Functions of a complex variable

Derivative: If $w = f(z)$ where z and w are complex numbers, the derivative $\frac{dw}{dz}$ at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \right]$$

provided that the limit exists as $z \to z_0$ along *any* path. If $f(z)$ has a derivative at a point z_0 and at all points in some neighbourhood of z_0 then $f(z)$ is said to be **analytic** at z_0. If $f(z)$ is analytic at all points in an (open) region R then $f(z)$ is said to be **analytic** in R.

Cauchy-Riemann equations: If $z = x + jy$ and $w = f(z) = u(x, y) + jv(x, y)$ where x, y, u and v are real variables, and $f(z)$ is analytic in some region R of the z plane, then the **Cauchy-Riemann equations** hold throughout R:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If these partial derivatives are continuous within R, the Cauchy-Riemann equations are sufficient conditions to ensure $f(z)$ is analytic. Furthermore, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

Singularities: If $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0 then z_0 is called a **singular point** of $f(z)$.

Laurent series: If $f(z)$ is analytic on concentric circles C_1 and C_2 of radii r_1 and r_2, centred at z_0, and also analytic throughout the annular region between the circles, then for each point z within the annulus, $f(z)$ may be represented by the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

in which c_n are complex constants. The series may be written

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z - z_0)^n + \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Poles: The first sum on the right is the **principal part**. If there are only a finite number of terms in the principal part e.g.

$$f(z) = \frac{c_{-m}}{(z - z_0)^m} + \ldots + \frac{c_{-1}}{(z - z_0)} + c_0 + c_1(z - z_0) + \ldots + c_m(z - z_0)^m + \ldots$$

in which $c_{-m} \neq 0$, then $f(z)$ has a singularity called a **pole of order** m at $z = z_0$. A pole of order 1 is called a **simple pole**. If there are infinitely many terms in the principal part, z_0 is called an **isolated essential singularity**. If the principal part is zero, then $f(z)$ has a **removable singularity** at $z = z_0$ and the Laurent series reduces to a Taylor series.

Residues: If $f(z)$ has a pole at $z = z_0$ then the coefficient, c_{-1}, of $\frac{1}{z-z_0}$ in the Laurent expansion is called the **residue** of $f(z)$ at $z = z_0$. The residue at a pole of order m is given by:

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}.$$

When evaluating the integrals which follow, the curve C is traversed in an anticlockwise sense.

Cauchy's theorem: If $f(z)$ is analytic within and on a simple closed curve C then $\oint_C f(z) dz = 0$.

Cauchy's integral formula: If $f(z)$ is analytic within and on a simple closed curve C, and if z_0 is any point within C then

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi j f(z_0).$$

Further

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi j}{n!} f^{(n)}(z_0).$$

The residue theorem: If $f(z)$ is analytic within and on a simple closed curve C apart from a finite number of poles inside C, then

$$\oint_C f(z) dz = 2\pi j \times [\text{ sum of residues of } f(z) \text{ at the poles inside } C].$$

Eigenvalues & Eigenvectors

An **eigenvector** of a square matrix A is a non-zero column vector X such that $AX = \lambda X$ where λ, (a scalar), is the corresponding **eigenvalue**. The eigenvalues are found by solving the **characteristic equation**

$$\det(A - \lambda I) = 0.$$

An $n \times n$ symmetric matrix A with real elements has only real eigenvalues and n independent eigenvectors. The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

The **modal matrix** corresponding to the $n \times n$ square matrix A is an $n \times n$ square matrix P whose columns are the eigenvectors of A. If n independent eigenvectors are used to form P then $P^{-1}AP$ is a diagonal matrix in which the diagonal entries are the eigenvalues of A taken in the same order that the eigenvectors were taken to form P.

