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in mathematics & statistics support

Complex numbers - motivating the study of complex numbers

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This leaflet will motivate the study of **complex numbers** by showing how to find the square roots of negative numbers such as

$$\sqrt{-7} \qquad \text{or} \qquad \sqrt{-9}$$

Before we do, let's have a think about **real numbers**.

A real number is any number which can be represented by a point on a number line. Such a number line is shown in Figure 1. Here we have marked on the number line whole numbers between -9 and $+9$. Every real number has a place on this line. For example, fractions, such as $-\frac{1}{2}$ and decimal fractions such as 3.5 are represented by the points shown. Numbers which cannot be represented as fractions, such as π, also have their place.

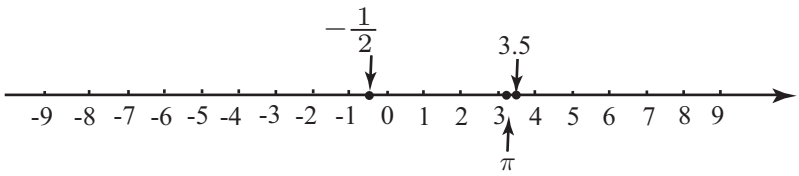


Figure 1. Every real number has a place on the number line

Consider what happens when we square any real number, that is, we multiply it by itself.

For example, squaring $+3$ the result is

$$3^2 = (+3) \times (+3) = +9$$

Squaring -3, the result is

$$(-3)^2 = (-3) \times (-3) = +9$$

In both cases the answer is positive.

If we take any real number and square it, the answer we get can never be negative.

It will turn out to be very useful to overcome this limitation: we introduce a new number, which will have the symbol i. The number i will have the special property that $\mathrm{i}^2 = -1$, that is, its square is negative! The number i cannot be a real number because its square is negative, and we have just seen that the square of a real number can never be negative. We say that the number i is **imaginary**. Having defined i^2 as -1 it follows that $\mathrm{i} = \sqrt{-1}$.

Key Point

i is an **imaginary** number such that $\mathrm{i}^2 = -1$

Equivalently, $\mathrm{i} = \sqrt{-1}$

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We now have a way of writing the square root of any negative number.

Example

Suppose we want to write an expression for $\sqrt{-9}$.

Noting that $-9 = 9 \times -1$, we can write

$$\begin{aligned}\sqrt{-9} &= \sqrt{9 \times -1} \\ &= \sqrt{9} \times \sqrt{-1} \\ &= 3 \times \sqrt{-1}\end{aligned}$$

Then, using the fact that $\sqrt{-1} = \mathrm{i}$, we have

$$\sqrt{-9} = 3\,\mathrm{i}$$

So we now have a way of formally writing down the square root of a negative number.

Example

Another example: suppose we want to find $\sqrt{-7}$. We write

$$\begin{aligned}\sqrt{-7} &= \sqrt{7 \times -1} \\ &= \sqrt{7} \times \sqrt{-1}\end{aligned}$$

Then, using the fact that $\sqrt{-1} = \mathrm{i}$, we have

$$\sqrt{-7} = \sqrt{7}\,\mathrm{i}$$

The whole idea of trying to write down square roots of negative numbers by introducing imaginary numbers may, at first sight, seem far-fetched and silly. But by doing this we open up a whole new world of numbers which have very important applications in engineering, physics and mathematics.

We can use the number i just as we would in other calculations:

Examples

Simplify the expressions: a) i^3, b) i^4.

Solution

a) $\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i}$. But $\mathrm{i}^2 = -1$ and so $\mathrm{i}^3 = -1 \times \mathrm{i} = -\mathrm{i}$.

b) $\mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = (-1) \times (-1) = +1$.

In the next unit we will look at how the imaginary number i can appear when we solve quadratic equations. Later units will explain that a **complex number** is one which contains the imaginary number i.

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