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The polar form of a complex number

complex10

In this unit we look at the **polar form** of a complex number. You will have already seen that a complex number takes the form $z = a + b\mathrm{i}$. This form is called **Cartesian form**. When we are given a complex number in Cartesian form it is straightforward to plot it on an Argand diagram and then find its modulus and argument.

Instead of starting with the Cartesian form, sometimes the modulus, r say, and argument, θ say, are given to us. When this happens we are dealing with the polar form. In polar form we write

$$z = r\angle\theta$$

This means that z is the complex number with modulus r and argument θ.

Polar form:

$$z = r\angle\theta$$

Example. Plot the complex number $z = 4\angle 40°$ on an Argand diagram and find its Cartesian form.

Solution. The Argand diagram in Figure 1 shows the complex number with modulus 4 and argument $40°$.

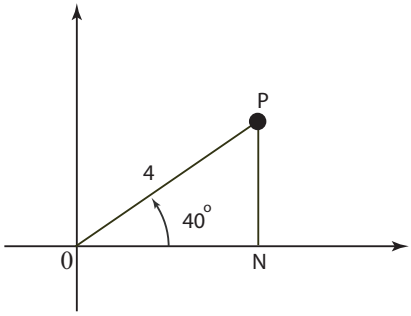


Figure 1. The complex number $z = 4\angle 40°$.

We can use trigonometry to find the Cartesian form:

$$\cos 40° = \frac{ON}{4}$$

so that

$$ON = 4\cos 40° = 3.06$$

Similarly,

$$\sin 40° = \frac{NP}{4}$$

$$NP = 4\sin 40° = 2.57$$

So the Cartesian form is $z = 3.06 + 2.57\mathrm{i}$.

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Having looked at a specific case, we will now look at the general case of a complex number $z = r\angle\theta$ as shown in Figure 2.

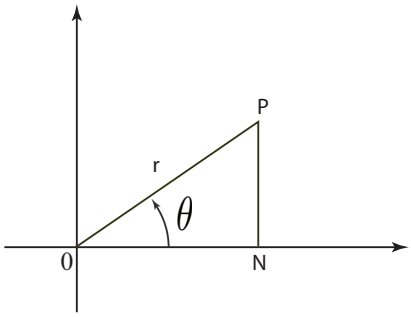


Figure 2. The complex number $z = r\angle\theta$.

To find the Cartesian form we need to find the length ON because this is the real part, and the length NP because this is the imaginary part.

$$\cos\theta = \frac{ON}{r} \qquad \text{so that} \qquad ON = r\cos\theta$$

$$\sin\theta = \frac{NP}{r} \qquad \text{so that} \qquad NP = r\sin\theta$$

We can now write down the Cartesian form: $z = r\cos\theta + r\sin\theta\,\mathrm{i}$.

Equivalent forms of the complex number z:

$$z = r\angle\theta = r(\cos\theta + \mathrm{i}\sin\theta)$$

Example. Convert the complex number $z = 6\angle 110°$ to Cartesian form.

Solution. The complex number is shown in Figure 3.

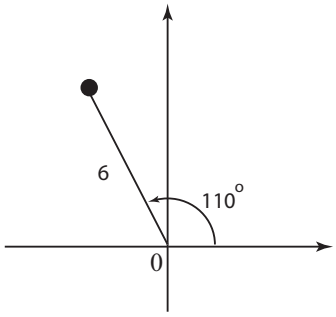


Figure 3. The complex number $z = 6\angle 110°$.

Using the previous result:

$$z = 6(\cos 110° + \mathrm{i}\sin 110°)$$

Using a calculator we find

$$z = -2.05 + 5.64\mathrm{i}$$

Comparing this solution with the diagram in Figure 3 shows that, as expected from the diagram, the real part is negative.

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