ae^{\frac{x^2}{2}}.$$

At $x = 0$, we are told that $y = 1$ and, hence, $A = 1$, giving

$$y = e^{\frac{x^2}{2}}.$$

But a table of values of x against y in the previous interval reveals the following:

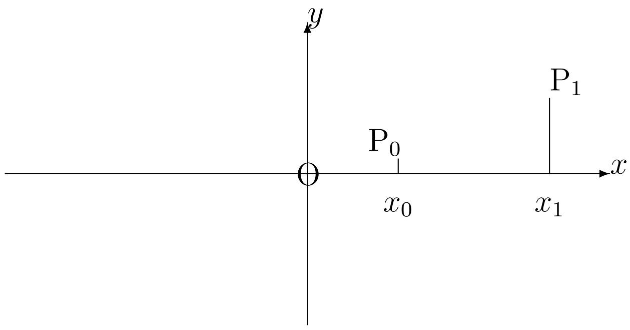
x	$e^{\frac{x^2}{2}}$
0	1
0.1	1.00501
0.2	1.0202
0.3	1.04603
0.4	1.08329
0.5	1.13315

There is thus an error in our approximate value of 0.0296, which is about 2.6%. Attempts to determine y for values of x which are greater than 0.5 would result in a very rapid growth of error.

17.6.2 EULER'S MODIFIED METHOD

In the previous method, we used the gradient to the solution curve at the point P_0 in order to find an approximate position for the point P_1, and so on up to P_n.

But the approximation turns out to be much better if, instead, we use the **average** of the two gradients at P_0 and P_1 for which we need use only x_0, y_0 and δx in order to calculate approximately.



The gradient, m_0, at P_0, is given by

$$m_0 = f(x_0, y_0).$$

The gradient, m_1, at P_1, is given approximately by

$$m_1 = f(x_0 + \delta x, y_0 + \delta y_0),$$

where $\delta y_0 = f(x_0, y_0)\delta x$.

Note:
We cannot call $y_0 + \delta y_0$ by the name y_1, as we did with the unmodified method, because this label is now reserved for the new and **better** approximation at $x = x_0 + \delta x$.

The average gradient, between P_0 and P_1, is given by

$$m_0^* = \frac{1}{2}(m_0 + m_1).$$

Hence, our approximation to y at the point P_1 is given by

$$y_1 = y_0 + m_0^* \delta x.$$

Similarly, we proceed from y_1 to y_2, and so on until we reach y_n.

In general, the intermediate values of y are given by

$$y_{i+1} = y_i + m_i^* \delta x.$$

EXAMPLE

Solve the example in the previous section using Euler's Modified method.

Solution

i	x_i	y_i	$m_i = f(x_i, y_i)$	$\delta y_i = f(x_i, y_i)\delta x$	$m_{i+1} = f(x_i + \delta x, y_i + \delta y_i)$	$m_i^* = \frac{1}{2}(m_i + m_{i+1})$	$y_{i+1} = y_i + m_i^* \delta x$
0	0	1	0	0	0.1	0.05	1.005
1	0.1	1.005	0.1005	0.0101	0.2030	0.1518	1.0202
2	0.2	1.0202	0.2040	0.0204	0.3122	0.2581	1.0460
3	0.3	1.0460	0.3138	0.0314	0.4310	0.3724	1.0832
4	0.4	1.0832	0.4333	0.0433	0.5633	0.4983	1.1330
5	0.5	1.1330	——	——	——	——	——

17.6.3 EXERCISES

1. (a) Taking intervals $\delta x = 0.2$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0,$$
and that $y(0) = 1$
Compare your solution with the exact solution given by
$$y = e^{-x}.$$

   (b) Taking intervals $\delta x = 0.1$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y}{x}$$
and that $y(0.5) = 0.5$.
Compare your solution with the exact solution given by
$$y = x^2 + \frac{x}{2}.$$

   (c) Taking intervals $\delta x = 0.2$, use Euler's unmodified method to determine $y(1)$, given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^{-x},$$
and that $y(0) = 0$.
Compare your solution with the exact solution given by
$$y = \sinh x.$$

(d) Given that $y(1) = 2$, use Euler's unmodified method to continue the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2},$$

to obtain values of y for values of x from $x = 1$ to $x = 1.5$, in steps of 0.1.

2. Repeat all parts of question 1 using Euler's modified method.

17.6.4 ANSWERS TO EXERCISES

1. (a) 0.33, 0.37, 11% low;

   (b) 1.45 1.50. 3% low;

   (c) 1.113, 1.175, 5% low;

   (d) 2.0000, 2.200, 2.431 2.697 3.001, 3.347.

2. (a) 0.371, 0.368, 0.8% high;

   (b) 1.495, 1.50, 0.33% low;

   (c) 1.175 , accurate to three decimal places;

   (d) 2.000, 2.216, 2.465, 2.751, 3.079, 3.452.

Note:
In questions 1(d) and 2(d), the actual values are 2.000, 2.245, 2.496, 2.784, 3.113 and 3.489, from the exact solution of the differential equation.

"JUST THE MATHS"

UNIT NUMBER

17.7

NUMERICAL MATHEMATICS 7
(Numerical solution)
of
(ordinary differential equations (B))

by

A.J.Hobson

UNIT 17.7 - NUMERICAL MATHEMATICS 7

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (B)

17.7.1 PICARD'S METHOD

This method of solving a differential equation approximately is one of successive approximation; that is, it is an **iterative** method in which the numerical results become more and more accurate, the more times it is used.

An approximate value of y (taken, at first, to be a constant) is substituted into the right hand side of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

The equation is then integrated with respect to x giving y in terms of x as a second approximation, into which given numerical values are substituted and the result rounded off to an assigned number of decimal places or significant figures.

The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

A hint on notation
Imagine, for example, that we wished to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

given that $y = y_0 = 7$ when $x = x_0 = 2$.

This ofcourse can be solved exactly to give

$$y = x^3 + C,$$

which requires that

$$7 = 2^3 + C.$$

Hence,

$$y - 7 = x^3 - 2^3;$$

or, in more general terms

$$y - y_0 = x^3 - x_0^3.$$

Thus,

$$\int_{y_0}^{y} \mathrm{d}y = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

In other words,

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

The rule, in future, therefore, will be to integrate both sides of the given differential equation with respect to x, from x_0 to x.

EXAMPLES

1. Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2,$$

and that $y = 0$ when $x = 0$, determine the value of y when $x = 0.3$, correct to four places of decimals.

Solution

To begin the solution, we proceed as follows:

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} (x + y^2) \mathrm{d}x,$$

where $x_0 = 0$.

Hence,

$$y - y_0 = \int_{x_0}^{x} (x + y^2) \mathrm{d}x,$$

where $y_0 = 0$.

That is,

$$y = \int_{0}^{x} (x + y^2) \mathrm{d}x.$$

(a) First Iteration

We do not know y in terms of x yet, so we replace y by the constant value y_0 in the function to be integrated.

The result of the first iteration is thus given, at $x = 0.3$, by

$$y_1 = \int_{0}^{x} x \mathrm{d}x = \frac{x^2}{2} \simeq 0.0450$$

(b) Second Iteration
Now we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_1^2 = x + \frac{x^4}{4}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20}.$$

The result of the second iteration is thus given by

$$y_2 = \frac{x^2}{2} + \frac{x^5}{20} \simeq 0.0451$$

at $x = 0.3$.

(c) Third Iteration
Now we use

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= x + y_2^2 \\ &= x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}.\end{aligned}$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

The result of the third iteration is thus given by

$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \simeq 0.0451 \quad at \quad x = 0.3$$

Hence, $y = 0.0451$, correct to four decimal places, at $x = 0.3$.

2. If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y}{x}$$

and $y = 2$ when $x = 1$, perform three iterations of Picard's method to estimate a value for y when $x = 1.2$. Work to four places of decimals throughout and state how accurate is the result of the third iteration.

Solution

(a) First Iteration

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $x_0 = 1$.

That is,

$$y - y_0 = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $y_0 = 2$.

Hence,

$$y - 2 = \int_{1}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x.$$

Replacing y by $y_0 = 2$ in the function being integrated, we have

$$y - 2 = \int_{1}^{x} \left(2 - \frac{2}{x}\right) \mathrm{d}x.$$

Therefore,

$$\begin{aligned} y &= 2 + [2x - 2\ln x]_1^x \\ &= 2 + 2x - 2\ln x - 2 + 2\ln 1 = 2(x - \ln x). \end{aligned}$$

The result of the first iteration is thus given by

$$y_1 = 2(x - \ln x) \simeq 2.0354,$$

when $x = 1.2$.

(b) Second Iteration

In this case we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_1}{x} = 2 - \frac{2(x - \ln x)}{x} = \frac{2\ln x}{x}.$$

Hence,

$$\int_{1}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{1}^{x} \frac{2lnx}{x}\mathrm{d}x.$$

That is,

$$y - 2 = \left[(\ln x)^2\right]_1^x = (\ln x)^2.$$

The result of the second iteration is thus given by

$$y_2 = 2 + (\ln x)^2 \simeq 2.0332,$$

when $x = 1.2$.

(c) Third Iteration

Finally, we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_2}{x} = 2 - \frac{2}{x} - \frac{(\ln x)^2}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \left[2 - \frac{2}{x} - \frac{(lnx)^2}{x}\right] \mathrm{d}x.$$

That is,

$$y - 2 = \left[2x - 2\ln x - \frac{(\ln x)^3}{3}\right]_1^x$$

$$= 2x - 2\ln x - \frac{(\ln x)^3}{3} - 2.$$

The result of the third iteration is thus given by

$$y_3 = 2x - 2\ln x - \frac{(\ln x)^3}{3} \simeq 2.0293,$$

when $x = 1.2$.

The results of the last two iterations are identical when rounded off to two places of decimals, namely 2.03. Hence, the accuracy of the third iteration is two decimal place accuracy.

17.7.2 EXERCISES

1. Use Picard's method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^x$$

at $x = 1$, correct to two significant figures, given that $y = 0$ when $x = 0$.

2. Use Picard's method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + \frac{y}{2}$$

at $x = 0.5$, correct to two decimal places, given that $y = 1$ when $x = 0$.

3. Given the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - xy,$$

where $y(0) = 0$, use Picard's method to obtain y as a series of powers of x which will give two decimal place accuracy in the interval $0 \leq x \leq 1$.

What is the solution when $x = 1$?

17.7.3 ANSWERS TO EXERCISES

1.

$$y(1) \simeq 2.7$$

2.

$$y(0.5) \simeq 1.33$$

3.

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \frac{x^9}{945} - ......$$

$$y(1) \simeq 0.72$$

"JUST THE MATHS"

UNIT NUMBER

17.8

NUMERICAL MATHEMATICS 8
(Numerical solution)
of
(ordinary differential equations (C))

by

A.J.Hobson

UNIT 17.8 - NUMERICAL MATHEMATICS 8

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (C)

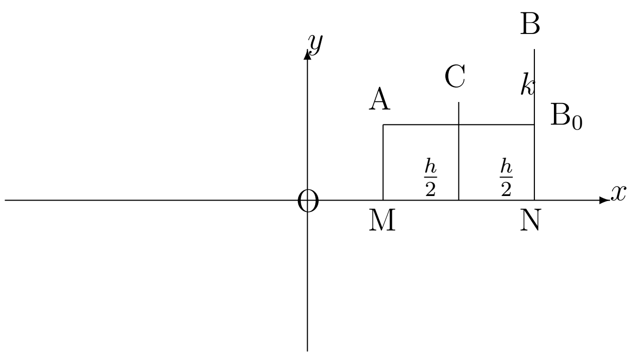
17.8.1 RUNGE'S METHOD

We solve, approximately, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y),$$

subject to the condition that $y = y_0$ when $x = x_0$.

Consider the **graph** of the solution, passing through the two points, $\mathrm{A}(x_0, y_0)$ and $\mathrm{B}(x_0 + h, y_0 + k)$.



We can say that

$$\int_{x_0}^{x_0+h} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

That is,

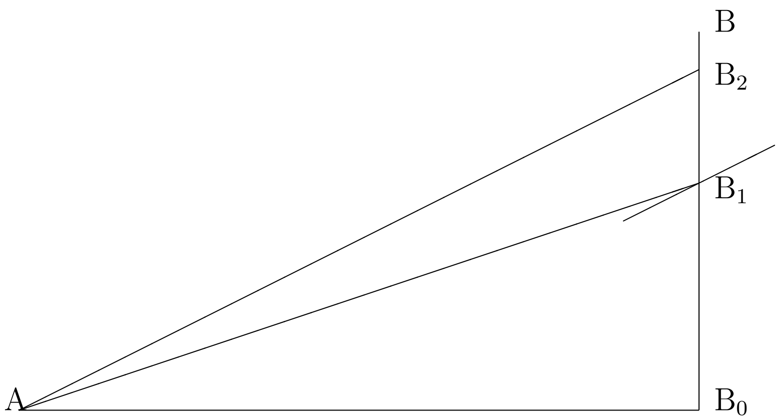
$$y_\mathrm{B} - y_\mathrm{A} = \int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

Suppose that C is the intersection with the curve of the perpendicular bisector of MN. Then, by Simpson's Rule (See Unit 17.3),

$$\int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x = \frac{h/2}{3}\left[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}\right].$$

(i) The value of $f(\mathrm{A})$
This is already given, namely, $f(x_0, y_0)$.

(ii) The Value of $f(\mathrm{B})$



In the diagram, if the tangent at A meets $\mathrm{B}_0\mathrm{B}$ in B_1, then the gradient at A is given by

$$\frac{\mathrm{B}_1\mathrm{B}_0}{\mathrm{AB}_0} = f(x_0, y_0).$$

Therefore,

$$\mathrm{B}_1\mathrm{B}_0 = \mathrm{AB}_0 f(x_0, y_0) = hf(x_0, y_0).$$

Calling this value k_1, as an initial approximation to k, we have

$$k_1 = hf(x_0, y_0).$$

As a rough approximation to the gradient of the solution curve passing through B, we now take the gradient of the solution curve passing through B_1. Its value is

$$f(x_0 + h, y_0 + k_1).$$

To find a better approximation, we assume that a straight line of gradient $f(x_0+h, y_0+k_1)$, drawn at A, meets $\mathrm{B}_0\mathrm{B}$ in B_2, a point nearer to B than B_1.

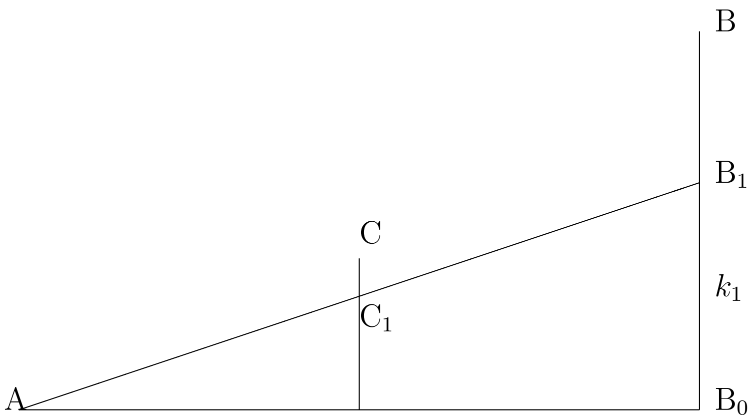
Letting $\mathrm{B}_0\mathrm{B}_2 = k_2$, we have

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

The co-ordinates of B_2 are (x_0+h, y_0+k_2) and the gradient of the solution curve through B_2 is taken as a closer approximation than before to the gradient of the solution curve through B. Its value is

$$f(x_0+h, y_0+k_2).$$

(iii) The Value of f(C)



Let C_1 be the intersection of the ordinate through C and the tangent at A. Then C_1 is the point,

$$\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right),$$

and the gradient at C_1 of the solution curve through C_1 is

$$f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right).$$

We take this to be an approximation to the gradient at C for the arc, AB.

We saw earlier that

$$y_{\mathrm{B}}-y_{\mathrm{A}}=\int_{x_0}^{x_0+h} f(x,y)\mathrm{d}x.$$

Therefore,

$$y_{\mathrm{B}}-y_{\mathrm{A}}=\frac{h}{6}[f(\mathrm{A})+\mathrm{f(B)}+4\mathrm{f(C)}].$$

That is,

$$y=y_0+\frac{h}{6}\left[f(x_0,y_0)+f(x_0+h,y_0+k_2)+4f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)\right].$$

PRACTICAL LAYOUT

If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

and $y = y_0$ when $x = x_0$, then the value of y when $x = x_0 + h$ is determined by the following sequence of calculations:

1. $k_1 = hf(x_0, y_0)$.
2. $k_2 = hf(x_0 + h, y_0 + k_1)$.
3. $k_3 = hf(x_0 + h, y_0 + k_2)$.
4. $k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$.
5. $k = \frac{1}{6}(k_1 + k_3 + 4k_4)$.
6. $y \simeq y_0 + k$.

EXAMPLE

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 3y$$

at $x = 0.1$, given that $y = 1$ when $x = 0$.

Solution

We use $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

1. $k_1 = 0.1(5 - 3) = 0.2$
2. $k_2 = 0.1(5 - 3[1.2]) = 0.14$
3. $k_3 = 0.1(5 - 3[1.14]) = 0.158$
4. $k_4 = 0.1(5 - 3[1.1]) = 0.17$
5. $k = \frac{1}{6}(0.2 + 0.158 + 4[0.17]) = 0.173$
6. $y \simeq 1.173$ at $x = 0.1$

Note:

It can be shown that the error in the result is of the order h^5; that is, the error is equivalent to some constant multiplied by h^5.

17.8.2 EXERCISES

1. Use Runge's Method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2$$

at $x = 0.3$, given that $y = 0$ when $x = 0$.

Work to four places of decimals throughout.

2. Use Runge's Method to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{y+x}$$

at $x = 1.1$, given that $y(1) = 1$.

Work to three places of decimals throughout.

3. Use Runge's Method with **successive** increments of $h = 0.1$ to find the solution at $x = 0.5$ of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y(0) = 1$. Work to four decimal places throughout.

Compare your results with those given by the exact solution

$$y = e^{\frac{1}{2}x^2}.$$

4. Use Runge's Method with $h = 0.2$ to determine the solution at $x = 1$ of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + e^{-x},$$

given that $y(0) = 0$. Work to four decimal places throughout.

