

Transposition of formulae

In mathematics, engineering and science, formulae are used to relate physical quantities to each other. They provide rules so that if we know the values of certain quantities, we can calculate the values of others. In this unit we discuss how formulae can be transposed, or transformed, or rearranged.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- transpose formulae in order to make other variables the subject of the formula

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1. Introduction

Consider the formula for the period, T , of a simple pendulum of length l :

$$T = 2\pi \sqrt{\frac{l}{g}}, \quad \text{where } l \text{ is the length of the pendulum}$$

Now, on Earth, we tend to regard g , the acceleration due to gravity, as being fixed. It varies a little with altitude but for most purposes we can regard it as a constant.

Just suppose we had a pendulum of a fixed length l , and we took it somewhere else, say the moon, or Mars. The gravity there would not be the same and so the value of g would be different from its value on Earth.

Suppose we wanted to measure g . One way would be to take the pendulum, set it swinging and measure the time, T , for one complete cycle. Once we have measured the time, we could then use that to calculate g . But before we could do this, we would need to know what g was, in terms of the rest of the symbols in the formula. So we need to re-arrange the formula so that it states " $g = ?$ ". We will show you how to do this rearrangement later in the video.

To rearrange, transform, or tranpose the formula, we need many of the techniques used to solve equations. So, the video '*Solving Linear Equations in One Variable*' might be very useful to have a look at.

2. Solving a simple linear equation

Before we look at rearranging more complicated formulae we recap by having a look at a simple linear equation. Suppose we wanted to solve

$$3x + 5 = 6 - 3(5 - 2x)$$

Our aim is to end up with an expression for x , that is ' $x = ?$ '. We start by expanding the brackets on the right.

$$3x + 5 = 6 - 15 + 6x$$

$$3x + 5 = 6x - 9$$

We then proceed to manipulate this to try to get all the terms involving x on to one side. We must preserve the balance in the original equation by doing exactly the same operations to both sides.

Subtracting $3x$ from both sides:

$$3x - 3x + 5 = 6x - 3x - 9$$

$$5 = 3x - 9$$

Adding 9 to both sides:

$$5 + 9 = 3x - 9 + 9$$

$$14 = 3x$$

and so

$$x = \frac{14}{3}$$

We have obtained an expression for x as required.

With practice the amount of working written down will reduce because you will be able to carry out many of the stages at the same time. Having gone through the steps of solving an equation, the same technique, particularly the idea of keeping the balance by *doing the same thing to both sides*, is what we are going to do when we look at the transformation of formulae.

3. Transposition of simple formulae

Example

Consider the formula $v = u + at$. Suppose we wish to transpose this formula to obtain one for t . Because we want to obtain t on its own we start by subtracting u from each side:

$$\begin{aligned} v &= u + at \\ v - u &= at \end{aligned}$$

We now divide everything on both sides by a .

$$\frac{v - u}{a} = \frac{at}{a} = t$$

and so finally $t = \frac{v - u}{a}$. We have transposed the formula to find an expression for t .

Example

Consider the formula $v^2 = u^2 + 2as$ and suppose we wish to transpose it to find u . We want to obtain u on its own and so we begin by subtracting $2as$ from each side.

$$\begin{aligned} v^2 &= u^2 + 2as \\ v^2 - 2as &= u^2 \end{aligned}$$

Finally, taking the square root of both sides:

$$u = \sqrt{v^2 - 2as}$$

Notice we need to take the square root of the whole term ($\sqrt{v^2 - 2as}$) in order to find u .

Example

Consider the formula $s = ut + \frac{1}{2}at^2$. Suppose we want to transpose it to find a .

Because we want a on its own, we begin by subtracting ut from both sides.

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ s - ut &= \frac{1}{2}at^2 \end{aligned}$$

Multiplying both sides by 2:

$$2(s - ut) = at^2$$

Dividing both sides by t^2 :

$$\frac{2(s - ut)}{t^2} = a$$

and so

$$a = \frac{2(s - ut)}{t^2}$$

Example

Suppose we wish to rearrange $y(2x + 1) = x + 1$ in order to find x .

Notice that x occurs both on the left and on the right. We need to try to get all the terms involving x together. We begin by expanding the brackets on the left:

$$y(2x + 1) = x + 1$$

$$2xy + y = x + 1$$

Subtracting x from both sides:

$$2xy - x + y = 1$$

The left-hand side now has two terms involving x . We can factorise these as follows:

$$x(2y - 1) + y = 1$$

Then subtracting y from both sides:

$$x(2y - 1) = 1 - y$$

and finally, dividing both sides by $(2y - 1)$

$$x = \frac{1 - y}{2y - 1}$$

Example

Suppose we wish to rearrange $\frac{y}{y + x} + 5 = x$ to find an expression for y .