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Vector Calculus

$$\text{grad} \equiv \nabla \qquad \text{div} \equiv \nabla\cdot \qquad \text{curl} \equiv \nabla\times$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\text{Laplacian} \equiv \nabla^2 \equiv \text{div(grad)} \equiv \nabla\cdot\nabla \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

If $\Phi(x,y,z)$ is a scalar field and $\mathbf{v}(x,y,z) = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ is a vector field

$$\text{grad}\,\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i}+\frac{\partial\Phi}{\partial y}\mathbf{j}+\frac{\partial\Phi}{\partial z}\mathbf{k} \qquad \text{a vector.}$$

$$\text{div}\,\mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z} \qquad \text{a scalar.}$$

$$\text{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \qquad \text{a vector.}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}.$$

$$\nabla^2\mathbf{v} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}).$$

Vector calculus identities:

$\text{grad}(\Phi\psi) = \Phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\Phi$
$\text{div}\,(\Phi\mathbf{a}) = \Phi\,\text{div}\,\mathbf{a} + \mathbf{a}\cdot\text{grad}\,\Phi$
$\text{curl}\,(\Phi\mathbf{a}) = \Phi\,\text{curl}\,\mathbf{a} + \text{grad}\,\Phi\times\mathbf{a}$
$\text{curl}\,\text{grad}\,\Phi = \mathbf{0}, \qquad \text{div}\,\text{curl}\,\mathbf{a} = 0$
$\text{curl}\,\text{curl}\,\mathbf{a} = \text{grad}\,\text{div}\,\mathbf{a} - \nabla^2\mathbf{a}$
$\text{grad}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\text{grad})\mathbf{a} + (\mathbf{a}\cdot\text{grad})\,\mathbf{b} + \mathbf{b}\times\text{curl}\,\mathbf{a} + \mathbf{a}\times\text{curl}\,\mathbf{b}$
$\text{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\text{curl}\,\mathbf{a} - \mathbf{a}\cdot\text{curl}\,\mathbf{b}$
$\text{curl}(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\text{grad})\mathbf{a} - (\mathbf{a}\cdot\text{grad})\mathbf{b} + \mathbf{a}\,\text{div}\,\mathbf{b} - \mathbf{b}\,\text{div}\,\mathbf{a}$

Green's theorem in the plane:

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Stokes' theorem:

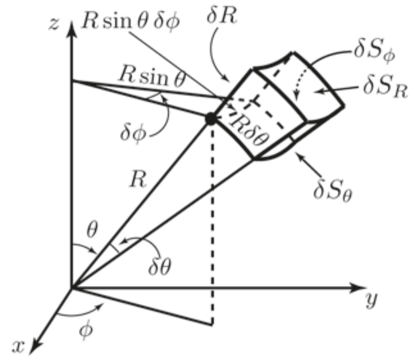
$$\oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{r} = \int_S \text{curl}\,\mathbf{v}\cdot\mathrm{d}\mathbf{S}.$$

The divergence theorem:

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{S} = \int_V \text{div}\,\mathbf{v}\,\mathrm{d}V.$$

Spherical polar coordinates

The diagram shows spherical polar coordinates (R,θ,ϕ).



$$\left.\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}\right\} \qquad \begin{aligned} R &\geq 0 \\ 0 \leq \theta &\leq \pi \\ 0 \leq \phi &< 2\pi \end{aligned}$$

If $\mathbf{v} = v_R\hat{\mathbf{e}}_R + v_\theta\hat{\mathbf{e}}_\theta + v_\phi\hat{\mathbf{e}}_\phi$:

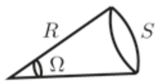
$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\hat{\mathbf{e}}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi.$$

$$\nabla\cdot\mathbf{v} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 v_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(v_\phi\right).$$

$$\nabla\times\mathbf{v} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_R & R\hat{\mathbf{e}}_\theta & R\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ v_R & Rv_\theta & R\sin\theta v_\phi \end{vmatrix}.$$

$$\begin{aligned}\nabla^2\Phi \;=\; & \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Phi}{\partial R}\right) + \\ & \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.\end{aligned}$$

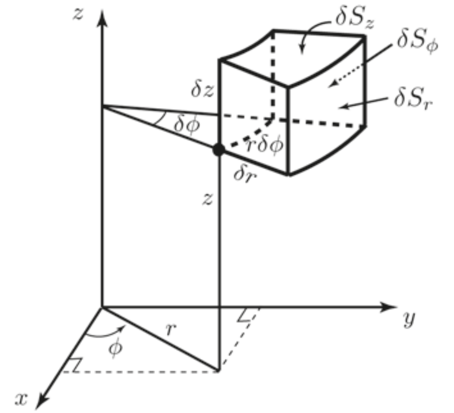
Volume element: $\delta V = R^2\sin\theta\,\delta R\,\delta\theta\,\delta\phi$.
Surface elements:
$\delta S_R = R^2\sin\theta\,\delta\theta\,\delta\phi$,
$\delta S_\theta = R\sin\theta\,\delta R\,\delta\phi$,
$\delta S_\phi = R\,\delta R\,\delta\theta$.