17.8.3 ANSWERS TO EXERCISES

1.

$$y(0.3) \simeq 0.0454$$

2.

$$y(1.1) \simeq 1.049$$

3.

$$y(0.5) \simeq 1.3318$$

4.

$$y(1) \simeq 1.1752$$

"JUST THE MATHS"

UNIT NUMBER

18.1

STATISTICS 1
(The presentation of data)

by

A.J.Hobson

UNIT 18.1 - STATISTICS 1 - THE PRESENTATION OF DATA

18.1.1 INTRODUCTION

(i) The collection of numerical information often leads to large masses of data which, if they are to be understood, or presented effectively, must be summarised and analysed in some way. This is the purpose of the subject of **"Statistics"**.

(ii) The source from which a set of data is collected is called a **"population"**. For example, a population of 1000 ball-bearings could provide data relating to their diameters.

(iii) Statistical problems may be either **"descriptive problems"** (in which all the data is known and can be analysed) or **"inference problems"** (in which data collected from a **"sample"** population is used to infer properties of a larger population). For example, the annual pattern of rainfall over several years in a particular place could be used to estimate the rainfall pattern in other years.

(iv) The variables measured in a statistical problem may be either **"discrete"** (in which case they may take only certain values) or **"continuous"** (in which case they make take any values within the limits of the problem itself. For example, the number of students passing an examination from a particular class of students is a discrete variable; but the diameter of ball-bearings from a stock of 1000 is a continuous variable.

(v) Various methods are seen in the commercial presentation of data but, in this series of Units, we shall be concerned with just two methods, one of which is tabular and the other graphical.

18.1.2 THE TABULATION OF DATA

(a) Ungrouped Data
Suppose we have a collection of measurements given by numbers. Some may occur only once, while others may be repeated several times.

If we write down the numbers as they appear, the processing of them is likely to be cumbersome. This is known as **"ungrouped (or raw) data"**, as, for example, in the following table which shows rainfall figures (in inches), for a certain location, in specified months over a 90 year period:

TABLE 1 - Ungrouped (or Raw) Data

18.6	13.8	10.4	15.0	16.0	22.1	16.2	36.1	11.6	7.8
22.6	17.9	25.3	32.8	16.6	13.6	8.5	23.7	14.2	22.9
17.7	26.3	9.2	24.9	17.9	26.5	26.6	16.5	18.1	24.8
16.6	32.3	14.0	11.6	20.0	33.8	15.8	15.2	24.0	16.4
24.1	23.2	17.3	10.5	15.0	20.2	20.2	17.3	16.6	16.9
22.0	23.9	24.0	12.2	21.8	12.2	22.0	9.6	8.0	20.4
17.2	18.3	13.0	10.6	17.2	8.9	16.8	14.2	15.7	8.0
17.7	16.1	17.8	11.6	10.4	13.6	8.4	12.6	8.1	11.6
21.1	20.5	19.8	24.8	9.7	25.1	31.8	24.9	20.0	17.6

(b) Ranked Data

A slightly more convenient method of tabulating a collection of data would be to arrange them in rank order, so making it easier to see how many times each number appears. This is known as **"ranked data"**.

The next table shows the previous rainfall figures in this form

TABLE 2 - Ranked Data

7.8	8.0	8.0	8.1	8.4	8.5	8.9	9.2	9.6	9.7
10.4	10.4	10.5	10.6	11.6	11.6	11.6	11.6	12.2	12.2
12.6	13.0	13.6	13.6	13.8	14.0	14.2	14.2	15.0	15.0
15.2	15.7	15.8	16.0	16.1	16.2	16.4	16.5	16.6	16.6
16.6	16.8	16.9	17.2	17.2	17.3	17.3	17.6	17.7	17.7
17.8	17.9	17.9	18.1	18.3	18.6	19.8	20.0	20.0	20.2
20.2	20.4	20.5	21.1	21.8	22.0	22.0	22.1	22.6	22.9
23.2	23.7	23.9	24.0	24.0	24.1	24.8	24.8	24.9	24.9
25.1	25.3	26.3	26.5	26.6	31.8	32.3	32.8	33.8	36.1

(c) Frequency Distribution Tables

Thirdly, it is possible to save a little space by making a table in which each individual item of the ranked data is written down once only, but paired with the number of times it occurs. The data is then presented in the form of a **"frequency distribution table"**.

TABLE 3 - Frequency Distribution Table

Value	Frequency	Value	Frequency	Value	Frequency
7.8	1	15.8	1	21.1	1
8.0	2	16.0	1	21.8	1
8.1	1	16.1	1	22.0	2
8.4	1	16.2	1	22.1	1
8.5	1	16.4	1	22.6	1
8.9	1	16.5	1	22.9	1
9.2	1	16.6	3	23.2	1
9.6	1	16.8	1	23.7	1
9.7	1	16.9	1	23.9	1
10.4	2	17.2	2	24.0	2
10.5	1	17.3	2	24.1	1
10.6	1	17.6	1	24.8	2
11.6	4	17.7	2	24.9	2
12.2	2	17.8	1	25.1	1
12.6	1	17.9	2	25.3	1
13.0	1	18.1	1	26.3	1
13.6	2	18.3	1	26.5	1
13.8	1	18.6	1	26.6	1
14.0	1	19.8	1	31.8	1
14.2	2	20.0	2	32.3	1
15.0	2	20.2	2	32.8	1
15.2	1	20.4	1	33.8	1
15.7	1	20.5	1	36.1	1

(d) Grouped Frequency Distribution Tables

For about forty or more items in a set of numerical data, it usually most convenient to group them together into between 10 and 25 **"classes"** of values, each covering a specified range or **"class interval"** (for example, $7.5 - 10.5$, $10.5 - 13.5$, $13.5 - 16.5$,......).

Each item is counted every time it appears in order to obtain the **"class frequency"** and each class interval has the same **"class width"**.

Too few classes means that the data is over-summarised, while too many classes means that there is little advantage in summarising at all.

Here, we use the convention that the lower boundary of the class is included while the upper boundary is excluded.

Each item in a particular class is considered to be approximately equal to the "class mid-point"; that is, the average of the two **"class boundaries"**.

A **"grouped frequency distribution table"** normally has columns which show the class intervals, class mid-points, class frequencies, and **"cumulative frequencies"**, the last of these being a running total of the frequencies themselves. There may also be a column of **"tallied frequencies"**, if the table is being constructed from the raw data without having first arranged the values in rank order.

TABLE 4 - Grouped Frequency Distribution

Class Interval	Class Mid-point	Tallied Frequency	Frequency	Cumulative Frequency
7.5 – 10.5	9	////| ////| //	12	12
10.5 – 13.5	12	////| ////|	10	22
13.5 – 16.5	15	////| ////| ////|	15	37
16.5 – 19.5	18	////| ////| ////| ////	19	56
19.5 – 22.5	21	////| ////| //	12	68
22.5 – 25.5	24	////| ////| ////	14	82
25.5 – 28.5	27	///	3	85
28.5 – 31.5	30		0	85
31.5 – 34.5	33	////	4	89
34.5 – 37.5	36	/	1	90

Notes:

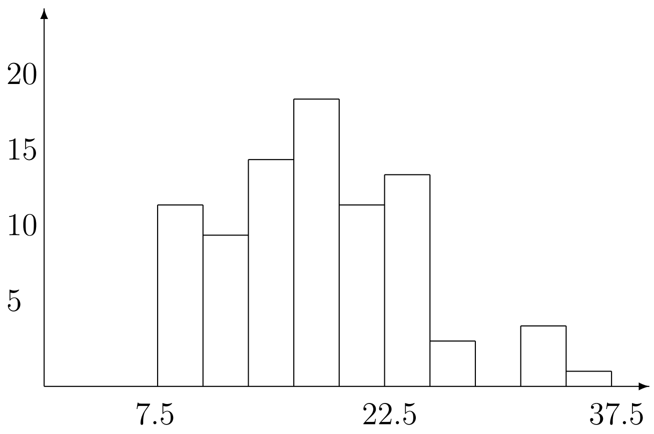
(i) The cumulative frequency shows, at a glance, how many items in the data are less than a specified value. In the above table, for example, 82 items are less than 25.5

(ii) It is sometimes more useful to use the ratio of the cumulative frequency to the total number of observations. This ratio is called the **"relative cumulative frequency"** and, in the above table, for example, the percentage of items in the data which are less than 25.5 is

$$\frac{82}{90} \times 100 \simeq 91\%$$

18.1.3 THE GRAPHICAL REPRESENTATION OF DATA

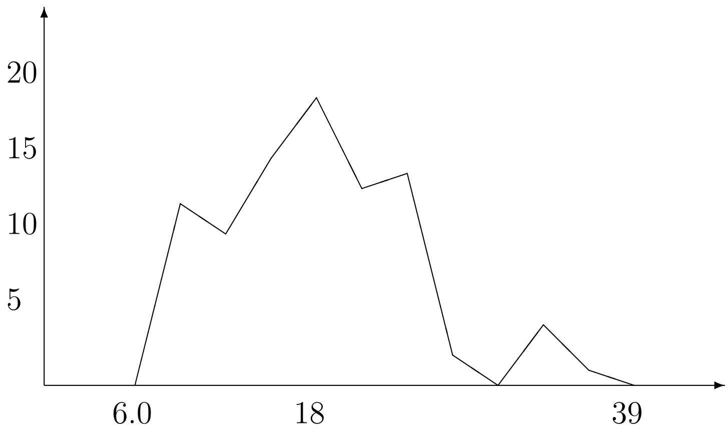
(a) The Histogram
A **"histogram"** is a diagram which is directly related to a grouped frequency distribution table and consists of a collection of rectangles whose height represents the class frequency (to some suitable scale) and whose breadth represents the class width.



The histogram shows, at a glance, not just the class intervals with the highest and lowest frequencies, but also how the frequencies are distributed.

In the case of examination results, for example, there is usually a group of high frequencies around the central class intervals and lower ones at the ends. Such an ideal situation would be called a **"Normal Distribution"**.

(b) The Frequency Polygon
Using the fact that each class interval may be represented, on average, by its class mid-point, we may plot the class mid-points against the class frequencies to obtain a display of single points. By joining up these points with straight line segments and including two extra class mid-points, we obtain a **"frequency polygon"**.



Notes:

(i) Although the frequency polygon officially plots only the class mid-points against their frequencies, it is sometimes convenient to read-off intermediate points in order to estimate additional data. For example, we might estimate that the value 11.0 occurred 11 times when, in fact, it did not occur at all.

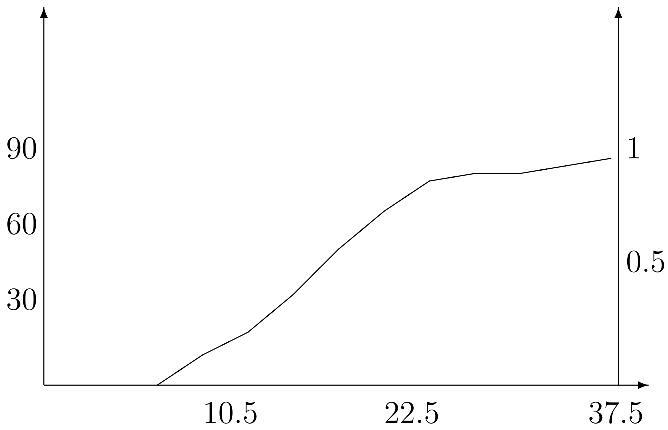
We may use this technique only for continuous variables.

(ii) Frequency polygons are more useful than histograms if we wish to compare two or more frequency distributions. A clearer picture is obtained if we plot them on the same diagram.

(iii) If the class intervals are made smaller and smaller while, at the same time, the total number of items in the data is increased more and more, the points of the frequency polygon will be very close together. The smooth curve joining them is called the **"frequency curve"** and is of greater use for estimating intermediate values.

(c) The Cumulative Frequency Polygon (or Ogive)
The earlier use of the cumulative frequency to estimate the number (or proportion) of values less than a certain amount may be applied graphically by plotting the upper class-boundary against the cumulative frequency, then joining up the points plotted with straight line segments. The graph obtained is called the **"cumulative frequency polygon"** or **"ogive"**.

We may also use a second vertical axis at the right-hand end of the diagram showing the relative cumulative frequency. The range of this axis will always be 0 to 1.



18.1.4 EXERCISES

1. State whether the following variables are discrete or continuous:
   (a) The denominators of a set of 10,000 rational numbers;
   (b) The difference between a fixed integer and each one of any set of real numbers;
   (c) The volume of fruit squash in a sample of 200 bottles taken from different firms;
   (d) The minimum plug-gap setting specified by the makers of 30 different cars;
   (e) The points gained by 80 competitors in a skating championship.

2. The following marks were obtained in an examination taken by 100 students:

Marks	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50
No. of Students	2	3	7	7	8

Marks	50 – 55	55 – 60	60 – 65	65 – 70	70 – 75
No. of Students	25	18	12	10	8

   (a) Draw a histogram for this data;
   (b) Draw a cumulative frequency polygon;
   (c) Estimate the mark exceeded by the top 25% of the students;
   (d) Suggest a pass-mark if 15% of the students are to fail.

3. In a test of 35 glue-laminated beams, the following values of the "spring constant" in kilopounds per inch were found:

   SPRING CONSTANT $\times$ 100

6.72	6.77	6.82	6.70	6.78	6.70	6.62
6.75	6.66	6.66	6.64	6.76	6.73	6.80
6.72	6.76	6.76	6.68	6.66	6.62	6.72
6.76	6.70	6.78	6.76	6.67	6.70	6.72
6.74	6.81	6.79	6.78	6.66	6.76	6.72

   (a) Arrange this data in rank order and hence construct a frequency distribution table;

   (b) Construct a grouped frequency distribution table with class intervals of length 0.02 starting with $6.61 - 6.63$ and include columns to show the class mid-points and the cumulative frequency.

4. The following data shows the diameters of 25 ball-bearings in cms.

0.386	0.391	0.396	0.380	0.397
0.376	0.384	0.401	0.382	0.404
0.390	0.404	0.380	0.390	0.388
0.381	0.393	0.387	0.377	0.399
0.400	0.383	0.384	0.390	0.379

   Construct a grouped frequency distribution table with class intervals $0.375 - 0.380$, $0.380 - 0.385$ etc. and construct

   (a) the histogram;

   (b) the frequency polygon;

   (c) the cumulative frequency polygon (ogive).

18.1.5 SELECTED ANSWERS TO EXERCISES

1. (a) discrete
   (b) continuous
   (c) continuous
   (d) discrete
   (e) discrete

2. (c) 63% is the mark exceeded by the top 25% of students
   (d) 43% needs to be the pass-mark if 15% of the students are to fail.

"JUST THE MATHS"

UNIT NUMBER

18.2

STATISTICS 2
(Measures of central tendency)

by

A.J.Hobson

UNIT 18.2 - STATISTICS 2

MEASURES OF CENTRAL TENDENCY

18.2.1 INTRODUCTION

Having shown, in Unit 18.1, how statistical data may be presented in a clear and concise form, we shall now be concerned with the methods of analysing the data in order to obtain the maximum amount of information from it.

In the previous Unit, it was stated that statistical problems may be either "descriptive problems" (in which all the data is known and can be analysed) or "inference problems" (in which data collected from a "sample" population is used to infer properties of a larger population).

In both types of problem, it is useful to be able to measure some value around which all items in the data may be considered to cluster. This is called **"a measure of central tendency"**; and we find it by using several types of average value as follows:

18.2.2 THE ARITHMETIC MEAN (BY CODING)

To obtain the **"arithmetic mean"** of a finite collection of n numbers, we may simply add all the numbers together and then divide by n. This elementary rule applies even if some of the numbers occur more than once and even if some of the numbers are negative.

However, the purpose of this section is to introduce some short-cuts (called **"coding"**) in the calculation of the arithmetic mean of large collections of data. The methods will be illustrated by the following example, in which the number of items of data is not over-large:

EXAMPLE

The solid contents, x, of water (in parts per million) was measured in eleven samples and the following data was obtained:

4520	4490	4500	4500
4570	4540	4520	4590
4520	4570	4520	

Determine the arithmetic mean, $\overline{x}$, of the data.

Solution

(i) Direct Calculation

By adding together the eleven numbers, then dividing by 11, we obtain

$$\overline{x} = 49840 \div 11 \simeq 4530.91$$

(ii) Using Frequencies

We could first make a frequency table having a column of <u>distinct</u> values x_i, $(i = 1, 2, 3, ..., 11)$, a column of frequencies f_i, $(i = 1, 2, 3......, 11)$ and a column of corresponding values $f_i x_i$.

The arithmetic mean is then calculated from the formula

$$\overline{x} = \frac{1}{11}\sum_{i=1}^{11} f_i x_i.$$

In the present example, the table would be

x_i	f_i	$f_i x_i$
4490	1	4490
4500	2	9000
4520	4	18080
4540	1	4540
4570	2	9140
4590	1	4590
	Total	49840

The arithmetic mean is then $\overline{x} = 49840 \div 11 \simeq 4530.91$ as before.

(iii) Reduction by a constant

With such large data-values, as in the present example, it can be convenient to reduce all of the values by a constant, k, before calculating the arithmetic mean.

It is easy to show that, by adding the constant, k, to the arithmetic mean of the reduced data, we obtain the arithmetic mean of the original data.

Proof:

For n values, $x_1, x_2, x_3, ..., x_n$, the arithmetic mean is given by

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n}.$$

If each value is reduced by a constant, k, the arithmetic mean of the reduced data is

$$\frac{(x_1 - k) + (x_2 - k) + (x_3 - k) + \ldots\ldots + (x_n - k)}{n} = \frac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n} - \frac{nk}{n} = \overline{x} - k.$$

(iv) Division by a constant

In a similar way to the previous paragraph, each value in a collection of data could by divided by a constant, k, before calculating the arithmetic mean.

This time, we may show that the arithmetic mean of the original data is obtained on multiplying the arithmetic mean of the reduced data by k.