We begin by multiplying every term on both sides by $(y + x)$ in order to remove the fractions:

$$y + 5(y + x) = x(y + x)$$

Next we multiply out the brackets:

$$y + 5y + 5x = xy + x^2$$

We try to get all the terms involving y onto the left-hand side. Subtracting xy from both sides:

$$6y - xy + 5x = x^2$$

Subtracting $5x$ from both sides, and taking out the common factor y we have

$$y(6 - x) = x^2 - 5x$$

Finally, dividing both sides by $6 - x$ we obtain

$$y = \frac{x^2 - 5x}{(6 - x)}$$

Exercise 1

Rearrange each of the following formulae to make the quantity shown the subject.

1. $v = u + at,$ u

2. $v^2 = u^2 + 2as,$ s

3. $s = vt - \frac{1}{2}at^2,$ a

4. $p = 2(w + h),$ h

5. $A = 2\pi r^2 + 2\pi rh,$ h

6. $E = \frac{1}{2}mv^2 + mgh,$ v

7. $E = \frac{1}{2}mv^2 + mgh,$ m

8. $a(3b - 1) = 2b + 2,$ b

9. $\frac{t}{2t - s} = 3s,$ t

10. $\frac{s}{2t - s} + 5 = 3t,$ s

4. The formula for the simple pendulum

We began with the formula $T = 2\pi\sqrt{\frac{l}{g}}$. Let us now try to rearrange this to find an expression for g .

We begin by squaring both sides of the equation in order to remove the square root.

$$T^2 = (2\pi)^2 \frac{l}{g}$$

To remove the fraction we multiply both sides by g :

$$T^2 g = (2\pi)^2 l$$

Dividing both sides by T^2 gives

$$g = \frac{(2\pi)^2 l}{T^2}$$

By observing the two square terms on the right, we note that this formula could be written, if we wish, in the equivalent form

$$g = \left(\frac{2\pi}{T}\right)^2 l$$

5. Further examples of useful formulae

Example - the lens formula

The so-called lens formula, which is used in optics, is given by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

Suppose we want to rearrange this formula to find u .

Because we want to isolate u we begin by subtracting $\frac{1}{v}$ from both sides.

$$\frac{1}{f} - \frac{1}{v} = \frac{1}{u}$$

The left-hand side fractions can be combined by expressing them over a common denominator

$$\frac{v - f}{fv} = \frac{1}{u}$$

Inverting both sides

$$\frac{fv}{v - f} = u$$

and so $u = \frac{fv}{v - f}$ as required.

Example

The formula $T = \frac{T_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$ arises in the study of relativity. Suppose we want to rearrange it to find an expression for $\frac{v}{c}$.

We begin by noticing that if we square both sides this will remove the square root term (i.e. the power $\frac{1}{2}$) on the right-hand side. So squaring:

$$T^2 = \frac{T_0^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

We remove the fraction by multiplying both sides by $\left(1 - \frac{v^2}{c^2}\right)$:

$$T^2 \left(1 - \frac{v^2}{c^2}\right) = T_0^2$$

Dividing both sides by T^2 :

$$\left(1 - \frac{v^2}{c^2}\right) = \frac{T_0^2}{T^2}$$

Adding $\frac{v^2}{c^2}$ to both sides gives

$$1 = \frac{T_0^2}{T^2} + \frac{v^2}{c^2}$$

Subtracting $\frac{T_0^2}{T^2}$ from both sides:

$$1 - \frac{T_0^2}{T^2} = \frac{v^2}{c^2}$$

Finally, taking the square root of both sides

$$\frac{v}{c} = \sqrt{1 - \frac{T_0^2}{T^2}}$$

as required.

Exercise 2

Rearrange each of the following formulae to make the quantity shown the subject.

1. $y = a + \frac{1}{x}, \quad x$

2. $y = a + \frac{1}{1-x}, \quad x$

3. $P = \frac{P_0}{1-r^2}, \quad r$

4. $m = k\sqrt{a(1-x)}, \quad x$

5. $V = \frac{V_0}{\sqrt{r^2-1}}, \quad r$

Answers

Exercise 1

1. $u = v - at$ 2. $s = \frac{v^2 - u^2}{2a}$ 3. $a = \frac{2(vt - s)}{t^2}$ 4. $h = \frac{1}{2}(p - 2w)$

5. $h = \frac{A - 2\pi r^2}{2\pi r}$ 6. $v = \sqrt{\frac{2(E - mgh)}{m}}$ 7. $m = \frac{2E}{v^2 + 2gh}$ 8. $b = \frac{2 + a}{3a - 2}$

9. $t = \frac{3s^2}{6s - 1}$ 10. $s = \frac{6t^2 - 10t}{3t - 4}$.

Exercise 2

1. $x = \frac{1}{y - a}$ 2. $x = 1 - \frac{1}{y - a}$ 3. $r = \sqrt{1 - \frac{P_0}{P}}$

4. $x = 1 - \frac{1}{a} \left(\frac{m}{k}\right)^2$ 5. $r = \sqrt{1 + \left(\frac{V_0}{V}\right)^2}$