mathcentre

Extending the table of derivatives

In this unit we continue to build up The Table of Derivatives using rules described in other units.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate trigonometric functions
- differentiate inverse trigonometric functions
- differentiate a^x.

Contents

1. Introduction

In this unit we construct several new entries for the Table of Derivatives using standard rules and results obtained previously. The table we will construct is shown here:

function $f(x)$	derivative $\frac{df}{dx}$ or $f'(x)$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\text{cosec}^2 x$
$\text{cosec } x = \frac{1}{\sin x}$	$-\text{cosec } x \cot x$
$\tan mx$	$m \sec^2 mx$
$\sec mx$	$m \sec mx \tan mx$
$\cot mx$	$-m\, \text{cosec}^2 mx$
$\text{cosec } mx$	$-m\, \text{cosec } mx \cot mx$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
a^x	$a^x \ln a$

Most of these new entries are derived in the following examples.

2. Examples

Example 1

Suppose we wish to differentiate $y = \tan x$. Note that we can write this as $y = \tan x = \frac{\sin x}{\cos x}$. Because this is a quotient we can use the quotient rule to perform the differentiation.

The quotient rule states:

$$\text{if} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So to differentiate $y = \frac{\sin x}{\cos x}$ we take $u = \sin x$ and $v = \cos x$. Then

$$\frac{du}{dx} = \cos x \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x$$

Then, applying the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \end{aligned}$$

A key trigonometric identity states that $\cos^2 x + \sin^2 x = 1$ and so this simplifies to

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

which can also be written as $\sec^2 x$. So the derivative of $\tan x$ is $\sec^2 x$.

Example 2

Suppose we wish to differentiate $y = \sec x$.

Note that we can write this as $y = \sec x = \frac{1}{\cos x}$. Because this is a quotient we can again use the quotient rule to perform the differentiation.

The quotient rule states:

$$\text{if} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So to differentiate $y = \frac{1}{\cos x}$ we take $u = 1$ and $v = \cos x$. Then

$$\frac{du}{dx} = 0 \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x$$

Then, applying the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos x(0) - 1(-\sin x)}{(\cos x)^2} \\ &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

this can be written as

$$\frac{dy}{dx} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$$

So the derivative of $\sec x$ is $\sec x \tan x$.

Example 3

Suppose we wish to differentiate $y = \tan mx$ where m is a constant.

We make a substitution to simplify this function. Suppose we let $u = mx$ so that $y = \tan u$. We can use the chain rule to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In this case, since $u = mx$ then $\frac{du}{dx} = m$.
Since $y = \tan u$, we have just seen that $\frac{dy}{du} = \sec^2 u$,
So, the chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \sec^2 u \times m \\ &= m \sec^2 mx\end{aligned}$$

Example 4

Suppose we wish to differentiate $y = \text{cosec}\, mx$ where m is a constant.

We make a substitution to simplify this function. Suppose we let $u = mx$ so that $y = \text{cosec}\, u$. We can again use the chain rule to find $\frac{dy}{dx}$: In this case, since $u = mx$ then $\frac{du}{dx} = m$.
Since $y = \text{cosec}\, u$, we note from the Table on page 2 that $\frac{dy}{du} = -\text{cosec}\, u \cot u$,
So, the chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= -\text{cosec}\, u \cot u \times m \\ &= -m\, \text{cosec}\, mx \cot mx \end{aligned}$$

Example 5

Suppose we wish to differentiate $y = \sin^{-1} x$.
We proceed by rewriting this as $\sin y = x$. We then differentiate both sides with respect to x:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

The right hand side is straightforward because the derivative of x with respect to x is just 1. The left hand side needs more care because we need to differentiate a function of y, that is $\sin y$, with respect to x. We do this implicitly as follows:

$$\begin{aligned} \frac{d}{dx}(\sin y) &= \frac{d}{dy}(\sin y) \times \frac{dy}{dx} \\ &= \cos y \frac{dy}{dx} \end{aligned}$$