Proof:

$$\frac{\frac{x_1}{k} + \frac{x_2}{k} + \frac{x_3}{k} + \ldots\ldots + \frac{x_n}{k}}{n} = \frac{\overline{x}}{k}.$$

In order to summarise the shortcuts used in the present example, the following table shows a combination of the use of frequencies and of the two types of reduction made to the data:

x_i	$x_i - 4490$	$x'_i = (x_i - 4490) \div 10$	f_i	$f_i x'_i$
4490	0	0	1	0
4500	10	1	2	2
4520	30	3	4	12
4540	50	5	1	5
4570	80	8	2	16
4590	100	10	1	10
			Total	45

The fictitious arithmetic mean, $\overline{x}' = \frac{45}{11} \simeq 4.0909$

The actual arithmetic mean, $\overline{x} \simeq (4.0909 \times 10) + 4490 \simeq 4530.91$

(v) The approximate arithmetic mean for a grouped distribution

For a large number of items of data, we may (without losing too much accuracy) take all items within a class interval to be equal to the class mid-point.

A calculation similar to that in the previous paragraph may then be performed if we reduce each mid-point by the first mid-point and divide by the class width (or other convenient number).

EXAMPLE

Calculate, approximately, the arithmetic mean of the data in TABLE 4 on page 4 of Unit 18.1

Solution

Class Interval	Class Mid-point x_i	$x_i - 9$	$(x_i - 9) \div 3 = x'_i$	Frequency f_i	$f_i x'_i$
$7.5 - 10.5$	9	0	0	12	0
$10.5 - 13.5$	12	3	1	10	10
$13.5 - 16.5$	15	6	2	15	30
$16.5 - 19.5$	18	9	3	19	57
$19.5 - 22.5$	21	12	4	12	48
$22.5 - 25.5$	24	15	5	14	70
$25.5 - 28.5$	27	18	6	3	18
$28.5 - 31.5$	30	21	7	0	0
$31.5 - 34.5$	33	24	8	4	32
$34.5 - 37.5$	36	27	9	1	9
			Totals	90	274

$$\text{Fictitious arithmetic mean} \quad \overline{x}' = \frac{274}{90} \simeq 3.0444$$

$$\text{Actual arithmetic mean} = 3.044 \times 3 + 9 \simeq 18.13.$$

Notes:

(i) By direct calculation from TABLE 1 in Unit 18.1, it may be shown that the arithmetic mean is 17.86 correct to two places of decimals; and this indicates an error of about 1.5%.

(ii) The arithmetic mean is widely used where samples are taken of a larger population. It usually turns out that two samples of the same population have arithmetic means which are close in value.

18.2.3 THE MEDIAN

Collections of data often include one or more values which are widely out of character with the rest; and the arithmetic mean can be significantly affected by such extreme values.

For example, the values 8,12,13,15,21,23 have an arithmetic mean of $\frac{92}{6} \simeq 15.33$; but the values 5,12,13,15,21,36 have an arithmetic mean of $\frac{102}{6} \simeq 17.00$.

A second type of average, not so much affected, is defined as follows:

DEFINITION

The **"median"** of a collection of data is the middle value when the data is arranged in rank order. For an even number of values in the collection of data, the median is the arithmetic mean of the centre two values.

EXAMPLES

1. For both 8,12,13,15,21,23 and 5,12,13,15,21,36, the median is given by

$$\frac{13+15}{2} = 14.$$

2. For a grouped distribution, the problem is more complex since we no longer have access to the individual values from the data.

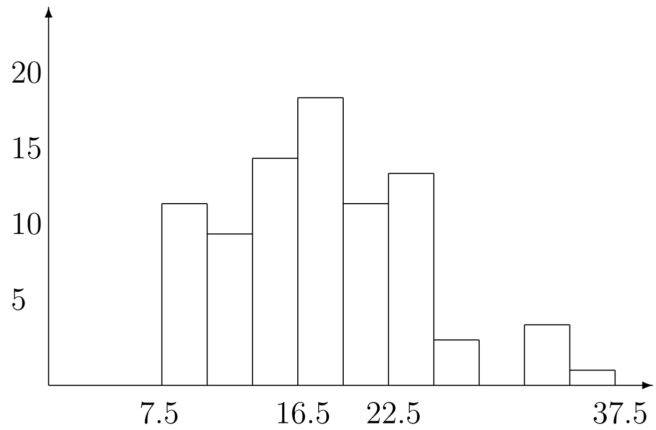
   However, the area of a histogram is directly proportional to the total number of values which it represents, since the base of all the rectangles are the same width and each height represents a frequency.

   We may thus take the median to be the value for which the vertical line through it divides the histogram into two equal areas.

   For non-symmetrical histograms, the median is often a better measure of central tendency than the arithmetic mean.

   Illustration

   Consider the histogram from Unit 18.1, representing rainfall figures over a 90 year period.



The total area of the histogram $= 90 \times 3 = 270$.

Half the area of the histogram $= 135$.

The area up as far as $16.5 = 3 \times 37 = 111$ while the area up as far as $19.5 = 3 \times 56 = 168$; hence the Median must lie between 16.5 and 19.5

The Median $= 16.5 + x$ where $18x = 135 - 111 = 24$ since 18 is the frequency of the class interval $16.5 - 19.5$

That is,

$$x = \frac{24}{18} = \frac{4}{3} \simeq 1.33,$$

giving a Median of 17.83

Notes:

(i) The median, in this case, is close to the arithmetic mean since the distribution is fairly symmetrical.

(ii) If a sequence of zero frequencies occurs, it may be necessary to take the arithmetic mean of two class mid-points, which are not consecutive to each other.

(iii) Another example of the advantage of median over arithmetic mean would be the average life of 100 electric lamps. To find the arithmetic mean, all 100 must be tested; but to find the median, the testing may stop after the 51st.

18.2.4 THE MODE

DEFINITIONS

1. For a collection of individual items of data, the **"mode"** is the value having the highest frequency.
2. In a grouped frequency distribution, the mid-point of the class interval with the highest frequency is called the **"crude mode"** and the class interval itself is called the **"modal class"**.

Note:
Like the median, the mode is not much affected by changes in the extreme values of the data. However, some distributions may have several different modes, which is a disadvantage of this measure of central tendency.

EXAMPLE

For the histogram discussed earlier, the mode is 18.0; but if the class interval, $22.5 - 25.5$, had 5 more members, then 24.0 would be a mode as well.

18.2.5 QUANTILES

To conclude this Unit, we shall define three more standard measurements which, in fact, extend the idea of a median; and we may recall that a median divides a collection of values in such a way that half of them fall on either side of it.

Collectively, these three new measurements are called **"quantiles"** but may be considered separately by their own names as follows:

(a) Quartiles

These are the three numbers dividing a ranked collection of values (or the area of a histogram) into 4 equal parts.

(b) Deciles

These are the nine numbers dividing a ranked collection of values (or the area of a histogram) into 10 equal parts.

(c) Percentiles

These are the ninety nine numbers dividing a ranked collection of values (or the area of a histogram) into 100 equal parts.

Note:
For collections of individual values, quartiles may need to be calculated as the arithmetic mean of two consecutive values.

EXAMPLES

1. (a) The 25th percentile = The 1st quartile.
   (b) The 5th Decile = The median.
   (c) The 85th Percentile = the point at which 85% of the values fall below it and 15% above it.
2. For the collection of values

$$5, 12, 13, 19, 25, 26, 30, 33,$$

   the quartiles are 12.5, 22 and 28.
3. For the collection of values

$$5, 12, 13, 19, 25, 26, 30,$$

   the quartiles are 12.5, 19 and 25.5

18.2.6 EXERCISES

1. The arithmetic mean of 75 observations is 52.6 and the arithmetic mean of 25 similar observations is 48.4; determine the Arithmetic Mean of all 100 observations.
2. Of 500 students, whose mean height is 67.8 inches, 150 are women. If the mean height of 150 women is 62.0 inches, what is the mean height of the men ?
3. By coding the following collection of data, determine the arithmetic mean correct to three places of decimals:

$$1.847, 1.843, 1.842, 1.847, 1.848, 1.841, 1.845$$

4. Using a histogram of the frequency distribution shown, determine
   (a) the arithmetic mean;
   (b) the median class;
   (c) the median;

(d) the modal class;

(e) the crude mode.

class interval	15 – 25	25 – 35	35 – 45	45 – 55	55 – 65	65 – 75
Frequency	4	11	19	14	0	2

5. The number of a certain component issued, per day, from stock over a 40 day period is given as follows:

83	80	91	81	88	82	87	97	83	99
75	85	72	92	84	90	87	78	93	98
86	80	93	86	88	83	82	101	89	82
85	95	80	89	84	92	76	81	103	94

Using class intervals 70 – 75, 75 – 80, 80 – 85 etc., draw up a frequency distribution table and construct a histogram.

From the histogram, determine the median and the 7th Decile.

18.2.7 ANSWERS TO EXERCISES

1.

$$51.55$$

2.

$$70.3 \text{ inches.}$$

3.

$$1.845$$

4. (a)

$$40.20$$

(b)

$$35 - 45.$$

(c)

$$40.26$$

(d)

$$35 - 45.$$

(e)

$$40.$$

5.

$$\text{Median} = 86.5, \quad 7\text{th} \quad \text{Decile} = 91.25$$

"JUST THE MATHS"

UNIT NUMBER

18.3

STATISTICS 3
(Measures of dispersion (or scatter))

by

A.J.Hobson

UNIT 18.3 - STATISTICS 3

MEASURES OF DISPERSION (OR SCATTER)

18.3.1 INTRODUCTION

Averages typify a whole collection of values, but they give little information about how the values are distributed within the whole collection.

For example, 99.9, 100.0, 100.1 is a collection which has an arithmetic mean of 100.0 and so is 99.0,100.0,101.0; but the second collection is more widely dispersed than the first.

It is the purpose of this Unit to examine two types of quantity which typify the distance of all the values in a collection from their arithmetic mean. They are known as measures of dispersion (or scatter) and the smaller these quantities are, the more clustered are the values around the arithmetic mean.

18.3.2 THE MEAN DEVIATION

If the n values x_1, x_2, x_3, ...,x_n have an arithmetic mean of $\overline{x}$, then $x_1 - \overline{x}$, $x_2 - \overline{x}$, $x_3 - \overline{x}$,......,$x_n - \overline{x}$ are called the **"deviations"** of x_1, x_2, x_3,......, x_n from the arithmetic mean.

Note:
The deviations add up to zero since

$$\sum_{i=1}^{n} (x_i - \overline{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x} = n\overline{x} - n\overline{x} = 0$$

DEFINITI0N
The **"mean deviation"** (*or, more accurately, the "mean absolute deviation*) is defined by the formula

$$\text{M.D.} = \frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}|$$

18.3.3 PRACTICAL CALCULATION OF MEAN DEVIATION

In calculating a mean deviation, the following short-cuts usually turn out to be useful, especially for larger collections of values:

(a) If a constant, k, is subtracted from each of the values x_i (i = 1,2,3...n), and also we use the "fictitious" arithmetic mean, $\overline{x}-k$, in the formula, then the mean deviation is unaffected.

Proof:

$$\frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}| = \frac{1}{n}\sum_{i=1}^{n} |(x_i - k) - (\overline{x} - k)|.$$

(b) If we divide each of the values x_i (i = 1,2,3,...n) by a positive constant, l, and also we use the "fictitious" arithmetic mean $\frac{\overline{x}}{l}$, then the mean deviation will be divided by l.

Proof:

$$\frac{1}{ln}\sum_{i=1}^{n} |x_i - \overline{x}| = \frac{1}{n}\sum_{i=1}^{n} \left|\frac{x_i}{l} - \frac{\overline{x}}{l}\right|.$$

Summary
If we code the data using both a subtraction by k and a division by l, the value obtained from the mean deviation formula needs to multiplied by l to give the correct value.

18.3.4 THE ROOT MEAN SQUARE (OR STANDARD) DEVIATION

A more common method of measuring dispersion, which ensures that negative deviations from the arithmetic mean do not tend to cancel out positive deviations, is to determine the arithmetic mean of their squares, and then take the square root.

DEFINITION
The **"root mean square deviation"** (*or "standard deviaton"*) is defined by the formula

$$\text{R.M.S.D.} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{x})^2}.$$

Notes:

(i) The root mean square deviation is usually denoted by the symbol, σ.

(ii) The quantity σ^2 is called the **"variance"**.

18.3.5 PRACTICAL CALCULATION OF THE STANDARD DEVIATION

In calculating a standard deviation, the following short-cuts usually turn out to be useful, especially for larger collections of values:

(a) If a constant, k, is subtracted from each of the values x_i (i = 1,2,3...n), and also we use the "fictitious" arithmetic mean, $\overline{x} - k$, in the formula, then σ is unaffected.

Proof:

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\ (x_i - \overline{x})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\ [(x_i - k) - (\overline{x} - k)]^2}.$$

(b) If we divide each of the values x_i (i = 1,2,3,...n) by a constant, l, and also we use the "fictitious" arithmetic mean $\frac{\overline{x}}{l}$, then σ will be divided by l.

Proof:

$$\frac{1}{l}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\ (x_i - \overline{x})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\ \left(\frac{x_i}{l} - \frac{\overline{x}}{l}\right)^2}.$$

Summary
If we code the data using both a subtraction by k and a division by l, the value obtained from the standard deviation formula needs to multiplied by l to give the correct value, σ.

(c) For the calculation of the standard deviation, whether by coding or not, a more convenient formula may be obtained by expanding out the expression $(x_i - \overline{x})^2$ as follows:

$$\sigma^2 = \frac{1}{n}\left[\sum_{i=1}^{n}\ x_i^2 - 2\overline{x}\sum_{i=1}^{n}\ x_i + \sum_{i=1}^{n}\ \overline{x}^2\right].$$

That is,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\overline{x}^2 + \overline{x}^2,$$

which gives the formula

$$\sigma = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - \overline{x}^2}.$$

Note:
In advanced statistical work, the above formulae for standard deviation are used only for descriptive problems in which we know every member of a collection of observations.

For inference problems, it may be shown that the standard deviation of a sample is always smaller than that of a total population; and the basic formula used for a sample is

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} (x_i - \overline{x})^2}.$$

18.3.6 OTHER MEASURES OF DISPERSION

We mention here, briefly, two other measures of dispersion:

(i) The Range

This is the difference between the highest and the smallest members of a collection of values.

(ii) The Coefficient of Variation

This is a quantity which expresses the standard deviation as a percentage of the arithmetic mean. It is given by the formula

$$\text{C.V.} = \frac{\sigma}{\overline{x}} \times 100.$$

EXAMPLE

The following grouped frequency distribution table shows the diameter of 98 rivets:

Class Intvl.	Cls. Mid Pt. x_i	Freq. f_i	Cum. Freq.	$(x_i - 6.61) \div 0.02 = x_i'$	$f_i x_i'$	$x_i'^2$	$f_i x_i'^2$	$f_i\lvert x_i' - \overline{x}'\rvert$
6.60 – 6.62	6.61	1	1	0	0	0	0	0.58
6.62 – 6.64	6.63	4	5	1	4	1	4	2.40
6.64 – 6.66	6.65	6	11	2	12	4	24	3.72
6.66 – 6.68	6.67	12	23	3	36	9	108	7.68
6.68 – 6.70	6.69	5	28	4	20	16	80	3.30
6.70 – 6.72	6.71	10	38	5	50	25	250	6.80
6.72 – 6.74	6.73	17	55	6	102	36	612	11.90
6.74 – 6.76	6.75	10	65	7	70	49	490	7.20
6.76 – 6.78	6.77	14	79	8	112	64	896	10.36
6.78 – 6.80	6.79	9	88	9	81	81	729	6.84
6.80 – 6.82	6.81	7	95	10	70	100	700	5.46
6.82 – 6.84	6.83	2	97	11	22	121	242	1.60
6.84 – 6.86	6.85	1	98	12	12	144	144	0.82
Totals		98			591		4279	68.66

Estimate the arithmetic mean, the standard deviation and the mean (absolute) deviation of these diameters.

Solution

$$\text{Fictitious arithmetic mean} = \frac{591}{98} \simeq 6.03$$

$$\text{Actual arithmetic mean} = 6.03 \times 0.02 + 6.61 \simeq 6.73$$

$$\text{Fictitious standard deviation} = \sqrt{\frac{4279}{98} - 6.03^2} \simeq 2.70$$

$$\text{Actual standard deviation} = 2.70 \times 0.02 \simeq 0.054$$

$$\text{Fictitious mean deviation} = \frac{68.66}{98} \simeq 0.70$$

$$\text{Actual mean deviation} \simeq 0.70 \times 0.02 \simeq 0.014$$

18.3.7 EXERCISES

1. For the collection of numbers

$$6.5,\ 8.3,\ 4.7,\ 9.2,\ 11.3,\ 8.5,\ 9.5,\ 9.2$$

   calculate (correct to two places of decimals) the arithmetic mean, the standard deviation and the mean (absolute) deviation.

2. Estimate the arithmetic mean, the standard deviation and the mean (absolute) deviation for the following grouped frequency distribution table:

Class Interval	10 – 30	30 – 50	50 – 70	70 – 90	90 – 110	110 – 130
Frequency	5	8	12	18	3	2

3. The following table shows the registered speeds of 100 speedometers at 30m.p.h.

Regd. Speed	Frequency
27.5 – 28.5	2
28.5 – 29.5	9
29.5 – 30.5	17
30.5 – 31.5	26
31.5 – 32.5	24
32.5 – 33.5	16
33.5 – 34.5	5
34.5 – 35.5	1

   Estimate the arithmetic mean, the standard deviation and the coefficient of variation.

18.3.8 ANSWERS TO EXERCISES

1. Arithmetic mean $\simeq 8.40$, standard deviation $\simeq 1.88$, mean deviation $\simeq 1.43$
2. Arithmetic mean $\simeq 65$, standard deviation $\simeq 24.66$, mean deviation $\simeq 20.21$
3. Arithmetic mean $\simeq 31.4$, standard deviation $\simeq 1.44$, coefficient of variation $\simeq 4.61$.

"JUST THE MATHS"

UNIT NUMBER

18.4

STATISTICS 4
(The principle of least squares)

by

A.J.Hobson

UNIT 18.4 - STATISTICS 4

THE PRINCIPLE OF LEAST SQUARES

18.4.1 THE NORMAL EQUATIONS

Suppose two variables, x and y, are known to obey a **"straight line law"** of the form $y = a + bx$, where a and b are constants to be found.

Suppose also that, in an experiment to test this law, we obtain n pairs of values, (x_i, y_i), where i = 1,2,3,...,n.

If the values x_i are **assigned** values, they are likely to be free from error, whereas the **observed** values, y_i will be subject to experimental error.

The principle underlying the straight line of **"best fit"** is that, in its most likely position, the sum of the squares of the y-deviations, from the line, of all observed points is a minimum.

