

Integration by parts

A special rule, **integration by parts**, is available for integrating products of two functions. This unit derives and illustrates this rule with a number of examples.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- state the formula for integration by parts
- integrate products of functions using integration by parts

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1. Introduction

Functions often arise as products of other functions, and we may be required to integrate these products. For example, we may be asked to determine

$$\int x \cos x \, dx .$$

Here, the integrand is the product of the functions x and $\cos x$. A rule exists for integrating products of functions and in the following section we will derive it.

2. Derivation of the formula for integration by parts

We already know how to differentiate a product: if

$$y = u v$$

then

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} .$$

Rearranging this rule:

$$u \frac{dv}{dx} = \frac{d(uv)}{dx} - v \frac{du}{dx} .$$

Now integrate both sides:

$$\int u \frac{dv}{dx} \, dx = \int \frac{d(uv)}{dx} \, dx - \int v \frac{du}{dx} \, dx .$$

The first term on the right simplifies since we are simply integrating what has been differentiated.

$$\int u \frac{dv}{dx} \, dx = u v - \int v \frac{du}{dx} \, dx .$$

This is the formula known as **integration by parts**.



Key Point

Integration by parts

$$\int u \frac{dv}{dx} \, dx = u v - \int v \frac{du}{dx} \, dx$$

The formula replaces one integral (that on the left) with another (that on the right); the intention is that the one on the right is a simpler integral to evaluate, as we shall see in the following examples.

3. Using the formula for integration by parts

Example

Find $\int x \cos x \, dx$.

Solution

Here, we are trying to integrate the product of the functions x and $\cos x$. To use the integration by parts formula we let one of the terms be $\frac{dv}{dx}$ and the other be u . Notice from the formula that whichever term we let equal u we need to differentiate it in order to find $\frac{du}{dx}$. So in this case, if we let u equal x , when we differentiate it we will find $\frac{du}{dx} = 1$, simply a constant. Notice that the formula replaces one integral, the one on the left, by another, the one on the right. Careful choice of u will produce an integral which is less complicated than the original.

Choose

$$u = x \quad \text{and} \quad \frac{dv}{dx} = \cos x.$$

With this choice, by differentiating we obtain

$$\frac{du}{dx} = 1.$$

Also from $\frac{dv}{dx} = \cos x$, by integrating we find

$$v = \int \cos x \, dx = \sin x.$$

(At this stage do not concern yourself with the constant of integration). Then use the formula

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx :$$

$$\begin{aligned} \int x \cos x \, dx &= x \sin x - \int (\sin x) \cdot 1 \, dx \\ &= x \sin x + \cos x + c \end{aligned}$$

where c is the constant of integration.

In the next Example we will see that it is sometimes necessary to apply the formula for integration by parts more than once.

Example

Find $\int x^2 e^{3x} \, dx$.

Solution

We have to make a choice and let one of the functions in the product equal u and one equal $\frac{dv}{dx}$. As a general rule we let u be the function which will become simpler when we differentiate it. In this case it makes sense to let

$$u = x^2 \quad \text{and} \quad \frac{dv}{dx} = e^{3x}.$$

Then

$$\frac{du}{dx} = 2x \quad \text{and} \quad v = \int e^{3x} dx = \frac{1}{3}e^{3x}.$$

Then, using the formula for integration by parts,

$$\begin{aligned} \int x^2 e^{3x} dx &= \frac{1}{3}e^{3x} \cdot x^2 - \int \frac{1}{3}e^{3x} \cdot 2x dx \\ &= \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}x e^{3x} dx. \end{aligned}$$

The resulting integral is still a product. It is a product of the functions $\frac{2}{3}x$ and e^{3x} . We can use the formula again. This time we choose

$$u = \frac{2}{3}x \quad \text{and} \quad \frac{dv}{dx} = e^{3x}.$$

Then

$$\frac{du}{dx} = \frac{2}{3} \quad \text{and} \quad v = \int e^{3x} dx = \frac{1}{3}e^{3x}.$$

So

$$\begin{aligned} \int x^2 e^{3x} dx &= \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}x e^{3x} dx \\ &= \frac{1}{3}x^2 e^{3x} - \left\{ \frac{2}{3}x \cdot \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x} \cdot \frac{2}{3} dx \right\} \\ &= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + c \end{aligned}$$

where c is the constant of integration. So we have done integration by parts twice to arrive at our final answer.

