Cramer's Rule

Introduction

Cramer's rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

1. Cramer's Rule - two equations

If we are given a pair of simultaneous equations

$$
\begin{aligned}
a_1 x + b_1 y &= d_1 \\
a_2 x + b_2 y &= d_2
\end{aligned}
$$

then x, and y can be found from

$$
x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}
$$

Example
Solve the equations

$$
\begin{aligned}
3x + 4y &= -14 \\
-2x - 3y &= 11
\end{aligned}
$$

Solution
Using Cramer's rule we can write the solution as the ratio of two determinants.

$$
x = \frac{\begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = 2, \qquad y = \frac{\begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{5}{-1} = -5
$$

The solution of the simultaneous equations is then $x = 2$, $y = -5$.

2. Cramer's rule - three equations

For the case of three equations in three unknowns: If

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3
\end{aligned}
$$

then x, y and z can be found from

$$
x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}
\qquad
y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}
\qquad
z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}
$$

Exercises

Use Cramer's rule to solve the following sets of simultaneous equations.

a)

$$
\begin{aligned}
7x + 3y &= 15 \\
-2x + 5y &= -16
\end{aligned}
$$

b)

$$
\begin{aligned}
x + 2y + 3z &= 17 \\
3x + 2y + z &= 11 \\
x - 5y + z &= -5
\end{aligned}
$$

Answers

a) $x = 3$, $y = -2$.   b) $x = 1$, $y = 2$, $z = 4$