Using partial differentiation, it may be shown that

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad - - - (1)$$

and

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \quad - - - (2).$$

The statements (1) and (2) are two simultaneous equations which may be solved for a and b.

They are called the **"normal equations"**.

A simpler notation for the normal equations is

$$\Sigma\ y = na + b\Sigma\ x$$

and

$$\Sigma xy = a\Sigma x + b\Sigma\ x^2.$$

By eliminating a and b in turn, we obtain the solutions

$$a = \frac{\Sigma\ x^2.\Sigma\ y - \Sigma x.\Sigma\ xy}{n\Sigma\ x^2 - (\Sigma\ x)^2} \quad \text{and} \quad b = \frac{n\Sigma\ xy - \Sigma\ x.\Sigma\ y}{n\Sigma\ x^2 - (\Sigma\ x)^2}.$$

With these values of a and b, the straight line $y = a + bx$ is called the **"regression line of y on x"**.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

x	y	xy	x^2
45	6.53	293.85	2025
42	6.30	264.60	1764
56	9.52	533.12	3136
48	7.50	360.00	2304
42	6.99	293.58	1764
35	5.90	206.50	1225
58	9.49	550.42	3364
40	6.20	248.00	1600
39	6.55	255.45	1521
50	8.72	436.00	2500
455	73.70	3441.52	21203

The regression line of y on x thus has equation $y = a + bx$, where

$$a = \frac{(21203)(73.70) - (455)(3441.52)}{(10)(21203) - (455)^2} \simeq -0.645$$

and

$$b = \frac{(10)(3441.52) - (455)(73.70)}{(10)21203) - (455)^2} \simeq 0.176$$

Thus, $y = 0.176x - 0.645$

18.4.2. SIMPLIFIED CALCULATION OF REGRESSION LINES

A simpler method of determining the regression line of y on x for a given set of data, is to consider a temporary change of origin to the point $(\overline{x}, \overline{y})$, where $\overline{x}$ is the arithmetic mean of the values x_i and $\overline{y}$ is the arithmetic mean of the values y_i.

RESULT

The regression line of y on x contains the point $(\overline{x}, \overline{y})$.

Proof:

From the first of the normal equations,

$$\frac{\Sigma\ y}{n} = a + b\frac{\Sigma\ x}{n}.$$

That is,

$$\overline{y} = a + b\overline{x}.$$

A change of origin to the point $(\overline{x}, \overline{y})$, with new variables X and Y, is associated with the formulae

$$X = x - \overline{x} \ \text{ and } \ Y = y - \overline{y}$$

and, in this system of reference, the regression line will pass through the origin.

Its equation is therefore

$$Y = BX,$$

where

$$B = \frac{n\Sigma\ XY - \Sigma\ X.\Sigma\ Y}{n\Sigma\ X^2 - (\Sigma\ X)^2}.$$

However,

$$\Sigma\ X = \Sigma\ (x - \overline{x}) = \Sigma\ x - \Sigma\ \overline{x} = n\overline{x} - n\overline{x} = 0$$

and

$$\Sigma\ Y = \Sigma\ (y - \overline{y}) = \Sigma\ y - \Sigma\ \overline{y} = n\overline{y} - n\overline{y} = 0.$$

Thus,

$$B = \frac{\Sigma\ XY}{\Sigma\ X^2}.$$

Note:
In a given problem, we make a table of values of x_i, y_i, X_i, Y_i, X_iY_i and X_i^2.

The regression line is then

$$y - \overline{y} = B(x - \overline{x}) \quad \text{or} \quad y = BX + (\overline{y} - B\overline{x});$$

though, there may be slight differences in the result obtained compared with that from the earlier method.

EXAMPLE

Determine the equation of the regression line of y on x for the following data which shows the Packed Cell Volume, $x$mm, and the Red Blood Cell Count, y millions, of 10 dogs:

x	45	42	56	48	42	35	58	40	39	50
y	6.53	6.30	9.52	7.50	6.99	5.90	9.49	6.20	6.55	8.72

Solution

The arithmetic mean of the x values is $\overline{x} = 45.5$

The arithmetic mean of the y values is $\overline{y} = 7.37$

This gives the following table:

x	y	$X = x - \overline{x}$	$Y = y - \overline{y}$	XY	X^2
45	6.53	-0.5	-0.84	0.42	0.25
42	6.30	-3.5	-1.07	3.745	12.25
56	9.52	10.5	2.15	22.575	110.25
48	7.50	2.5	0.13	0.325	6.25
42	6.99	-3.5	-0.38	1.33	12.25
35	5.90	-10.5	-1.47	15.435	110.25
58	9.49	12.5	2.12	26.5	156.25
40	6.20	-5.5	-1.17	6.435	30.25
39	6.55	-6.5	-0.82	5.33	42.25
50	8.72	4.5	1.35	6.075	20.25
455	73.70			88.17	500.5

Hence,

$$B = \frac{88.17}{500.5} \simeq 0.176$$

and so the regression line has equation

$$y = 0.176x + (7.37 - 0.176 \times 45.5).$$

That is,

$$y = 0.176x - 0.638$$

18.4.3 EXERCISES

1. For the following tables, determine the regression line of y on x, assuming that $y = a+bx$.

(a)

x	0	2	3	5	6
y	6	-1	-3	-10	-16

(b)

x	0	20	40	60	80
y	54	65	75	85	96

(c)

x	1	3	5	10	12
y	58	55	40	37	22

2. To determine the relation between the normal stress and the shear resistance of soil, a shear-box experiment was performed, giving the following results:

Normal Stress, x p.s.i.	11	13	15	17	19	21
Shear Stress, y p.s.i.	15.2	17.7	19.3	21.5	23.9	25.4

If $y = a + bx$, determine the regression line of y on x.

3. Fuel consumption, y miles per gallon, at speeds of x miles per hour, is given by the following table:

x	20	30	40	50	60	70	80	90
y	18.3	18.8	19.1	19.3	19.5	19.7	19.8	20.0

Assuming that

$$y = a + \frac{b}{x},$$

determine the most probable values of a and b.

18.4.4 ANSWERS TO EXERCISES

1. (a)

$$y = 6.46 - 3.52x;$$

(b)

$$y = 54.20 + 0.52x;$$

(c)

$$y = 60.78 - 2.97x.$$

2.

$$y = 4.09 + 1.03x.$$

3.

$$a \simeq -42 \quad \text{and} \quad b \simeq 20.$$

"JUST THE MATHS"

UNIT NUMBER

19.1

PROBABILITY 1
(Definitions and rules)

by

A.J.Hobson

UNIT 19.1 - PROBABILITY 1 - DEFINITIONS AND RULES

19.1.1 INTRODUCTION

To introduce the definition of probability, suppose 30 high-strength bolts became mixed with 25 ordinary bolts by mistake, all of the bolts being identical in appearance.

We would like to know how sure we can be that, in choosing a bolt, it will be a high-strength one. Phrases like "quite sure" or "fairly sure" are useless, mathematically, and we define a way of measuring the certainty.

We know that, in 55 simultaneous choices, 30 will be of high strength and 25 will be ordinary; so we say that, in one choice, there is a $\frac{30}{55}$ chance of success; that is, approximately, a 0.55 chance of success.

Obviously, in one single choice, we haven't any idea what the result will be; but experience has proved that, in a significant number of choices, just over half will most likely give a high-strength bolt.

Such predictions can be used, for example, to estimate the cost of mistakes on a production line.

DEFINITION 1.
The various occurrences which are possible in a statistical problem are called **"events"**.
If we are interested in one particular event, it is termed **"successful"** when it occurs and **"unsuccessful"** when it does not.

ILLUSTRATION

If, in a collection of 100 bolts, there are 30 high-strength, 25 ordinary and 45 low-strength, then we have three possible events according to which type is chosen.
We can make 100 **"trials"** and, in each trial, one of three events will occur.

DEFINITION 2.
If, in n possible trials, a successful event occurs s times, then the number $\frac{s}{n}$ is called the **"probability of success in a single trial"**. It is also known as the **"relative frequency of success"**.

ILLUSTRATIONS

1. From a bag containing 7 black balls and 4 white balls, the probability of drawing a white ball is $\frac{4}{11}$.

2. In tossing a perfectly balanced coin, the probability of obtaining a head is $\frac{1}{2}$.
3. In throwing a die, the probability of getting a six is $\frac{1}{6}$.
4. If 50 chocolates are identical in appearance, but consist of 15 soft-centres and 35 hard-centres, the probability of choosing a soft-centre is $\frac{15}{50} = 0.3$.

19.1.2 APPLICATION OF PROBABILITY TO GAMES OF CHANCE

If a competitor in a game of chance has a probability, p, of winning, and the prize money is £m, then £mp is considered to be a fair price for entry to the game.

The quantity mp is known as the **"expectation"** of the competitor.

19.1.3 EMPIRICAL PROBABILITY

So far, all the problems discussed on probability have been "descriptive"; that is, we know all the possible events, the number of successes and the number of failures. In other problems, called **"inference"** problems, it is necessary either (a) to take **"samples"** in order to infer facts about a total **"population'** (for example, a public census or an investigation of moon-rock); or (b) to rely on past experience (for example past records of heart deaths, road accidents, component failure).

If the probability of success, used in a problem, has been inferred by samples or previous experience, it is called **"empirical probability"**.

However, once the probability has been calculated, the calculations are carried out in the same way as for descriptive problems.

19.1.4 TYPES OF EVENT

DEFINITION 3.
If two or more events are such that not more than one of them can occur in a single trial, they are called "mutually exclusive".

ILLUSTRATION

Drawing an Ace or drawing a King from a pack of cards are mutually exclusive events; but drawing an Ace and drawing a Spade are not mutually exclusive events.

DEFINITION 4.
If two or more events are such that the probability of any one of them occurring is not affected by the occurrence of another, they are called **"independent"** events.

ILLUSTRATION

From a pack of 52 cards (that is, Jokers removed), the event of drawing and immediately replacing a red card will have a probability of $\frac{26}{52} = 0.5$; and the probability of this occurring a second time will be exactly the same. They are independent events.

However, two successive events of drawing a red card **without** replacing it are **not** independent. If the first card drawn is red, the probability that the second is red will be $\frac{25}{51}$; but, if the first card drawn is black, the probability that the second is red will be $\frac{26}{51}$

19.1.5 RULES OF PROBABILITY

1. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r mutually exclusive events, then the probability that some **one** of the r events will occur is

   $$p_1 + p_2 + p_3 + ... + p_r.$$

   ILLUSTRATION

   Suppose a bag contains 100 balls of which 1 is red, 2 are blue and 3 are black. The probability of choosing any one of these three colours will be

   $$0.06 = 0.01 + 0.02 + 0.03$$

   However, the probability of drawing a spade or an ace from a pack of 52 cards will not be $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$ but $\frac{16}{52}$ since there are just 16 cards which are either a spade or an ace.

2. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r independent events, then the probability that **all** will occur in a single trial is

   $$p_1.p_2.p_3......p_r.$$

   ILLUSTRATION

   Suppose there are three bags, each containing white, red and blue balls. Suppose also that the probabilities of drawing a white ball from the first bag, a red ball from the second bag and a blue ball from the third bag are respectively p_1, p_2 and p_3. The probability of making these three choices in succession is $p_1.p_2.p_3$ because they are independent events.

   However, if three cards are drawn, without replacing, from a pack of 52 cards, the probability of drawing a 3, followed by an ace, followed by a red card will not be $\frac{4}{52}.\frac{4}{52}.\frac{26}{52}$.

19.1.6 CONDITIONAL PROBABILITIES

For dependent events, the multiplication rule requires a knowledge of the **new** probabilities of successive events in the trial, after the previous ones have been dealt with. These are called **"conditional probabilitities"**.

EXAMPLE

From a box, containing 6 white balls and 4 black balls, 3 balls are drawn at random without replacing them. What is the probability that there will be 2 white and 1 black ?

Solution

The cases to consider, together with their probabilities are as follows:

(a) White, White, Black..............Probability $= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720} = \frac{1}{6}$.

(b) Black, White, White..............Probability $= \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

(c) White, Black, White..............Probability $= \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

The probability of any one of these three outcomes is therefore

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

19.1.7 EXERCISES

1. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is either an Ace or a picture card ?
2. If a die is rolled, what is the probability that the roll yields either a 3 or a 4 ?
3. In a single throw of two dice, what is the probability that a 9 or a doublet will be thrown ?
4. Ten balls, numbered 1 to 10, are placed in a bag. One ball is drawn and not replaced; and then a second ball is drawn. What are the probabilities that
   (a) the balls numbered 3 and 7 are drawn;
   (b) neither of these two balls are drawn ?
5. On a gaming machine, there are three reels with ten digits 0,1,2,3,4,5,6,7,8,9 plus a star on each reel. When a coin is inserted, and the machine started, the three reels revolve independently before coming to rest.

(a) What is the probabiity of getting a particular sequence of numbers ?
(b) What is the probability of getting three stars ?
(c) What is the probability of getting the same number on each reel ?

6. Three balls in succession are drawn, without replacement, from a bag containing 8 black, 8 white and 8 red balls. If a prize of £5 is awarded for drawing no black balls, what is the expectation ?

7. Three persons A,B and C take turns to throw three dice once. If the first one to throw a total of 11 is awarded a prize of £200, what are the expectations of A,B and C ?

19.1.8 ANSWERS TO EXERCISES

1. $\frac{4}{13}$.
2. $\frac{1}{3}$.
3. $\frac{5}{18}$.
4. (a)
$$\frac{1}{45};$$
(b)
$$\frac{28}{45}.$$
5. (a)
$$\frac{1}{11^3} \simeq 0.00075;$$
(b)
$$\frac{1}{11^3} \simeq 0.00075;$$
(c)
$$\frac{10}{11^3} \simeq 0.0075$$
6. The expectation is £1.38
7. For A,B and C, the expectations are £25, £21.88 and £19.14 respectively since the probabilities are $\frac{1}{8}$, $\frac{7}{8^2}$ and $\frac{7}{8^3}$ respectively.

"JUST THE MATHS"

UNIT NUMBER

19.2

PROBABILITY 2
(Permutations and combinations)

by

A.J.Hobson

UNIT 19.2 - PROBABILITY 2 - PERMUTATIONS AND COMBINATIONS

19.2.1 INTRODUCTION

In Unit 19.1, we saw that, in the type of problem known as "descriptive", we can work out the probability that an event will occur by counting up the total number of possible trials and the number of successful ones amongst them. But this can often be a tedious process without the results of the work which is included in the present Unit.

DEFINITION 1.
Each different arrangement of all or part of a set of objects is called a **"permutation"**.

DEFINITION 2.
Any set which can be made by using all or part of a given collection of objects, without regard to order, is called a **"combination"**.

EXAMPLES

1. Nine balls, numbered 1 to 9, are put into a bag, then emptied into a channel which guides them into a line of pockets. What is the probability of obtaining a particular nine digit number ?

   Solution

   We require the total number of arrangements of the nine digits and this is evaluated as follows:

   There are nine ways in which a digit can appear in the first pocket and, for each of these ways, there are then eight choices for the second pocket. Hence, the first two pockets can be filled in $9 \times 8 = 72$ ways.

   Continuing in this manner, the total number of arrangements will be

   $$9 \times 8 \times 7 \times 6 \times ...... \times 3 \times 2 \times \times 1 = 362880 = T, \text{ (say)}.$$

   This is the number of permutations of the ten digits and the required probability is therefore equal to $\frac{1}{T}$.

2. A box contains five components of identical appearance but different qualities. What is the probability of choosing a pair of components from the highest two qualities ?

   Solution

   Method 1.

   Let the components be A, B, C, D, E in order of descending quality.

The choices are
AB AC AD AE
BC BD BE
CD CE
DE,

giving ten choices.

These are the various combinations of five objects, two at a time; hence, the probability for AB = $\frac{1}{10}$.

Method 2.

We could also use the ideas of conditional probability as follows:

The probability of drawing A = $\frac{1}{5}$.
The probability of drawing B without replacing A = $\frac{1}{4}$.
The probability of drawing A and B **in either order** =

$$2 \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{10}.$$

19.2.2 RULES OF PERMUTATIONS AND COMBINATIONS

1. The number of permutations of all n objects in a set of n is

$$n(n-1)(n-2)\text{....................}3.2.1$$

which is denoted for short by the symbol n! It is called **"n factorial"**.

This rule was demonstrated in Example 1 of the previous section.

2. The number of permutations of n objects r at a time is given by

$$n(n-1)(n-2)\text{....................}(n-r+1) = \frac{n!}{(n-r)!}$$

Explanation

The first object can be chosen in any one of n different ways.
For each of these, the second object can then be chosen in $n-1$ ways.
For each of these, the third object can then be chosen in $n-2$ ways.
...
...
...
For each of these, the r-th object can be chosen in $n-(r-1) = n-r+1$ ways.

Note:
In Example 2 of the previous section, the number of permutations of five components two at a time is given by

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5.4.3.2.1}{3.2.1} = 20.$$

This is double the number of choices we obtained for any two components out of five because, in a permutation, the order matters.

3. The number of combinations of n objects r at a time is given by

$$\frac{n!}{(n-r)!r!}$$

Explanation

This is very much the same problem as the number of permutations of n objects r at a time; but, as permutations, a particular set of objects will be counted $r!$ times. In the case of combinations, such a set will be counted only once, which reduces the number of possibilities by a factor of $r!$.

In Example 2 of the previous section, it is precisely the number of combinations of five objects two at a time which is being calculated. That is,

$$\frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5.4.3.2.1}{3.2.1.2.1} = 10,$$

as before.

Note:
A traditional notation for the number of permutations of n objects r at a time is ${}^{n}P_{r}$. That is,

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

A traditional notation for the number of combinations of n objects r at a time is ${}^{n}C_{r}$. That is,

$${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$$

EXAMPLES

1. How many four digit numbers can be formed from the numbers 1,2,3,4,5,6,7,8,9 if no digit can be repeated ?

   Solution

   This is the number of permutations of 9 objects four at a time; that is,

$$\frac{9!}{5!} = 9.8.7.6. = 3,024.$$

2. In how many ways can a team of nine people be selected from twelve ?

   Solution

   The required number is

$$^{12}C_9 = \frac{12!}{3!9!} = \frac{12.11.10}{3.2.1} = 220.$$

3. In how many ways can we select a group of three men and two women from five men and four women ?

   Solution

   The number of ways of selecting three men from five men is

$$^{5}C_3 = \frac{5!}{2!3!} = \frac{5.4.}{2.1} = 10.$$

   For each of these ways, the number of ways of selecting two women from four women is

$$^{4}C_2 - \frac{4!}{2!2!} = \frac{4.3}{2.1} = 6.$$

   The total number of ways is therefore $10 \times 6 = 60$.