Solid angles: Consider part of a sphere of radius R. If the area cut off on the surface is S, the **solid angle** at the centre is $\Omega = \frac{S}{R^2}$ steradians. The solid angle at the apex of a cone of semi-vertical angle θ is $\Omega = 2\pi(1-\cos\theta)$.

Cylindrical polar coordinates

The diagram shows cylindrical polar coordinates (r,ϕ,z).



$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}\right\} \qquad \begin{aligned} r &\geq 0 \\ 0 \leq \phi &< 2\pi \\ -\infty < z &< \infty \end{aligned}$$

If $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\phi\hat{\mathbf{e}}_\phi + v_z\hat{\mathbf{e}}_z$:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z.$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(v_\phi) + \frac{\partial v_z}{\partial z}.$$

$$\nabla\times\mathbf{v} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ v_r & rv_\phi & v_z \end{vmatrix}.$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}.$$

Volume element: $\delta V = r\,\delta r\,\delta\phi\,\delta z$.
Surface elements:
$\delta S_r = r\,\delta\phi\,\delta z$,
$\delta S_\phi = \delta r\,\delta z$,
$\delta S_z = r\,\delta r\,\delta\phi$.

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Fourier Series

Fourier series:
If $f(t)$ is periodic with period T its Fourier series is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right)$$

or equivalently, if $\omega = 2\pi/T$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t).$$

a_n and b_n are called the **Fourier coefficients**, given by

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos\frac{2n\pi t}{T}\mathrm{d}t, \qquad \text{for } n = 0, 1, 2, 3 \ldots$$

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin\frac{2n\pi t}{T}\mathrm{d}t, \qquad \text{for } n = 1, 2, 3 \ldots$$

where d can be chosen to have any value.
If $f(t)$ is odd, $a_n \equiv 0$ and $f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$.
If $f(t)$ is even, $b_n \equiv 0$ and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$.

Parseval's theorem:

$$\frac{2}{T}\int_0^T (f(t))^2\mathrm{d}t = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Complex form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{j2n\pi t/T}, \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-j2n\pi t/T}\mathrm{d}t.$$

Half-range sine series: Given $f(t)$ for $0 < t < \frac{T}{2}$, its odd periodic extension has period T and Fourier series given by

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}.$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\mathrm{d}t \qquad \text{for } n = 1, 2, 3 \ldots$$

Half-range cosine series: Given $f(t)$ for $0 < t < \frac{T}{2}$, its even periodic extension has period T and Fourier series given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T}.$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2n\pi t}{T}\mathrm{d}t \qquad \text{for } n = 0, 1, 2, 3 \ldots$$

The Fourier transform

The **Fourier transform** of $f(t)$ is $F(\omega)$ defined by

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{d}t = F(\omega).$$

The **inverse Fourier transform** is given by

$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\mathrm{d}\omega = f(t).$$

function $f(t)$	Fourier transform $F(\omega)$
$Au(t)e^{-\alpha t},\ \alpha > 0$	$\frac{A}{\alpha + j\omega}$
$\begin{cases} 1 & -\alpha \le t \le \alpha \\ 0 & \text{otherwise} \end{cases}$	$\frac{2\sin\omega\alpha}{\omega}$
A constant	$2\pi A\delta(\omega)$
$u(t)A$	$A\left(\pi\delta(\omega) - \frac{j}{\omega}\right)$
$\delta(t)$	1
$\delta(t-a)$	$e^{-j\omega a}$
$\cos at$	$\pi(\delta(\omega + a) + \delta(\omega - a))$
$\sin at$	$\frac{\pi}{j}(\delta(\omega - a) - \delta(\omega + a))$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
$\frac{1}{t}$	$-j\pi\mathrm{sgn}(\omega)$
$\mathrm{e}^{-\alpha\|t\|},\ \alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$