(If necessary you should refer to the unit on implicit differentiation in order to understand this process.) Putting these results together we find

$$\cos y \frac{dy}{dx} = 1$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We now try to write the right hand side in terms of x. We can do this using the identity

$$\cos^2 y + \sin^2 y = 1$$

so that

$$\cos y = \sqrt{1 - \sin^2 y}$$

We take only the positive square root. This is because studying the graph of $y = \sin^{-1} x$ shows that its gradient, and hence $\frac{dy}{dx}$, is positive.
So

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

Finally, recall that $\sin y = x$ so that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

So the derivative of $y = \sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$.
A similar argument can be used to find the derivative of $y = \cos^{-1} x$ and this is left as an exercise.

Example 6

Suppose $y = \tan^{-1} x$ and we wish to find $\frac{dy}{dx}$. We proceed by rewriting this as $\tan y = x$. We then differentiate both sides with respect to x:

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

The right hand side is straightforward because the derivative of x with respect to x is just 1. The left hand side needs more care because we need to differentiate a function of y, that is $\tan y$, with respect to x. We do this implicitly as follows:

$$\begin{aligned} \frac{d}{dx}(\tan y) &= \frac{d}{dy}(\tan y) \times \frac{dy}{dx} \\ &= \sec^2 y \frac{dy}{dx} \end{aligned}$$

Putting these results together we find

$$\sec^2 y \frac{dy}{dx} = 1$$

Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1 + \tan^2 y} \\ &= \frac{1}{1 + x^2} \end{aligned}$$

Here we have made use of the trigonometric identity $1 + \tan^2 y = \sec^2 y$. So the derivative of $y = \tan^{-1} x$ is $\frac{1}{1+x^2}$.

Example 7

Suppose we want to differentiate $y = a^x$. We proceed by taking logarithms of both sides:

$$\ln y = \ln a^x = x \ln a$$

using the laws of logarithms.
Differentiating both sides with respect to x gives

$$\frac{d}{dx} \ln y = \ln a$$

since $\ln a$ is a constant. Now, using the chain rule,

$$\frac{d}{dx} \ln y = \frac{d}{dy} \ln y \times \frac{dy}{dx}$$

so

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

Then

$$\frac{dy}{dx} = y \ln a = a^x \ln a$$

Exercises

1. Use the extended table of derivatives in Section 1 to find the derivative of each of the following:

a)	$\tan 3x$	b)	$\cos^{-1} x$	c)	5^x	d)	$\cot 5x$
e)	$\cot 3x$	f)	2^x	g)	$\text{cosec}4x$	h)	$\tan^{-1} x$
i)	$\sec 3x$	j)	$\tan 4x$	k)	1^x	l)	$\sec\left(\frac{1}{2}x\right)$

2. By writing $\cot x = \dfrac{\cos x}{\sin x}$ and using the quotient rule find the derivative of $\cot x$.

3. By writing $\text{cosec}x = \dfrac{1}{\sin x}$ and using the quotient rule find the derivative of $\text{cosec}x$.

4. By implicitly differentiating $\cos y = x$ determine the derivative of $\cos^{-1} x$.

5. Use the chain rule to find the derivative of $\tan^{-1}\left(\dfrac{x}{a}\right)$.

Answers

1. a) $3\sec^2 3x$ b) $-\dfrac{1}{\sqrt{1-x^2}}$ c) $5^x \ln 5$ d) $-5\text{cosec}^2 5x$ e) $-3\text{cosec}^2 3x$
f) $2^x \ln 2$ g) $-4\text{cosec}4x \cot 4x$ h) $\dfrac{1}{1+x^2}$ i) $3\sec 3x \tan 3x$ j) $4\sec^2 4x$
k) 0 l) $\dfrac{1}{2}\sec\left(\dfrac{1}{2}x\right)\tan\left(\dfrac{1}{2}x\right)$

2. $-\text{cosec}^2 x$

3. $-\text{cosec}x \cot x$

4. $-\dfrac{1}{\sqrt{1-x^2}}$

5. $\dfrac{a}{a^2+x^2}$