4. What is the probability that one of four bridge players will obtain a thirteen card suit ?

   Solution

   The number of possible suits for each player is

$$N = {}^{52}C_{13} = \frac{52!}{39!13!}$$

   The probabilty that any one of the four players will obtain a thirteen card suit is thus

$$4 \times \frac{1}{N} = \frac{4.(39!)(13!)}{52!} \simeq 6.29 \times 10^{-10}.$$

5. A coin is tossed six times. Determine the probablity of obtaining exactly four heads.

   Solution

   In a single throw of the coin, the probability of a head (and of a tail) is $\frac{1}{2}$.

   Secondly, the probability that a particular four out of six throws will be heads, **and** the other two tails, will be

   $$\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\left[\frac{1}{2}.\frac{1}{2}\right] = \frac{1}{2^6} = \frac{1}{64}.$$

   Finally, the number of choices of four throws from six throws is

   $$^6C_4 = \frac{6!}{2!4!} = 15.$$

   Hence, the required probability of exactly four heads is $\frac{15}{64}$.

19.2.3 PERMUTATIONS OF SETS WITH SOME OBJECTS ALIKE

INTRODUCTORY EXAMPLE

Suppose twelve switch buttons are to be arranged in a row, and there are two red buttons, three yellow and seven green. How many possible distinct patterns can be formed ?

Solution

If all twelve buttons were of a different colour, there would be 12! possible arrangements.
If we now colour two switches red, there will be only half the number of arrangements since every pair of positions previously held by them would have counted 2! times; that is, twice.
If we then colour another three switches yellow, the positions previously occupied by them would have counted 3! times; that is, 6 times, so we reduce the number of arrangements further by a factor of 6.
Similarly, by colouring another seven switches green, we reduce the number of arrangements further by a factor of 7!.

Hence, the final number of arrangements will be

$$\frac{12!}{2!3!7!} = 7920.$$

This example illustrates another standard rule that, if we have n objects of which r_1 are alike of one kind, r_2 are alike of another, r_3 are alike of another ....and r_k are alike of another, then the number of permutations of these n objects is given by

$$\frac{n!}{r_1!r_2!r_3!...r_k!}$$

19.2.4 EXERCISES

1. Evaluate the following:
   (a) ${}^6P_3$; (b) ${}^6C_3$; (c) ${}^7P_4$; (d) ${}^7C_4$.
2. Verify that ${}^nC_{n-r} = {}^nC_r$ in the following cases:
   (a) $n = 7$ and $r = 2$; (b) $n = 10$ and $r = 3$; (c) $n = 5$ and $r = 2$.
3. How many four digit numbers can be formed from the digits 1,3,5,7,8,9 if none of the digits appears more than once in each number ?
4. In how many ways can three identical jobs be filled by 12 different people ?
5. From a bag containing 7 black balls and 5 white balls, how many sets of 5 balls, of which 3 are black and two are white, can be drawn ?
6. In how many seating arrangements can 8 people be placed around a table if there are 3 who insist on sitting together ?
7. A commitee of 10 is to be selected from 6 lawyers 8 engineers and 5 doctors. If the committe is to consist of 4 lawyers, 3 engineers and 3 doctors, how many such committees are possible ?
8. An electrical engineer is faced with 8 brown wires and 9 blue wires. If he is to connect 4 brown wires and 3 blue wires to 7 numbered terminals, in how many ways can this be done ?
9. A railway coach has 12 seats facing backwards and 12 seats facing forwards, In how many ways can 10 passengers be seated if 2 refuse to face forwards and 4 refuse to face backwards ?

19.2.5 ANSWERS TO EXERCISES

1. (a) 120; (b) 20; (c) 840; (d) 35.
2. (a) 21; (b) 120; (c) 10.
3. $$^6P_4 = 360.$$
4. $$^{12}C_3 = 220.$$
5. $$^7C_3 \times {}^5C_2 = 350.$$
6. $$3! \times 8 \times 5! = 5760.$$
7. $$^6C_4 \times {}^8C_3 \times {}^5C_3 = 8400.$$
8. $$^8C_4 \times {}^9C_3 \times 7! = 29,635,200.$$
9. $$^{12}P_2 \times {}^{12}P_4 \times {}^{18}P_4 = 115,165,670,400.$$

“JUST THE MATHS”

UNIT NUMBER

19.3

PROBABILITY 3
(Random variables)

by

A.J.Hobson

UNIT 19.3 - PROBABILITY 3 - RANDOM VARIABLES

19.3.1 DEFINING RANDOM VARIABLES

(i) The kind of experiments discussed in the theory of probability are usually what are known as **"random experiments"**.

For example, in an experiment which involves throwing a die, suppose that the die is "unbiased". This means that it is just as likely to show one face as any other.

Similarly, drawing 6 numbers out of a possible 45 for a lottery is a random experiment, provided it is just as likely for one number to be drawn as any other.

In general, an experiment is a random experiment if there is more than one possible outcome (or event) and any one of those possible outcomes may occur. We assume that the outcomes are mutually exclusive (see Unit 19.1, section 19.1.4).

The probabilities of the possible outcomes of a random experiment form a collection called the **"probability distribution"** of the experiment. These probabilities need not be the same as one another.

The complete list of possible outcomes is called the **"sample space"** of the experiment.

(ii) In a random experiment, each of the possible outcomes may, for convenience, be associated with a certain numerical value called a **"random variable"**. This variable, which we shall call x in general, makes it possible to refer to an outcome without having to use a complete description of it.

In tossing a coin, for instance, we might associate a head with the number 1 and a tail with the number 0; then we could state the probabilities of either a head or a tail being obtained by means of the formulae

$$P(x=1)=0.5 \ \text{ and } \ P(x=1)=0.5$$

Note:
There is no restriction on the way we define the values of x; it would have been just as correct to associate a head with -1 and a tail with 1. But it is customary to assign the values of random variables as logically as possible. For example, in discussing the probability that two 6's would be obtained in 5 throws of a dice, we could sensibly use $x = 1, 2, 3, 4, 5$ and 6, respectively, for the results that a 1,2,3,4,5 and 6 would be thrown.

(iii) It is usually necessary to distinguish between random variables which are **"discrete"** and those which are **"continuous"**.

Discrete random variables may take only certain specified values, while continuous random variables may take any value with in a certain specified range.

Examples of discrete random variables include those associated with the tossing of coins, the throwing of dice and numbers of defective components in a batch from a production line.

Examples of continuous variables include those associated with persons' height or weight, lifetimes of manufactured components and rainfall figures.

Note:
For a random variable, x, the associated probabilities form a function, $P(x)$, called the **"probability function"**.

19.3.2 PROBABILITY DISTRIBUTION AND PROBABILITY DENSITY FUNCTIONS

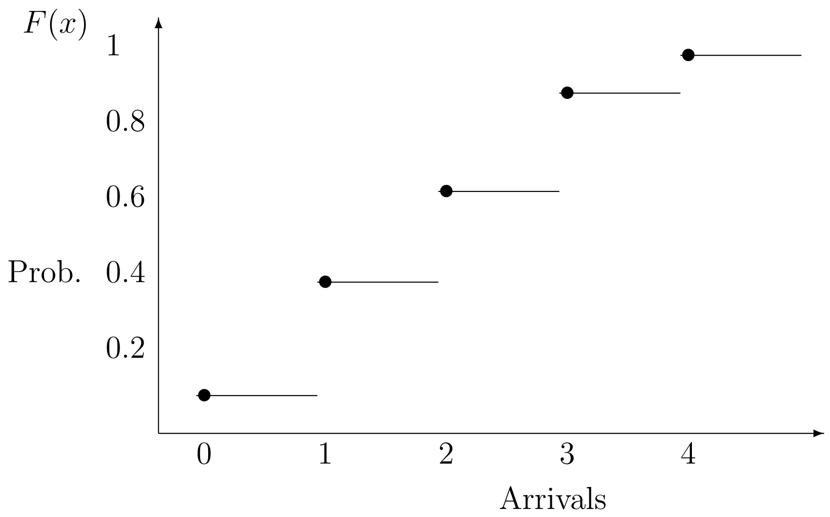
(a) Probability Distribution Functions

A **"probability distribution function"**, which will normally be denoted here by $F(x)$, is the relationship between a random variable, x, and the probability of obtaining any outcome for which the random variable can take values up to and including x. In other words, it is the probability, $P(\leq x)$, that the random variable for the outcome is less than or equal to x.

(i) Probability distribution functions for discrete variables

By way of illustration, suppose that the number of ships arriving at a container terminal during any one day can be 0,1,2,3 or 4, with respective probabilites 0.1, 0.3, 0.35, 0.2 and 0.05. The probabilites for outcomes other than those specified is taken to be zero.

The graph of the probability distribution function is as follows:



The value of the probability distribution function at a value, x, of the random variable is the sum of the probabilities to the left of, and including, x.

In view of the discontinuous nature of the graph, we have used "bullet" marks to indicate which end of each horizontal line belongs to the graph.

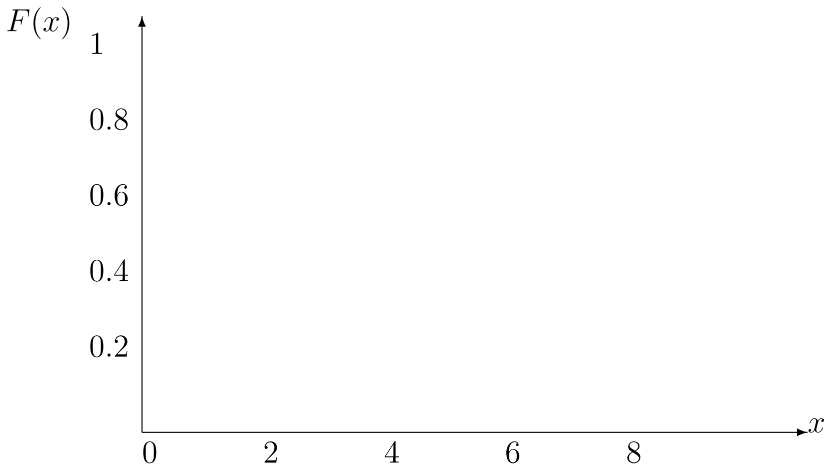
(ii) Probability distribution functions for continuous variables

For a continuous random variable, the probability distribution function is defined in a similar way as for a discrete variable. It measures (as before) the probability that the value of the random variable is less than or equal to x.

By way of illustration, we shall quote, here, the example of an **"exponential distribution"** in which it may be shown that the lifetime of a certain electronic component (in thousands of hours) is represented by a probability distribution function

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

The graph of the probability distribution function is as follows:



(b) Probability Density Functions

In the case of continuous random variables, a second function, $f(x)$ called the **"probability density function"** is defined to be the derivative, with respect to x, of the probability distribution function, $F(x)$.

That is,

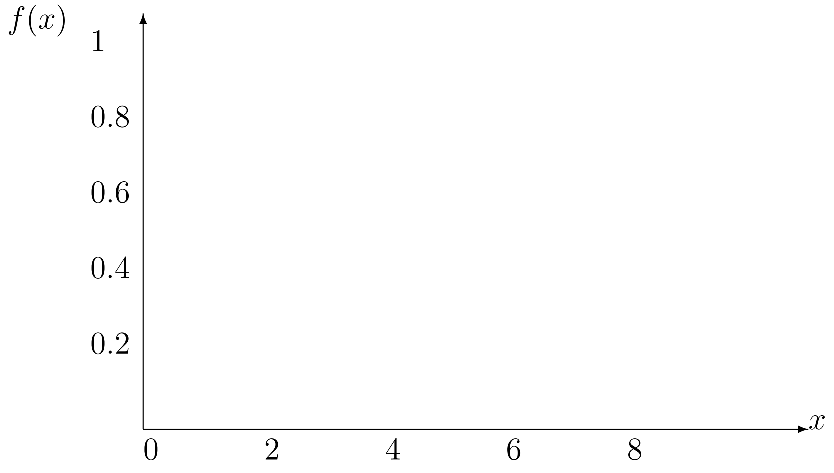
$$f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)].$$

The probability density function measures the **concentration** of possible values of x.

In the previous example, the probability density function is therefore given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0 \end{cases}$$

The graph of the probability density function is as follows:



We may observe that most components have short lifetimes, while a small number can survive for much longer.

(c) Properties of probability distribution and probability density functions

The following properties are a consequence of the appropriate definitions:

(i)

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

Proof:

It is impossible for a random variable to have a value less than $-\infty$ and it is certain to have a value less than ∞.

(ii) If $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

Proof:

The outcomes of an experiment with random variable values up to and including x_2 includes those outcomes with random variable values up to and including x_1 so that $F(x_2)$ is as least as great as $F(x_1)$.

Note:
Results **(i)** and **(ii)** imply that, for any value of x, the probability distribution function is either constant or increasing between 0 and 1.

(iii) The probability that an outcome will have a random variable value, x, within the range $x_1 < x \leq x_2$ is given by the expression

$$F(x_2) - F(x_1).$$

Proof:

From, the outcomes of an experiment with random variable values up to and including x_2, suppose we exclude those outcomes with random variable values up to and including x_1. The residue will be those outcomes with random variable values which lie within the range $x_1 < x \leq x_2$.

Thus, the difference between the values of the probability distribution function at two particular points is the probability that the value of the random variable will either lie between those two points or will be equal to the higher of the two.

Note:
For a continuous random variable, this is also equal to the area under the graph of the probability density function between the two given points, by virtue of the definition that $f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}[F(x)]$.

That is,

$$\int_{x_1}^{x_2} f(x)\ \mathrm{d}x.$$

(iv)

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1.$$

Proof:

The total area under the probability density function must be 1 since the random variable must have a value somewhere.

EXAMPLE

For the distribution of component lifetimes (in thousands of hours) given earlier by

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0, \end{cases}$$

determine the proportion of components which last longer than 3000 hours but less than or equal to 6000 hours.

Solution

The probability that components have lifetimes up to and including 3000 hours is given by

$$F(3) = 1 - e^{-\frac{3}{2}}.$$

The probability that components have lifetimes up to and including 600 hours is given by

$$F(6) = 1 - e^{-\frac{6}{2}} = 1 - e^{-3}.$$

The probability that components last longer than 3000 hours but less than or equal to 6000 hours is thus given by

$$F(6) - F(3) = e^{-\frac{3}{2}} - e^{-3} \simeq 0.173$$

The required proportion is thus approximately one in six.

19.3.3 EXERCISES

1. A coin is tossed three times and the random variable, x, represents the number of heads minus the number of tails.

   Construct a definition for the probability distribution function, $F(x)$,

   (a) if the coin is 'fair' (perfectly balanced);

   (b) if the coin is biased so that a head is twice as likely to occur as a tail.

2. Construct a definition for the probability distribution function, $F(x)$, for the sum, x, of numbers obtained when a pair of dice is tossed.

3. A certain assembly process is such that the probability of success at each attempt is 0.2. The probability, $P(x)$ that x independent attempts are needed to achieve success is given by

$$P(x) \equiv (0.2)(0.8)^{x-1} \quad x = 1,\ 2,\ 3, \ldots$$

Plot a grap of the probability distribution function, $F(x)$, and determine the probability that success will be achieved by

(a) less than four independent attempts;

(b) more than three but less than or equal to five independent attempts.

4. The probability density function of a random variable, x, is given by

$$f(x) \equiv \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4; \\ 0 & \text{elswhere,} \end{cases}$$

where c is a constant.

Determine

(a) the value of c;

(b) the probability distribution function, $F(x)$;

(c) the probability that $x > 1$.

5. The shelf life (in hours) of a certain perishable packaged food is a random variable, x, with probability density function, $f(x)$ given by

$$f(x) \equiv \begin{cases} 20000(x+100)^{-3} & \text{when } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probabilities that one of these packages will have a shelf life of

(a) at least 200 hours;

(b) at most 200 hours;

(c) more than 80 hours but less than or equal to 120 hours.

19.3.4 ANSWERS TO EXERCISES

1. (a)

$$F(x) \equiv \begin{cases} \frac{1}{8} & \text{for } -3 \leq x < -1; \\ \frac{1}{2} & \text{for } -1 \leq x < 1; \\ \frac{7}{8} & \text{for } 1 \leq x < 3; \\ 1 & \text{for } x \geq 3. \end{cases}$$

(b)

$$F(x) \equiv \begin{cases} \frac{1}{27} & \text{for } -3 \leq x < -1; \\ \frac{7}{27} & \text{for } -1 \leq x < 1; \\ \frac{19}{27} & \text{for } 1 \leq x < 3; \\ 1 & \text{for } x \geq 3. \end{cases}$$

2.

$$F(x) \equiv \begin{cases} \frac{1}{36} & \text{for } 2 \leq x < 3; \\ \frac{1}{12} & \text{for } 3 \leq x < 4; \\ \frac{1}{6} & \text{for } 4 \leq x < 5: \\ \frac{5}{18} & \text{for } 5 \leq x < 6; \\ \frac{5}{12} & \text{for } 6 \leq x < 7; \\ \frac{7}{12} & \text{for } 7 \leq x < 8. \\ \frac{13}{18} & \text{for } 8 \leq x < 9; \\ \frac{5}{6} & \text{for } 9 \leq x < 10; \\ \frac{11}{12} & \text{for } 10 \leq x < 11; \\ \frac{35}{36} & \text{for } 11 \leq x < 12; \\ 1 & \text{for } x \geq 12. \end{cases}$$

3. (a) 0.488 (b) 0.3123

4. (a) $c = \frac{1}{4}$;

(b)

$$F(x) = \begin{cases} 0 & \text{when } x < 0; \\ \frac{1}{2}\sqrt{x} & \text{when } 0 \leq x \leq 4; \\ 1 & \text{when } x > 4. \end{cases}$$

(c) $\frac{1}{2}$

5. (a) $\frac{1}{9}$; (b) $\frac{3}{4}$; (c) 0.102

"JUST THE MATHS"

UNIT NUMBER

19.4

PROBABILITY 4
(Measures of location and dispersion)

by

A.J.Hobson

UNIT 19.4 - PROBABILITY 4
MEASURES OF LOCATION AND DISPERSION

19.4.1 COMMON TYPES OF MEASURE

We include, here three common measures of location (or central tendency), and one common measure of dispersion (or scatter), used in the discussion of probability distributions.