Linearity:

$$\mathcal{F}\{f + g\} = \mathcal{F}\{f\} + \mathcal{F}\{g\}, \qquad \mathcal{F}\{kf\} = k\mathcal{F}\{f\}.$$

Shift theorems: If $F(\omega)$ is the Fourier transform of $f(t)$

$$\mathcal{F}\{\mathrm{e}^{jat}f(t)\} = F(\omega - a), \qquad a \text{ constant}.$$

$$\mathcal{F}\{f(t - \alpha)\} = \mathrm{e}^{-j\alpha\omega}F(\omega), \qquad \alpha \text{ constant}.$$

Differentiation: The Fourier transform of the nth derivative, $f^{(n)}(t)$, is $(j\omega)^n F(\omega)$.

Duality: If $F(\omega)$ is the Fourier transform of $f(t)$ then

the Fourier transform of $F(t) = 2\pi \times f(-\omega)$.

Convolution and correlation:
The Fourier transform of $f(t) * g(t)$ is $F(\omega)G(\omega)$ where

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\lambda)g(t - \lambda)\,\mathrm{d}\lambda = g(t) * f(t).$$

The Fourier transform of $f(t) \star g(t)$ is $F(\omega)G(-\omega)$ where

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\lambda)g(\lambda - t)\,\mathrm{d}\lambda.$$

Discrete Fourier transform (dft)

Given a sequence of N terms

$$\{g[0], g[1], g[2], \ldots, g[N-1]\}$$

its discrete Fourier transform (dft) is the sequence

$$\{G[0], G[1], G[2], \ldots, G[N-1]\}$$

where

$$G[k] = \sum_{n=0}^{N-1} g[n]\mathrm{e}^{-2jnk\pi/N}.$$

Further

$$g[n] = \frac{1}{N}\sum_{k=0}^{N-1} G[k]\mathrm{e}^{2jnk\pi/N}.$$

Maclaurin & Taylor Series

Maclaurin Series:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

Taylor series (one variable):

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$

Taylor series (two variables): For a function $f(x, y)$ of two variables

$$\begin{aligned} f(x,y) &= f(a,b) + \frac{1}{1!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)f(a,b) \\ &+ \frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^2 f(a,b) + \ldots \\ &+ \frac{1}{r!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^r f(a,b) + \ldots \end{aligned}$$

Stationary points in two variables: For $z = f(x, y)$, stationary points (a, b) are located by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Define $\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ at (a, b).
The type of stationary point is given by:

$\Delta < 0$	saddle point.
$\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$	minimum point.
$\Delta > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$	maximum point.

The Laplace transform

The **Laplace transform** of $f(t)$ is $F(s)$ defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t.$$

function $f(t), t \geqslant 0$	Laplace transform $F(s)$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$
$t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
$t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$u(t)$ unit step	$\frac{1}{s}$
$\delta(t)$ impulse function	1
$\delta(t-a)$	e^{-sa}
$f(t)$ periodic	$\frac{\int_0^T e^{-st} f(t)\mathrm{d}t}{1-e^{-sT}}$
$t^n f(t)$	$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$

Linearity:

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}, \qquad \mathcal{L}\{kf\} = k\mathcal{L}\{f\}.$$

Shift theorems: If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{e^{-at} f(t)\} = F(s+a).$$

$$\mathcal{L}\{u(t-d) f(t-d)\} = e^{-sd} F(s) \qquad d > 0.$$

$u(t)$ is the unit step or Heaviside function.

