Engineering Maths First Aid Kit

7.7

The exponential form

Introduction

In addition to the cartesian and polar forms of a complex number there is a third form in which a complex number may be written - the **exponential form**. In this leaflet we explain this form.

1. Euler's relations

Two important results in complex number theory are known as **Euler's relations**. These link the exponential function and the trigonometric functions. They state:

Euler's relations:

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

The derivation of these relations is beyond the scope of this leaflet. By firstly adding, and then subtracting, Euler's relations we can obtain expressions for the trigonometric functions in terms of exponential functions. Try this!

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

2. The exponential form of a complex number

Using the polar form, a complex number with modulus r and argument θ may be written

$$z = r(\cos\theta + j\sin\theta)$$

It follows immediately from Euler's relations that we can also write this complex number in **exponential form** as $z = re^{j\theta}$.

Exponential form

$$z = r\,e^{j\theta}$$

When using this form you should ensure that all angles are measured in radians and not degrees.

Example

State the modulus and argument of the following complex numbers:

a) $z = 5e^{j\pi/6}$, b) $z = 0.01e^{0.02j}$, c) $3e^{-j\pi/2}$, d) $5e^2$.

Solution
In each case compare the given number with the standard form $z = re^{j\theta}$ to identify the modulus r and the argument θ.

a) The modulus and argument of $5e^{j\pi/6}$ are 5 and $\frac{\pi}{6}$ respectively.

b) The modulus and argument of $0.01e^{0.02j}$ are 0.01 and 0.02 respectively.

c) The modulus and argument of $3e^{-j\pi/2}$ are 3 and $-\frac{\pi}{2}$ respectively.

d) The number $5e^2$ is purely real, and can be evaluated using a calculator. Its modulus is 36.95 and its argument is zero.

Example
Find the real and imaginary parts of $z = 5e^{2j}$.

Solution
Recall that $e^{j\theta} = \cos\theta + j\sin\theta$. Then

$$\begin{aligned} 5e^{2j} &= 5(\cos 2 + j\sin 2) \\ &= 5\cos 2 + (5\sin 2)j \end{aligned}$$

The real part is $5\cos 2$ which equals -2.08. The imaginary part is $5\sin 2$, that is 4.55 (to 2 dp).

Example
Express the number $z = 3 + 3j$ in exponential form.

Solution
To express a number in exponential form we must first find its modulus and argument. The modulus of $3 + 3j$ is $\sqrt{3^2 + 3^2} = \sqrt{18}$. The complex number lies in the first quadrant of the Argand diagram and so its argument θ is given by $\theta = \tan^{-1}\frac{3}{3} = \frac{\pi}{4}$. Thus

$$z = 3 + 3j = \sqrt{18}e^{j\pi/4}$$

Exercises
1. State the modulus and argument of each of the following complex numbers:

a) $5e^{0.3j}$, b) $4e^{-j2\pi/3}$, c) $e^{2\pi j}$, d) $0.35e^{-0.2j}$.

2. Express each of the following in the form $re^{j\theta}$.

a) $3\angle(\pi/3)$, b) $\sqrt{2}\angle(\pi/4)$, c) $3\angle(-\pi/4)$, d) $5\angle 0$, e) $17\angle(\pi/2)$.

3. Express each of the following in the form $a + bj$.

a) $13e^{j\pi/3}$, b) $13e^{-j\pi/3}$, c) $4e^{2\pi j}$, d) $7e^{0.2j}$.

4. Show that e^{1+3j} is equal to e^1e^{3j}. Hence deduce $e^{1+3j} = -2.69 + 0.38j$.

Answers
1. a) 5, 0.3 radians, b) 4, $-2\pi/3$ radians, c) $1, 2\pi$ radians, d) 0.35, -0.2 radians.

2. a) $3e^{j\pi/3}$, b) $\sqrt{2}e^{j\pi/4}$, c) $3e^{-j\pi/4}$, d) $5e^0 = 5$, e) $17e^{j\pi/2}$.

3. a) $6.5 + 11.3j$, b) $6.5 - 11.3j$, c) 4, d) $6.86 + 1.39j$.