(a) The Mean

(i) For Discrete Random Variables

If the values $x_1,\ x_2,\ x_3,\ \ldots\ .\ ,\ x_n$ of a discrete random variable, x, have probabilities $P_1,\ P_2,\ P_3,\ \ldots\ .\ , P_n$, respectively, then P_i represents the expected frequency of x_i divided by the total number of possible outcomes. For example, if the probability of a certain value of x is 0.25, then there is a one in four chance of its occurring.

The arithmetic mean, μ, of the distribution may therefore be given by the formula

$$\mu = \sum_{i=1}^{n} x_i P_i.$$

(ii) For Continuous Random Variables

In this case, it is necessary to use the probability density function, $f(x)$, for the distribution which is the rate of increase of the probability distribution function, $F(x)$.

For a small interval, δx of x-values, the probability that any of these values occurs is approximately $f(x)\delta x$, which leads to the formula

$$\mu = \int_{-\infty}^{\infty} x f(x)\ \mathrm{d}x.$$

(b) The Median

(i) For Discrete Random Variables

The median provides an estimate of the middle value of x, taking into account the frequency at which each value occurs. More precisely, it is a value, m, of the random variable, x, for which

$$P(x \leq m) \geq \frac{1}{2} \ \textbf{and}\ \ P(x \geq m) \geq \frac{1}{2}.$$

The median for a discrete random variable may not be unique (see Example 1, on page 3).

(ii) For Continuous Random Variables

The median for a continuous random variable is a value of the random variable, x, for which there are equal chances of x being greater than or less than the median itself. More precisely, it may be defined as the value, m, for which $P(x \leq m) = F(m) = \frac{1}{2}$.

Note:
Other measures of location are sometimes used, such as **"quartiles"**, **"deciles"** and **"percentiles"**, which divide the range of x values into four, ten and one hundred equal parts, respectively. For example, the third quartile of a distribution function, $F(x)$, may be defined as a value, q_3, of the random variable, x, such that

$$F(q_3) = \frac{3}{4}.$$

(c) The Mode

The mode is a measure of the most likely value occurring of the random variable, x.

(i) For Discrete Random Variables

In this case, the mode is any value of x with the highest probability, and, again, it may not be unique (see Example 1, on page 3).

(ii) For Continuous Random Variables

In this case, we require a value of x for which the probability density function (measuring the concentration of x values) has a maximum.

(d) The Standard Deviation

The most common meaure of dispersion (or scatter) for a probability distribution is the **"standard deviation"**, σ.

(i) For Discrete Random Variables

In this case, the standard deviation is defined by the formula

$$\sigma = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 P(x)}.$$

(ii) For Continuous Random Variables

In this case, the standard deviation is defined by the formula

$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x)\ \mathrm{d}x},$$

where $f(x)$ denotes the probability density function.

Each measures the dispersion of the x values around the mean, μ.

Note:
σ^2 is known as the **"variance"** of the probability distribution.

EXAMPLES

1. Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviation for a simple toss of an unbiased die.

   Solution

   (a) The mean is given by

   $$\mu = \sum_{i=1}^{6} i \times \frac{1}{6} = \frac{22}{6} = 3.5$$

   (b) Both 3 and 4 on the die fit the definition of a median, since

   $$P(x \le 3) = \frac{1}{2}, \quad P(x \ge 3) = \frac{2}{3}$$

   and

   $$P(x \le 4) = \frac{2}{3}, \quad P(x \ge 4) = \frac{1}{2}.$$

   (c) All six outcomes count as a mode since they all have a probability of $\frac{1}{6}$.

   (d) The standard deviation is given by

   $$\sigma = \sqrt{\sum_{i=1}^{6} \frac{1}{6}(i-3.5)^2} \simeq 2.917$$

2. Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviation for the distribution function

$$F(x) \equiv \begin{cases} 1 - e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ 0 & \text{when } x < 0. \end{cases}$$

Solution

First, we need the probability density function, $f(x)$, which is given by

$$f(x) \equiv \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{when } x \geq 0; \\ \\ 0 & \text{when } x < 0 \end{cases}$$

Hence,

(a)

$$\mu = \int_0^\infty \frac{1}{2} x e^{-\frac{x}{2}} \, \mathrm{d}x,$$

which, on integration by parts, gives

$$\mu = \left[-xe^{-\frac{x}{2}}\right]_0^\infty + \int_0^\infty e^{-\frac{x}{2}} \, \mathrm{d}x = \left[-2e^{-\frac{x}{2}}\right]_0^\infty = 2.$$

(b) The median is the value, m, for which

$$F(m) = \frac{1}{2}.$$

That is,

$$1 - e^{-\frac{m}{2}} = \frac{1}{2},$$

giving

$$-\frac{m}{2} = \ln\left[\frac{1}{2}\right];$$

and, hence, $m \simeq 1.386$.

(c) The mode is zero, since the maximum value of the probability density function occurs when $x = 0$.

(d) The standard deviation is given by

$$\sigma^2 = \int_0^\infty \frac{1}{2}(x-2)^2 e^{-\frac{x}{2}}\ \mathrm{d}x,$$

which, on integration by parts, gives

$$\sigma^2 = -\left[(x-2)^2 e^{-\frac{x}{2}}\right]_0^\infty + \int_0^\infty 2(x-2)e^{-\frac{x}{2}}\ \mathrm{d}x$$

$$= 4 - \left[4(x-2)e^{-\frac{x}{2}}\right]_0^\infty\ 4e^{-\frac{x}{2}}\ \mathrm{d}x = 4.$$

Thus $\sigma = 2$.

19.4.2 EXERCISES

1. A probability function, $P(x)$, is defined by the following table:

x	0	1	2	3	4	5	6
$P(x)$	0.17	0.29	0.27	0.16	0.07	0.03	0.01

Determine (a) the mean, (b) the median, (c) the mode and (d) the standard deviaton of this distribution.

2. A certain assembly process is such that the probability of success at each attempt is 0.2. The probability, $P(x)$ that x independent attempts are needed to achieve success is given by

$$P(x) \equiv (0.2)(0.8)^{x-1} \quad x = 1,\ 2,\ 3, \ldots$$

Determine (a) the mean, (b) the median and (c) the standard deviation of this distribution.

3. The running distance (in thousands of kilometres) which car owners achieve from a certain type of tyre is a random variable with probability density function, $f(x)$, where

$$f(x) \equiv \begin{cases} \frac{1}{30}e^{-x/30} & \text{when } x > 0; \\ 0 & \text{when } x \leq 0, \end{cases}$$

Determine

(a) the probability that one of these tyres will last at most 19000km;

(b) (i) the mean, (ii) the median and (iii) the standard deviation of the distribution.

4. The probability density function, $f(x)$, of a random variable, x, is defined by

$$f(x) \equiv \begin{cases} 30x^2(1-x)^2 & \text{when } 0 < x < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Determine the probability that x will take a value within two standard deviations of its mean.

5. The probability density function, $f(x)$, of a random variable, x, is given by

$$f(x) \equiv \begin{cases} \frac{1}{12}xe^{-x^2/12} & \text{when } x > 0; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Determine (a) the median and (b) the mode for this distribution. Show also, (c), that the mean, μ, is given by

$$\mu = \int_0^\infty e^{-x^2/12}\, \mathrm{d}x.$$

19.4.3 ANSWERS TO EXERCISES

1. (a) Mean = 1.8; (b) median = 2; (c) mode = 1; (d) standard deviation = 1.34
2. (a) Mean = 5; (b) median = 3; (c) standard deviation = 4.47
3. (a) 0.47; (b) (i) mean = 30; (ii) median = 20.8; (iii) standard deviation = 30.
4. 0.969
5. (a) Median = $\sqrt{12\ln 2}$; (b) mode = $\sqrt{6}$.

"JUST THE MATHS"

UNIT NUMBER

19.5

PROBABILITY 5
(The binomial distribution)

by

A.J.Hobson

UNIT 19.5 - PROBABILITY 5 - THE BINOMIAL DISTRIBUTION

19.5.1 INTRODUCTION AND THEORY

In this Unit, we shall be concerned, firstly, with probability problems having only **two** events (which are mutually exclusive and independent), although many trials may be possible. For example, the pairs of events could be "up and down", "black and white", "good and bad", and, in general, "successful and unsuccessful".

Statement of the problem

Suppose that the probability of success in a single trial is unaffected when successive trials are carried out (that is, we have independent events). Then what is the probability that, in n successive trials, **exactly** r will be successful ?

General Analysis of the problem

Let us build up the solution in simple stages:

(a) If p is the probability of success in a single trial, then the probability of failure is $1-p = q$, say.

(b) In the following table, let S stand for success and let F stand for failure. The table shows the possible results of one, two or three trials and their corresponding probabilities:

NO. OF TRIALS	POSSIBLE RESULTS	RESPECTIVE PROBABILITIES
1	F,S	q, p
2	FF,FS,SF,SS	q^2, qp, pq, p^2
3	FFF,FFS,FSF,FSS, SFF,SFS,SSF,SSS	$q^3, q^2p, q^2p, qp^2,$ q^2p, qp^2, qp^2, p^3

(c) **Summary**

(i) In **one** trial, the probabilities that there will be exactly 0 or exactly 1 successes are the respective terms of the expression

$$q + p.$$

(ii) In **two** trials, the probabilities that there will be exactly 0, exactly 1 or exactly 2 successes are the respective terms of the expression

$$q^2 + 2qp + p^2; \quad \text{that is,} \quad (q + p)^2.$$

(iii) In **three** trials, the probabilities that there will be exactly 0, exactly 1, exactly 2 or exactly 3 successes are the respective terms of the expression

$$q^3 + 3q^2p + 3qp^2 + p^3; \quad \text{that is,} \quad (q + p)^3.$$

(iv) In **any number**, n, of trials, the probabilities that there will be exactly 0, exactly 1, exactly 2, exactly 3, ... or exactly n successes are the respective terms in the binomial expansion of the expression

$$(q + p)^n.$$

(d) **MAIN RESULT**:

The probability that, in n trials, there will be exactly r successes, is the term containing p^r in the binomial expansion of $(q + p)^n$.

It can be shown that this is the value of

$$^nC_r q^{n-r} p^r.$$

EXAMPLES

1. Determine the probability that, in 6 tosses of a coin, there will be exactly 4 heads.

   Solution

   $q = 0.5, \quad p = 0.5, \quad n = 6, \quad r = 4.$

   Hence, the required probability is given by

   $$^6C_4.(0.5)^2.(0.5)^4 = \frac{6!}{2!4!}.\frac{1}{4}.\frac{1}{16} = \frac{15}{64}.$$

2. Determine the probability of obtaining the most probable number of heads in 6 tosses of a coin.

   Solution

   The most probable number of heads is given by $\frac{1}{2} \times 6 = 3$.

   The probability of obtaining exactly 3 heads is given by

   $$^6C_3.(0.5)^3(0.5)^3 = \frac{6!}{3!3!}.\frac{1}{8}.\frac{1}{8} = \frac{20}{64} \simeq 0.31$$

3. Determine the probability of obtaining exactly 2 fives in 7 throws of a die.

   Solution

   $q = \frac{5}{6}, \quad p = \frac{1}{6}, \quad n = 7, \quad r = 2.$

   Hence, the required probability is given by

$$^{7}C_{2}.\left(\frac{5}{6}\right)^{5}.\left(\frac{1}{6}\right)^{2} = \frac{7!}{5!2!}.\left(\frac{5}{6}\right)^{5}.\left(\frac{1}{6}\right)^{2} \simeq 0.234$$

4. Determine the probability of throwing at most 2 sixes in 6 throws of a die.

   Solution

   The phrase "at most 2 sixes" means exactly 0, or exactly 1, or exactly 2. Hence, we add together the first three terms in the expansion of $(q+p)^6$, where $q = \frac{5}{6}$ and $p = \frac{1}{6}$. It may be shown that

$$(q+p)^6 = q^6 + 6q^5p + 15q^4p^2 + ...$$

   and, by substituting for q and p, the sum of the first three terms turns out to be

$$\frac{21875}{23328} \simeq 0.938$$

5. It is known that 10% of certain components manufactured are defective. If a random sample of 12 such components is taken, what is the probability that at least 9 are defective ?

   Solution

   Firstly, we note how the information suggests that removal of components for examination does not affect the probability of 10%. This is reasonable, since our sample is almost certainly very small compared with all components in existence.

   Secondly, the probability of <u>success</u> in this example is 0.1, even though it refers to <u>defective</u> items, and hence the probability of failure is 0.9.

   Thirdly, using $p = 0.1, \quad q = 0.9, \quad n = 12$, we require the probabilities (added together) of exactly 9, 10, 11 or 12 defective items and these are the <u>last</u> four terms in the expansion of $(q+p)^n$.

   That is,

$$^{12}C_9.(0.9)^3.(0.1)^9 + {}^{12}C_{10}.(0.9)^2.(0.1)^{10} + {}^{12}C_{11}.(0.9).(0.1)^{11} + (0.1)^{12} \simeq 1.658 \times 10^{-7}.$$

Note:

The use of the **"binomial distribution"** becomes very tedious when the number of trials is large and two other standard distributions (called the **"normal distribution"** and the **"Poisson distribution"**) can sometimes be used.

19.5.2 EXERCISES

1. A coin is tossed six times. What is the probability of getting exactly four heads ?
2. What is the probability of throwing at least four 7's in five throws of a pair of dice ?
3. In a roll of five dice, what is the probability of getting exactly four faces alike ?
4. If three dice are thrown, determine the probability that
   (a) all three will show the number 4;
   (b) all three will be alike;
   (c) two will show the number 4 and the third, something else;
   (d) all three will be different;
   (e) only two will be alike.
5. Hospital records show that 10% of the cases of a certain disease are fatal. If five patients suffer from this disease, determine the probability that
   (a) all will recover;
   (b) at least three will die;
   (c) exactly three will die;
   (d) a particular three will die and the others survive.
6. A particular student can solve, on average, half of the problems given to him. In order to pass the course, he is required to solve seven out of ten problems on an examination paper. What is the probability that he will pass ?

19.5.3 ANSWERS TO EXERCISES

1.

$$15 \times (0.5)^4 \times 0.5 \simeq 0.47$$

2.

$$\left(\frac{1}{6}\right)^5 + 5 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right) \simeq 0.0032$$

3.

$$6 \times 5 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right) \simeq 0.019$$

4. (a)
$$\left(\frac{1}{6}\right)^3 \simeq 0.0046;$$
(b)
$$3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right) \simeq 0.069;$$
(c)
$$6 \times \left(\frac{1}{6}\right)^3 \simeq 0.028;$$
(d)
$$1 - 6\left[\left(\frac{1}{6}\right)^3 + 3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)\right] \simeq 0.56;$$
(e)
$$6 \times 6 \times \left(\frac{1}{6}\right)^3 \simeq 0.168$$

5. (a)
$$(0.9)^5;$$
(b)
$$(0.1)^5 + 5 \times (0.9) \times (0.1)^4 + 10 \times (0.9)^2 \times (0.1)^3 \simeq 0.0086;$$
(c)
$$10 \times (0.9)^2 \times (0.1)^3 \simeq 0.0081;$$
(d)
$$\frac{10 \times (0.9)^2 \times (0.1)^3}{{}^5C_3} \simeq 0.00081$$

6.
$$(1 + 10 + 45 + 120) \times (0.5)^{10} \simeq 0.17$$

"JUST THE MATHS"

UNIT NUMBER

19.6

PROBABILITY 6
(Statistics for the binomial distribution)

by

A.J.Hobson

UNIT 19.6 - PROBABILITY 6

STATISTICS FOR THE BINOMIAL DISTRIBUTION

19.6.1 CONSTRUCTION OF HISTOGRAMS

Elementary discussion on the presentation of data, in the form of frequency tables, histograms etc., usually involves experiments which are actually carried out.

But we illustrate now how the binomial distribution may be used to estimate the results of a certain kind of experiment <u>before</u> it is performed.

EXAMPLE

For four coins, tossed 32 times, construct a histogram showing the expected number of occurrences of 0,1,2,3,4..... heads.

Solution

Firstly, in a single toss of the four coins, the probability of head (or tail) for each coin is $\frac{1}{2}$.

The terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^4$ give the probabilities of exactly 0,1,2,3 and 4 heads, respectively.

The expansion is

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4.$$

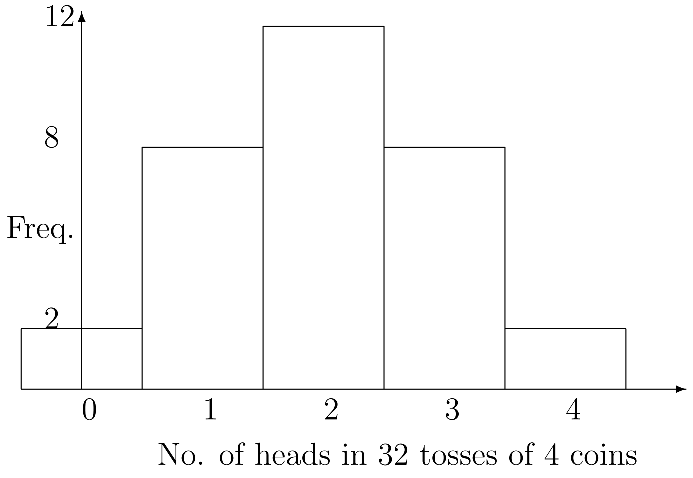
That is,

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4(1+4+6+4+1),$$

showing that the probabilities of 0,1,2,3 and 4 heads in a single toss of four coins are $\frac{1}{16}$, $\frac{1}{4}$, $\frac{6}{16}$, $\frac{1}{4}$, and $\frac{1}{16}$, respectively.

Therefore, in 32 tosses of four coins, we may expect 0 heads, twice; 1 head, 8 times; 2 heads, 12 times; 3 heads, 8 times and 4 heads, twice.

The following histogram uses class-intervals for which each member is, in fact, situated at the mid-point:



Notes:

(i) The only reason that the above histogram is symmetrical in shape is that the probability of success and failure are equal to each other, so that the terms of the binomial expansion are, themselves, symmetrical.