Laplace transform of derivatives and integrals:

$$\mathcal{L}\{f'\} = sF(s) - f(0).$$

$$\mathcal{L}\{f''\} = s^2 F(s) - sf(0) - f'(0).$$

$$\mathcal{L}\left\{\int_0^t f(t)\mathrm{d}t\right\} = \frac{1}{s} F(s).$$

The convolution theorem:

The Laplace transform of $f(t) * g(t)$ is $F(s)G(s)$ where

$$f(t) * g(t) = \int_0^t f(t-\lambda) g(\lambda)\,\mathrm{d}\lambda = g(t) * f(t).$$

The z transform

Given a sequence, $f[k]$, $k = 0, 1, 2\ldots$, the (one-sided) z **transform** of $f[k]$, is $F(z)$ defined by

$$F(z) = \mathcal{Z}\{f[k]\} = \sum_{k=0}^{\infty} f[k] z^{-k}.$$

sequence $f[k]$	z transform $F(z)$
$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$	1
$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$	$\frac{z}{z-1}$
k	$\frac{z}{(z-1)^2}$
e^{-ak}	$\frac{z}{z-e^{-a}}$
a^k	$\frac{z}{z-a}$
ka^k	$\frac{az}{(z-a)^2}$
k^2	$\frac{z(z+1)}{(z-1)^3}$
$\sin ak$	$\frac{z \sin a}{z^2 - 2z\cos a + 1}$
$\cos ak$	$\frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$
$e^{-ak} \sin bk$	$\frac{ze^{-a} \sin b}{z^2 - 2ze^{-a}\cos b + e^{-2a}}$
$e^{-ak} \cos bk$	$\frac{z^2 - ze^{-a} \cos b}{z^2 - 2ze^{-a}\cos b + e^{-2a}}$
$e^{-bk} f[k]$	$F(e^b z)$
$kf[k]$	$-z\frac{\mathrm{d}}{\mathrm{d}z} F(z)$

Linearity: If $f[k]$ and $g[k]$ are two sequences and c is a constant

$$\mathcal{Z}\{f[k] + g[k]\} = \mathcal{Z}\{f[k]\} + \mathcal{Z}\{g[k]\}.$$

$$\mathcal{Z}\{cf[k]\} = c\mathcal{Z}\{f[k]\}.$$

First shift theorem:

$$\mathcal{Z}\{f[k+1]\} = zF(z) - zf[0].$$

$$\mathcal{Z}\{f[k+2]\} = z^2 F(z) - z^2 f[0] - zf[1].$$

Second shift theorem:

$$\mathcal{Z}\{f[k-i]u[k-i]\} = z^{-i} F(z), \qquad i = 1, 2, 3\ldots$$

where $F(z)$ is the z transform of $f[k]$ and $u[k]$ is the unit step sequence.

Convolution: $\mathcal{Z}\{f[k] * g[k]\} = F(z)G(z)$.

where

$$f[k] * g[k] = \sum_{m=0}^{k} f[m] g[k-m].$$

Numerical Integration

Simpson's rule: for n even, and $h = \frac{x_n - x_0}{n}$,

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n).$$

Truncation error $\approx -\frac{(x_n - x_0) h^4 f^{(4)}(\zeta)}{180}$.

n point **Gauss-Legendre** formula:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} w_i f(x_i).$$

n	x_i	w_i
2	± 0.577350	1.000000
3	± 0.774597	0.555556
	0.0	0.888889
4	± 0.861136	0.347855
	± 0.339981	0.652145
5	± 0.906180	0.236927
	0.0	0.568889
	± 0.538469	0.478629

Ordinary differential equations

To solve $\dfrac{dy}{dx} = f(x, y)$:

Euler's method:

$$y_{r+1} = y_r + hf(x_r, y_r).$$

Modified Euler method:

$$y_{r+1}^{(p)} = y_r + hf_r \qquad f_{r+1}^{(p)} = f(x_{r+1}, y_{r+1}^{(p)}).$$

$$y_{r+1}^{(c)} = y_r + \frac{h}{2}(f_r + f_{r+1}^{(p)}).$$

Runge-Kutta method:

$$k_1 = hf(x_r, y_r), \quad k_2 = hf(x_r + \frac{h}{2}, y_r + \frac{k_1}{2}).$$

$$k_3 = hf(x_r + \frac{h}{2}, y_r + \frac{k_2}{2}), \quad k_4 = hf(x_r + h, y_r + k_3).$$

$$y_{r+1} = y_r + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$