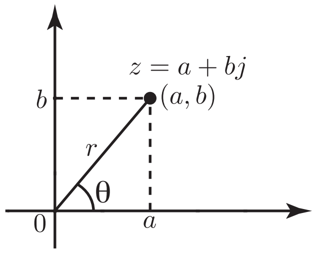
The form $r(\cos\theta + j\sin\theta)$

Introduction.

Any complex number can be written in the form $z = r(\cos\theta + j\sin\theta)$ where r is its modulus and θ is its argument. This leaflet explains this form.

1. The form $r(\cos\theta + j\sin\theta)$

Consider the figure below which shows the complex number $z = a + bj = r\angle\theta$.



Using trigonometry we can write

$$\cos\theta = \frac{a}{r} \qquad \text{and} \qquad \sin\theta = \frac{b}{r}$$

so that, by rearranging,

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

We can use these results to find the real and imaginary parts of a complex number given in polar form:

if $z = r\angle\theta$, the real and imaginary parts of z are:

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta, \qquad \text{respectively}$$

Using these results we can then write $z = a + bj$ as

$$\begin{aligned} z = a + bj &= r\cos\theta + jr\sin\theta \\ &= r(\cos\theta + j\sin\theta) \end{aligned}$$

This is an alternative way of expressing the complex number with modulus r and argument θ.

$$z = a + bj = r\angle\theta = r(\cos\theta + j\sin\theta)$$

Example
State the modulus and argument of a) $z = 9(\cos 40° + j \sin 40°)$, b) $z = 17(\cos 3.2 + j \sin 3.2)$.

Solution
a) Comparing the given complex number with the standard form $r(\cos\theta + j\sin\theta)$ we see that $r = 9$ and $\theta = 40°$. The modulus is 9 and the argument is $40°$.

b) Comparing the given complex number with the standard form $r(\cos\theta + j\sin\theta)$ we see that $r = 17$ and $\theta = 3.2$ radians. The modulus is 17 and the argument is 3.2 radians.

Example
a) Find the modulus and argument of the complex number $z = 5j$.

b) Express $5j$ in the form $r(\cos\theta + j\sin\theta)$.

Solution
a) On an Argand diagram the complex number $5j$ lies on the positive vertical axis a distance 5 from the origin. Thus $5j$ is a complex number with modulus 5 and argument $\frac{\pi}{2}$.

b)

$$z = 5j = 5(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$$

Using degrees we would write

$$z = 5j = 5(\cos 90° + j\sin 90°)$$

Example
a) State the modulus and argument of the complex number $z = 4\angle(\pi/3)$.

b) Express $z = 4\angle(\pi/3)$ in the form $r(\cos\theta + j\sin\theta)$.

Solution
a) Its modulus is 4 and its argument is $\frac{\pi}{3}$.

b) $z = 4(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3})$.

Noting $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ the complex number can be written $2 + 2\sqrt{3}j$.

Exercises
1. By first finding the modulus and argument express $z = 3j$ in the form $r(\cos\theta + j\sin\theta)$.

2. By first finding the modulus and argument express $z = -3$ in the form $r(\cos\theta + j\sin\theta)$.

3. By first finding the modulus and argument express $z = -1 - j$ in the form $r(\cos\theta + j\sin\theta)$.

Answers
1. $3(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$, 2. $3(\cos\pi + j\sin\pi)$,

3. $\sqrt{2}(\cos(-135°) + j\sin(-135°)) = \sqrt{2}(\cos 135° - j\sin 135°)$.