(ii) Since the widths of the class-intervals in the above histogram are 1, the areas of the rectangles are equal to their heights. Thus, for example, the total area of the first three rectangles represents the expected number of times of obtaining at most 2 heads in 32 tosses of 4 coins.

19.6.2 MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

THEOREM

If p is the probability of success of an event in a single trial and q is the probability of its failure, then the binomial distribution, giving the expected frequencies of 0,1,2,3,.....n successes in n trials, has a mean of np and a standard deviation of $\sqrt{npq}$, irrespective of the number of times the experiment is to be carried out.

Proof:

(a) The Mean

From the binomial expansion formula,

$$(q+p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 + \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ldots + nqp^{n-1} + p^n.$$

Hence, if the n trials are made N times, the average number of successes is equal to the following expression, multiplied by N, then divided by N:

$$0 \times q^n + 1 \times nq^{n-1}p + 2 \times \frac{n(n-1)}{2!}q^{n-2}p^2 +$$

$$3 \times \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ .\ .(n-1) \times nqp^{n-1} + np^n.$$

That is, the mean is

$$np\left(q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2}q^{n-3}p^2 + \ \ldots + (n-1)qp^{n-2} + p^{n-1}\right)$$

$$= np(q+p)^{n-1} = np \ \text{ since } \ q+p=1.$$

(b) The Standard Deviation

For the standard deviation, we observe that, if f_r is the frequency of r successes when the n trials are conducted N times, then

$$f_r = N\frac{n!}{(n-r)!r!}q^{n-r}p^r.$$

We use this, first, to establish a result for

$$\sum_{r=0}^{n} r^2 f_r.$$

For example,

$0^2 f_0 = 0.Nq^n = 0 = 0.f_0.$

$1^2 f_1 = 1.Nnq^{n-1}p = 1.f_1.$

$2^2 f_2 =$

$$2Nn(n-1)q^{n-2}p^2 = Nn(n-1)q^{n-2}p^2 + Nn(n-1)p^2q^{n-2}$$

$$= 2f_2 + Nn(n-1)p^2q^{n-2};$$

$3^2 f_3 =$

$$3N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3 = N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3 + Nn(n-1)p^2(n-2)q^{n-3}p$$

$$= 3f_3 + Nn(n-1)p^2(n-2)q^{n-3}p;$$

$4^2 f_4 =$

$$4N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4 = N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4 + Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2$$

$$= 4f_4 + Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2;$$

and, in general, when $r \geq 2$,

$r^2 f_r =$

$$N\frac{n(n-1)(n-2)....(n-r+1)}{(r-1)!} + Nn(n-1)p^2q^{n-r}p^r = rf_r + Nn(n-1)p^2\frac{(n-2)!}{(n-r)!(r-2)!}q^{n-r}p^{r-2}.$$

This result, together with those for $0^2.f_0$ and $1^2 f_1$, shows that

$$\sum_{r=0}^{n} r^2 f_r = \sum_{r=0}^{n} rf_r + Nn(n-1)p^2\sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!}q^{n-r}p^{r-2}.$$

That is,

$$\sum_{r=0}^{n} r^2 f_r = Nnp + Nn(n-1)p^2(q+p)^{n-2} = Nnp + Nn(n-1)p^2,$$

since $q+p=1$.

It was also established, in Unit 18.3, that the standard deviation of a set, $x_1, x_2, x_3, \ldots x_m$, of m observations, with a mean value of $\overline{x}$, is given by the formula

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} x_i^2 - \overline{x}^2,$$

which, in the present case, may be written

$$\sigma^2 = \frac{1}{N}\sum_{r=0}^{n} r^2 f_r - \frac{1}{N^2}\left(\sum_{r=0}^{n} r f_r\right)^2.$$

Hence,

$$\sigma^2 = \frac{1}{N}\left(Nnp + Nn(n-1)p^2\right) - \frac{1}{N^2}(Nnp)^2,$$

which gives

$$\sigma^2 = np + n^2p^2 - np^2 - n^2p^2 = np(1-p) = npq;$$

and so,

$$\sigma = \sqrt{npq}.$$

ILLUSTRATION

For direct calculation of the mean and the standard deviation for the data in the previous coin-tossing problem, we may use the following table, in which x_i denotes numbers of heads and f_i denotes the corresponding expected frequencies:

x_i	f_i	$f_i x_i$	$f_i x_i^2$
0	2	0	0
1	8	8	8
2	12	24	48
3	8	24	72
4	2	8	32
Totals	32	64	160

The mean is given by

$$\overline{x} = \frac{64}{32} = 2 \quad \text{(obviously)},$$

which agrees with $np = 4 \times \frac{1}{2}$.

The standard deviation is given by

$$\sigma = \sqrt{\left[\frac{160}{32} - 2^2\right]} = 1,$$

which agrees with $\sqrt{npq} = \sqrt{4 \times \frac{1}{2} \times \frac{1}{2}}$.

Note:
If the experiment were carried out N times instead of 32 times, all values in the last three columns of the above table would be multiplied by a factor of $\frac{N}{32}$, which would then cancel out in the remaining calculations.

EXAMPLE

Three dice are rolled 216 times. Construct a binomial distribution and show the frequencies of occurrence for 0,1,2 and 3 sixes.

Evaluate the mean and the standard deviation of the distribution.

Solution

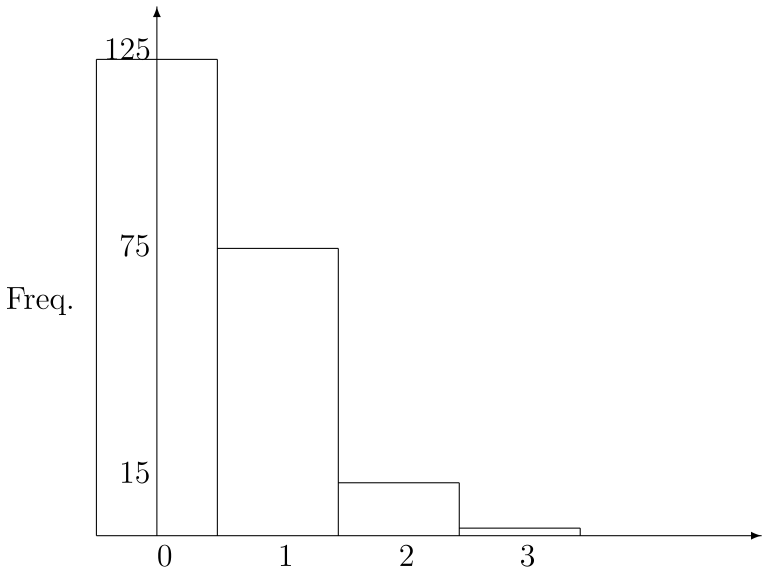
First of all, the probability of success in obtaining a six with a single throw of a die is $\frac{1}{6}$, and the corresponding probability of failure is $\frac{5}{6}$.

For a single throw of three dice, we require the expansion

$$\left(\frac{1}{6}+\frac{5}{6}\right)^3 \equiv \left(\frac{1}{6}\right)^3 + 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3,$$

showing that the probabilities of 0,1,2 and 3 sixes are $\frac{125}{216}$, $\frac{75}{216}$, $\frac{15}{216}$ and $\frac{1}{216}$, respectively.

Hence, in 216 throws of the three dice we may expect 0 sixes, 125 times; 1 six, 75 times; 2 sixes, 15 times and 3 sixes, once. The corresponding histogram is as follows:



From the previous Theorem, the mean value is

$$3 \times \frac{1}{6} = \frac{1}{2}$$

and the standard deviation is

$$\sqrt{3 \times \frac{1}{6} \times \frac{5}{6}} = \frac{\sqrt{15}}{6}.$$

19.6.3 EXERCISES

1. Four dice are rolled 81 times. If less than 5 on a die is considered to be a success, and everything else a failure,

   (a) draw the corresponding histogram for the expected frequencies of success;

   (b) determine the expected number of times of obtaining at least three successes among the four dice;

   (c) shade the area of the histogram which is a measure of the result in (c);

   (d) calculate the mean and the standard deviation of the frequency distribution in (a).

2. In a seed-viability test, 450 seeds were placed on a filter-paper in 90 rows of 5. The number of seeds that germinated in each row were counted, and the results were as follows:

No. of seeds germinating per row	0	1	2	3	4	5
Observed Frequency of rows	0	1	11	30	38	10

   If the germinating seeds were distributed, at random, among the rows, we would expect a binomial distribution with an index of $n = 5$.

   Determine

   (a) the average number of seeds germinating per row;

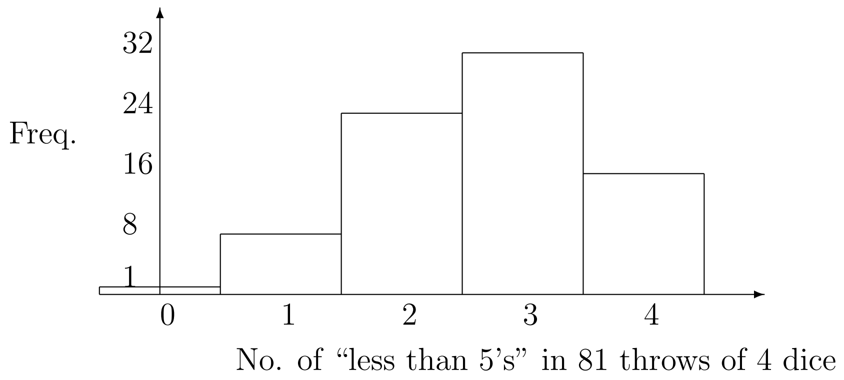
   (b) the probability of a single seed germinating;

   (c) the expected frequencies of rows for each number of seeds germinating;

   Draw the histogram for the expected frequencies and the histogram for the observed frequencies.

19.6.4 ANSWERS TO EXERCISES

1. (a) The histogram is as follows:



   (b) Expected frequency of 3 or 4 successes $= 32 + 16 = 48$;

   (c) Shade the last two rectangles on the right of the histogram;

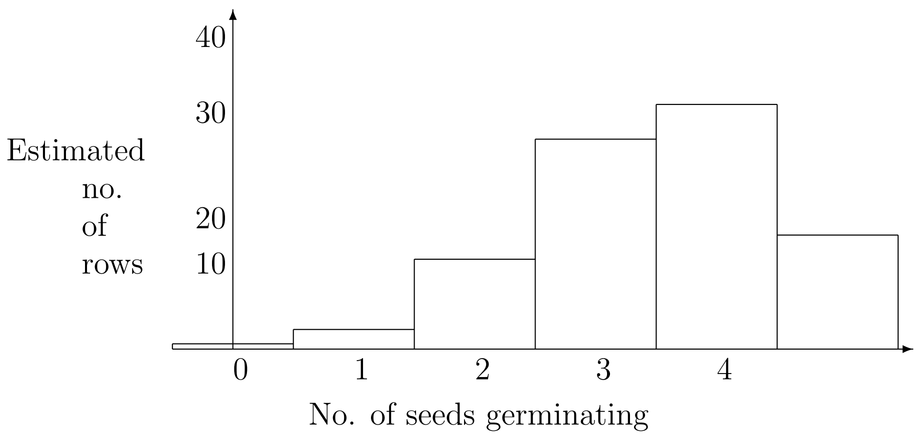
   (d) Mean $= 4 \times \frac{2}{3} = \frac{8}{3}$ and Standard Deviation $= \sqrt{4 \times \frac{2}{3} \times \frac{1}{3}} = \frac{2\sqrt{2}}{3}$.

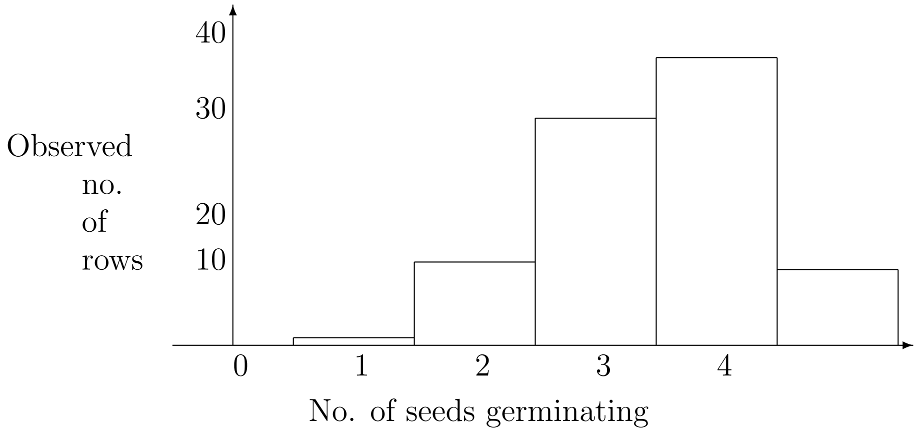
2. (a) Average number of seeds germinating per row is $\frac{325}{90} = 3.50$;

   (b) Probability of a single seed germinating is $\frac{3.5}{5} = \frac{7}{10}$;

   (c) Expected frequencies for 0,1,2,3,4,5 seeds are 0.2187, 2.5515, 11.9070, 27.7830, 32.4135, 15.1263 respectively.

   (d) The histograms are as follows:




40
30
Observed no. of rows
20
10
0
1
2
3
4
No. of seeds germinating

"JUST THE MATHS"

UNIT NUMBER

19.7

PROBABILITY 7
(The Poisson distribution)

by

A.J.Hobson

UNIT 19.7 - PROBABILITY 7

THE POISSON DISTRIBUTION

19.7.1 THE THEORY

We recall that, in a binomial distribution of n trials, the probability, P_r, that an event occurs exactly r times out of a possible n is given by

$$P_r = \frac{n!}{(n-r)!r!}p^r q^{n-r},$$

where p is the probability of success in a single trial and $q = 1 - p$ is the probability of failure.

Now suppose that n is very large compared with r and that p is very small compared with 1.

Then,

(a)

$$\frac{n!}{(n-r)!} = n(n-1)(n-2).....(n-r+1) \simeq n^r.$$

ILLUSTRATION

If $n = 120$ and $r = 3$, then

$$\frac{n!}{(n-r)!} = \frac{120!}{117!} = 120 \times 119 \times 118 \simeq 120^3.$$

(b)

$$q^r = (1-p)^r \simeq 1,$$

so that

$$q^{n-r} \simeq q^n = (1-p)^n.$$

We may deduce that

$$P_r \simeq \frac{n^r p^r (1-p)^n}{r!} = \frac{(np)^r (1-p)^n}{r!}$$

or

$$P_r \simeq \frac{(np)^r}{r!}\left[1 - np + \frac{n(n-1)}{2!}p^2 - \frac{n(n-1)(n-2)}{3!}p^3 + \ldots\ldots\right]$$

$$\simeq \frac{(np)^r}{r!}\left[1 - np + \frac{(np)^2}{2!} - \frac{(np)^3}{3!} + \ldots\ldots\right].$$

Hence,

$$P_r \simeq \frac{(np)^r}{r!}e^{-np}.$$

The number, np, in this formula is of special significance, being the average number of successes to be expected a single set of n trials.

If we denote np by μ, we obtain the **"Poisson distribution"** formula,

$$P_r \simeq \frac{\mu^r e^{-\mu}}{r!}.$$

Notes:

(i) Although the formula has been derived from the binomial distribution, as an approximation, it may also be used in its own right, in which case we drop the approximation sign.

(ii) The Poisson distribution is more use than the binomial distribution when n is a very large number, the binomial distribution requiring the tedious evaluation of its various coefficients.

(iii) The Poisson distribution is of particular use when the average frequency of occurrence of an event is known, but not the number of trials.

EXAMPLES

1. The number of cars passing over a toll-bridge during the time interval from 10a.m. until 11a.m. is 1,200.

   (a) Determine the probability that not more than 4 cars will pass during the time interval from 10.45a.m. until 10.46a.m.

   (b) Determine the probability that 5 or more cars pass during the same interval.

   Solution

   The number of cars which pass in 60 minutes is 1200, so that the average number of cars passing, per minute, is $20 = \mu$.

   (a) The probability that not more than 4 cars pass in a one-minute interval is the sum of the probabilities for 0,1,2,3 and 4 cars.

   That is,

$$\sum_{r=0}^{4} \frac{(20)^r e^{-20}}{r!} =$$

$$\left[\frac{(20)^0}{0!} + \frac{(20)^1}{1!} + \frac{(20)^2}{2!} + \frac{(20)^3}{3!} + \frac{(20)^4}{4!}\right] e^{-20} =$$

$$8221e^{-20} \simeq 1.69 \times 10^{-5}.$$

   (b) The probability that 5 or more cars will pass in a one-minute interval is the probability of failure in (a). In other words,

$$1 - \sum_{r=0}^{4} \frac{(20)^r e^{-20}}{r!} =$$

$$1 - 8221e^{-20} \simeq 0.99998$$

2. A company finds that, on average, there is a claim for damages which it must pay 7 times in every 10 years. It has expensive insurance to cover this situation.

   The premium has just been increased, and the firm is considering letting the insurance lapse for 12 months as it can afford to meet a single claim.

Assuming a Poisson distribution, what is the probability that there will be at least two claims during the year ?

Solution

Using $\mu = \frac{7}{10} = 0.7$, the probability that there will be at <u>most</u> one claim during the year is given by

$$P_0 + P_1 = e^{-0.7} + e^{-0.7}0.7 = e^{0.7}(1 + 0.7).$$

The probability that there will be at <u>least</u> two claims during the year is given by

$$1 - e^{-0.7}(1 + 0.7) \simeq 0.1558$$

3. There is a probability of 0.005 that a welding machine will produce a faulty joint when it is operated. The machine is used to weld 1000 rivets. Determine the probability that at least three of these are faulty.

   Solution

   First, we have $\mu = 0.005 \times 1000 = 5$.

   Hence, the probability that at <u>most</u> two will be faulty is given by

$$P_0 + P_1 + P_2 = e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right].$$

   That is,

$$e^{-5}[1 + 5 + 12.5] \simeq 0.125$$

   Hence, the probability that at least three will be faulty is approximately

$$1 - 0.125 = 0.875$$

19.7.2 EXERCISES

1. The probability that a glass fibre will shatter during an experiment is believed to follow a Poisson distribution.

   In a certain apparatus, it was found that, on average, 7 glass fibres shattered.

   Determine the probability that, during a single demonstration of the experiment,

   (a) two glass fibres will shatter;

   (b) at least one glass fibre will shatter.

2. A major airline operates 350 flights per day throughout the world. The probability that a flight will be delayed for more than one hour, for any reason, is 0.7%. If more than four flights suffer such delays in one day, the implications for route organisation and crewing become serious. Calling such a day a “bad day”, determine the probabilities that

   (a) any particular day is a bad day;

   (b) at most two bad days occur in one week;

   (c) more than 50 bad days occur in a year of 365 days.

   State your answers correct to three significant figures.

3. It is known that 3% of bolts made by a certain machine are defective. If the bolts are packaged in boxes of 50, determine the probability that a given box will contain 4 defectives.

4. If 0.04% of cars break down while driving through a certain tunnel, determine the probability that at most 2 break down out of 2000 cars entering the tunnel on a given day. State your answer correct to three significant figures.

19.7.3 ANSWERS TO EXERCISES

1. (a) 2.28×10^{-3}; (b) 0.0676
2. (a) 0.102; (b) 0.128; (c) 0.011
3. 0.047 using $\mu = 1.5$
4. 0.952 using $\mu = 0.8$

"JUST THE MATHS"

UNIT NUMBER

19.8

PROBABILITY 8
(The normal distribution)

by

A.J.Hobson

UNIT 19.8 - PROBABILITY 8

THE NORMAL DISTRIBUTION

19.8.1 LIMITING POSITION OF A FREQUENCY POLYGON

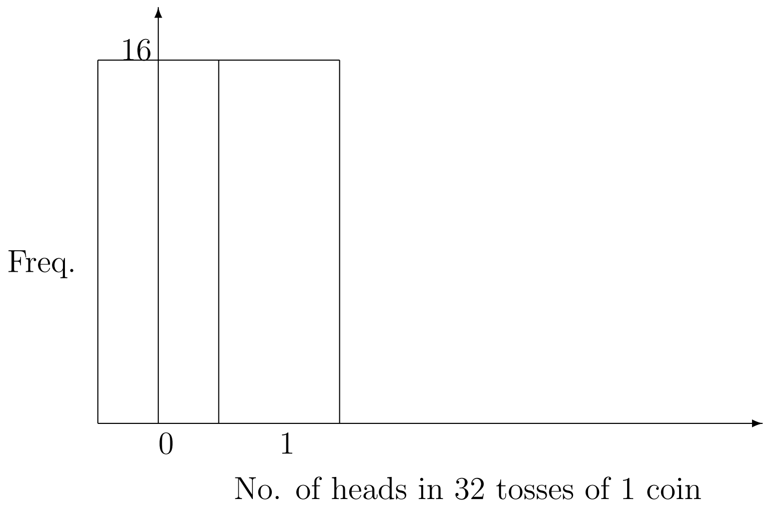
The distribution considered here is also appropriate to examples where the number of trials is large and, hence, the calculation of frequencies and probabilities, using the binomial distribution, would be inconvenient.

We shall introduce the **"normal distribution"** by considering the histograms of the binomial distribution for a toss of 32 coins as the number of coins increases.

The probability of obtaining a head is $\frac{1}{2}$ and the probability of obtaining a tail is also $\frac{1}{2}$.

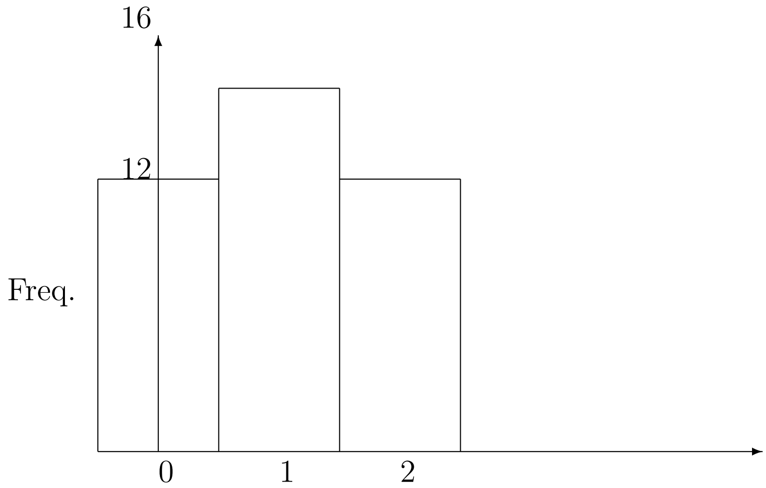
(i) One Coin

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^1 = 32\left(\frac{1}{2}+\frac{1}{2}\right) = 16 + 16.$$



(ii) Two Coins

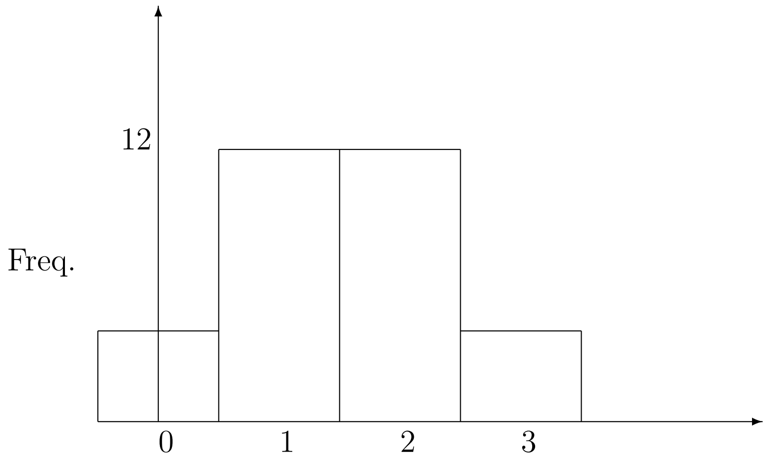
$$32\left(\frac{1}{2}+\frac{1}{2}\right)^2 = 32\left(\left[\frac{1}{2}\right]^2 + 2\left[\frac{1}{2}\right]\left[\frac{1}{2}\right] + \left[\frac{1}{2}\right]^2\right) = 8+16+8.$$



No. of heads in 32 tosses of 2 coins

(iii) Three Coins

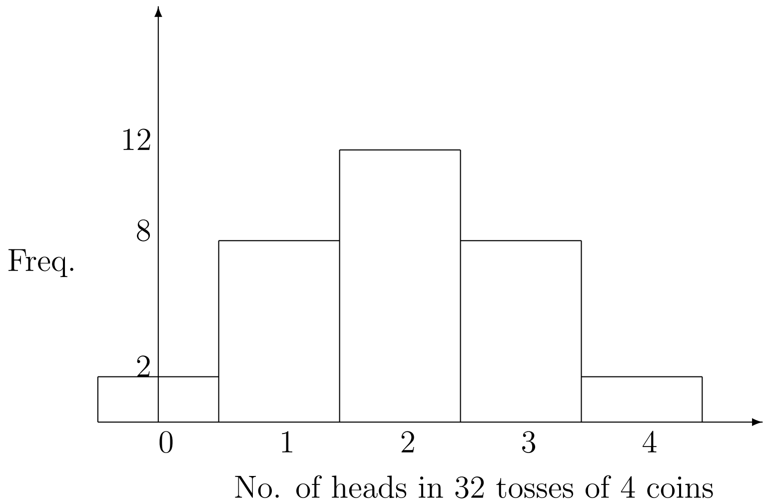
$$32\left(\frac{1}{2}+\frac{1}{2}\right)^3 = 32\left(\left[\frac{1}{2}\right]^3 + 3\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right] + 3\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^2 + \left[\frac{1}{2}\right]^3\right) = 4+12+12+4.$$



No. of heads in 32 tosses of 3 coins

(iv) Four Coins

$$32\left(\frac{1}{2}+\frac{1}{2}\right)^4 = 32\left(\left[\frac{1}{2}\right]^4 + 4\left[\frac{1}{2}\right]^3\left[\frac{1}{2}\right] + 6\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right]^2 + 4\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^3 + \left[\frac{1}{2}\right]^4\right) = 2+8+12+8+2.$$



It is apparent that, as the number of coins increases, the frequency polygon (consisting of the straight lines joining the midpoints of the tops of each rectangle) approaches a symmetrical bell-shaped curve.

This, of course, is true only when the histogram itself is either symmetrical or nearly symmetrical.

DEFINITION

As the number of trials increases indefinitely, the limiting position of the frequency polygon is called the **"normal frequency curve"**.

THEOREM

In a binomial distribution for N samples of n trials each, where the probability of success in a single trial is p, it may be shown that, as n increases indefinitely, the frequency polygon approaches a smooth curve, called the **"normal curve"**, whose equation is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}.$$

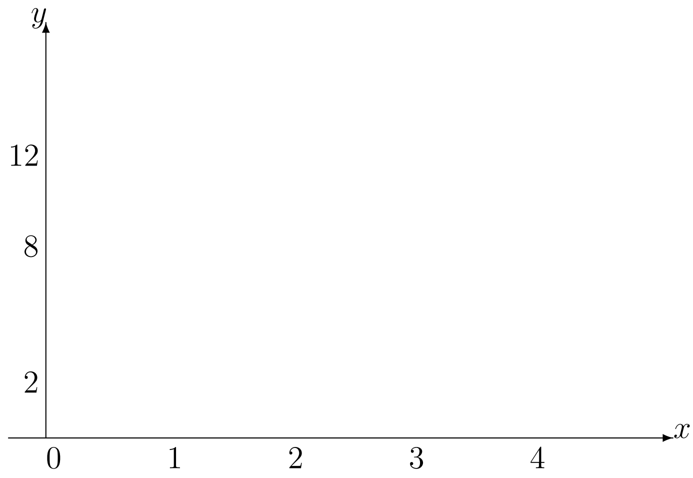
In this equation

$\overline{x}$ is the mean of the binomial distribution $= np$;

σ is the standard deviation of the binomial distribution $= \sqrt{np(1-p)}$;

y is the frequency of occurence of the value, x.

For example, the histogram for 32 tosses of 4 coins approximates to the following normal curve:



Notes:

(i) We omit the proof of the Theorem since it is beyond the scope of the present text.

(ii) The larger the value of n, the better is the level of approximation.

(iii) The normal curve is symmetrical about the straight line $x = \overline{x}$, since the value of y is the same at $x = \overline{x} \pm h$ for any number, h.

(iv) If the relative frequency (or probability) with which the value, x, occurs is denoted by P, then $P = y/N$ and the above relationship can be written

$$P = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}},$$

the graph of which is called the **"normal probability curve"**.

(v) Symmetrical curves are easier to deal with if the vertical axes of co-ordinates is the line of symmetry.
The normal probability curve can be simplified if we move the origin to the point $(\overline{x}, 0)$ and plot $P\sigma$ on the vertical axis instead of P.

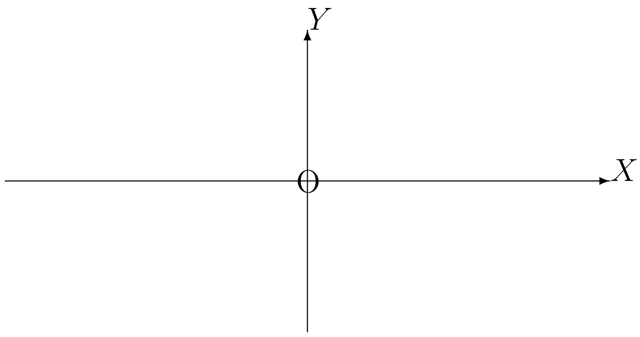
Letting $P\sigma = Y$ and $\frac{x-\overline{x}}{\sigma} = X$ the equation of the normal probability curve becomes

$$Y = \frac{1}{\sqrt{2\pi}}e^{-\frac{X^2}{2}},$$

which represents the **"standard normal probability curve"**.

From any point on it, we may obtain the values of the original P and x values by using the formulae

$$x = \sigma X + \overline{x} \quad \text{and} \quad P = \frac{Y}{\sigma}.$$



(vi) If the probability of success, p, in a single trial is **not** equal to, or approximately equal to, $\frac{1}{2}$, then the distribution given by the normal frequency curve and the two subsequent curves will be a poor approximation and is seldom used for such cases.

19.8.2 AREA UNDER THE NORMAL CURVE

For the histogram of a binomial distribution, corresponding to values of x, suppose that $x = a$ and $x = b$ are the values of x at the base-centres of two particular rectangles, where $b > a$ and all rectangles have width 1.

Then, the area of the histogram from $x = a - \frac{1}{2}$ to $x = b + \frac{1}{2}$ represents the number of times which we can expect values of x, between $x = a$ and $x = b$ inclusive, to occur.

Consequently, for a large number of trials, we may use the area under the normal curve between $x = a - \frac{1}{2}$ and $x = b + \frac{1}{2}$.

In a similar way, the **probability** that x will lie between $x = a$ and $x = b$ is represented by the area under the normal probability curve from $x = a - \frac{1}{2}$ and $x = b + \frac{1}{2}$. We note that the total area under this curve must be 1, since it represents the probability that **any** value of x will occur (a certainty).

In order to make use of a standard normal probabilty curve for the same purpose, the conversion formulae from x to X and P to Y must be used.

Note:
Tables are commercially available for the area under a standard normal probability curve; and, in using such tables, the above conversions will usually be necessary. A sample table is given in an appendix at the end of this unit.

EXAMPLE

If 12 dice are thrown, determine the probability, using the normal probability curve approximation, that 7 or more dice will show a 5.

Solution

For this example, we use $p = \frac{1}{6}$, $q = \frac{5}{6}$, $n = 12$ and we need the area under the normal probability curve from $x = 6.5$ to $x = 12.5$.

The mean of the binomial distribution, in this case, is $\overline{x} = 12 \times \frac{1}{6} = 2$.

The standard deviation is $\sigma = \sqrt{2 \times \frac{1}{6} \times \frac{5}{6}} \simeq \sqrt{1.67} \simeq 1.29$.

The required area under the standard normal probability curve will be that lying between

$$X = \frac{6.5 - 2}{1.29} \simeq 3.49 \text{ and } X = \frac{12.5 - 2}{1.29} \simeq 8.14$$

In practice, we take the whole area to the right of $X = 3.49$ since the area beyond $X = 8.14$ is negligible.

Also, the total area to the right of $X = 0$ is 0.5; and, hence, the required area is 0.5 minus the area from $X = 0$ to $X = 3.49$.

From tables, the required area is $0.5 - 0.4998 = 0.0002$ and this is the probability that, when 12 dice are thrown, 7 or more will show a 5.

Note:
If we had required the probability that 7 or fewer dice show a 5, we would have needed the area under the normal probability curve from $x = -0.5$ to $x = 7.5$. This is equivalent to taking the whole of the area under the standard normal probability curve which lies to the left of

$$X = \frac{7.5 - 2}{1.29} \simeq 4.26$$

19.8.3 NORMAL DISTRIBUTION FOR CONTINUOUS VARIABLES

So far, the variable, x, considered in a Normal distribution has been "discrete"; that is, x has been able to take only the specific values 0,1,2,3.....etc.

Here, we consider the situation when x is a "continuous" variable; that is, it may take any value within a certain range appropriate to the problem under consideration.

For a large number of observations of a continuous variable, the corresponding histogram need not have rectangles of class-width 1 but of some other number, say c.

In this case, it may be shown that the normal curve approximation to the histogram has equation

$$y = \frac{Nc}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}.$$

The smaller is the value of c, the larger is the number of rectangles and the better is the approximation supplied by the curve.

If we wished to calculate the number of x-values lying between $x = a$ and $x = b$ (where $b > a$), we would need to calculate the area of the histogram from $x = a$ to $x = b$ inclusive, then **divide by** c, since the base-width is no longer 1.

We conclude that the number of these x-values approximates to the area under the normal curve from $x = a$ to $x = b$; and similarly, the area under the normal probability curve, from $x = a$ to $x = b$ gives an estimate for the probability that values of x between $x = a$ and $x = b$ will occur.

EXAMPLE

Given a normal distribution of a continuous variable, x, with $N = 2000$, $\overline{x} = 20$ and $\sigma = 5$, determine

(a) the number of x-values lying between 12 and 22;

(b) the number of x-values larger than 30.

Solution

(a) The area under the normal probability curve between $x = 12$ and $x = 22$ is the area under the standard normal probability curve from

$$X = \frac{12 - 20}{5} = -1.6 \;\; \text{to} \;\; X = \frac{22 - 20}{5} = 0.4$$

and, from tables, this is $0.4452 + 0.1554 = 0.6006$

Hence, the required number of values is approximately $0.6006 \times 2000 \simeq 1201$.

(b) The total area under the normal probability curve to the right of $x = 30$ is the area under the standard normal probability curve to the right of

$$X = \frac{30 - 20}{5} = 2$$

and, from tables, this is 0.0227

Hence, the required number of values is approximately $0.0227 \times 2000 \simeq 45$.

19.8.4 EXERCISES

1. Use a normal probability curve approximation to determine the probability that, in a toss of 9 coins, 3 to 6 heads are shown.
2. A coin is tossed 100 times. Use a normal probability curve approximation to determine the probability of obtaining
   (a) exactly 50 heads;
   (b) 60 or more heads.

3. Assume that one half of the people in a certain community are regular viewers of television. Of 100 investigators, each interviewing 10 individuals, how many would you expect to report that 3 people or fewer were regular viewers ? (Use a normal probability curve approximation).
4. A manufacturer knows that, on average, 2% of his products are defective. Using a normal probability curve approximation, determine the probability that a batch of 100 components will contain exactly 5 defectives.
5. If the average life of a certain make of storage battery is 30 months, with a standard deviation of 6 months, what percentage can be expected to last from 24 to 36 months, assuming that their lifetimes follow a normal distribution.
6. If the heights of 10,000 university students closely follow a normal distribution, with a mean of 69.0 inches and a standard deviation of 2.5 inches, how many of these students would you expect to be at least 6 feet in height ?
7. In a certain trade, the average wage is £7.20 per hour and the standard deviation is 90p. If the wages are assumed to follow a normal distribution, what percentage of the workers receive wages between £6.00 and £7.00 per hour ?

19.8.5 ANSWERS TO EXERCISES

1. $P \simeq 0.8614$
2. (a) 0.0796
   (b) 0.0287
3. 17.
4. 0.0305
5. 68.26
6. 1151.
7. 32.11

APPENDIX

AREA UNDER THE STANDARD NORMAL PROBABILITY CURVE

The area given is that from $X = 0$ to a given value, X_1

X_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1046	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4773	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4983	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4989	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4995	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998	0.4988
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.5000	0.5000	0.5000