Remember that to apply the formula you have to be able to integrate the function you call $\frac{dv}{dx}$. This can cause problems — consider the next Example.

Example

Find $\int x \ln|x| dx$.

Solution

Remember the formula:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

It would be natural to choose $u = x$ so that when we differentiate it we get $\frac{du}{dx} = 1$. However this choice would mean choosing $\frac{dv}{dx} = \ln|x|$ and we would need to be able to integrate this. This integral is not a known standard form. So, in this Example we will choose

$$u = \ln|x| \quad \text{and} \quad \frac{dv}{dx} = x$$

from which

$$\frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad v = \int x dx = \frac{x^2}{2}.$$

Then, applying the formula

$$\begin{aligned} \int x \ln|x| dx &= \frac{x^2}{2} \ln|x| - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln|x| - \int \frac{x}{2} dx \\ &= \frac{x^2}{2} \ln|x| - \frac{x^2}{4} + c \end{aligned}$$

where c is the constant of integration.

Example

Find $\int \ln|x| dx$.

Solution

We can use the formula for integration by parts to find this integral if we note that we can write $\ln|x|$ as $1 \cdot \ln|x|$, a product. We choose

$$\frac{dv}{dx} = 1 \quad \text{and} \quad u = \ln|x|$$

so that

$$v = \int 1 dx = x \quad \text{and} \quad \frac{du}{dx} = \frac{1}{x}.$$

Then,

$$\begin{aligned} \int 1 \cdot \ln|x| dx &= x \ln|x| - \int x \cdot \frac{1}{x} dx \\ &= x \ln|x| - \int 1 dx \\ &= x \ln|x| - x + c \end{aligned}$$

where c is a constant of integration.

Example

Find $\int e^x \sin x \, dx$.

Solution

Whichever terms we choose for u and $\frac{dv}{dx}$ it may not appear that integration by parts is going to produce a simpler integral. Nevertheless, let us make a choice:

$$\frac{dv}{dx} = \sin x \quad \text{and} \quad u = e^x$$

so that

$$v = \int \sin x \, dx = -\cos x \quad \text{and} \quad \frac{du}{dx} = e^x.$$

Then,

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \cdot -\cos x - \int -\cos x \cdot e^x \, dx \\ &= -\cos x \cdot e^x + \int e^x \cos x \, dx. \end{aligned}$$

We now integrate by parts again choosing

$$\frac{dv}{dx} = \cos x \quad \text{and} \quad u = e^x$$

so that

$$v = \int \cos x \, dx = \sin x \quad \text{and} \quad \frac{du}{dx} = e^x.$$

So

$$\begin{aligned} \int e^x \sin x \, dx &= -\cos x \cdot e^x + \left\{ e^x \sin x - \int \sin x \cdot e^x \, dx \right\} \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx. \end{aligned}$$

Notice that the integral we have ended up with is exactly the same as the one we started with.

Let us call this I . That is $I = \int e^x \sin x \, dx$.

So

$$I = e^x \sin x - e^x \cos x - I$$

from which

$$2I = e^x \sin x - e^x \cos x$$

and

$$I = \frac{1}{2} (e^x \sin x - e^x \cos x).$$

So

$$\int e^x \sin x \, dx = \frac{1}{2} (e^x \sin x - e^x \cos x) + c$$

where c is the constant of integration.

Exercises

1. Evaluate the following integrals:

$$\begin{aligned} \text{(a)} \quad & \int x \sin x \, dx & \text{(b)} \quad & \int x \cos 4x \, dx & \text{(c)} \quad & \int x e^{-x} \, dx & \text{(d)} \quad & \int x^2 \cos x \, dx \\ \text{(e)} \quad & \int 2x^2 e^x \, dx & \text{(f)} \quad & \int x^2 \ln |x| \, dx & \text{(g)} \quad & \int \tan^{-1} x \, dx & \text{(h)} \quad & \int \sin^{-1} x \, dx \\ \text{(i)} \quad & \int e^x \cos x \, dx & \text{(j)} \quad & \int \sin^3 x \, dx & & & & \text{(Hint: write } \sin^3 x \text{ as } \sin^2 x \sin x.) \end{aligned}$$

2. Calculate the value of each of the following:

$$\begin{aligned} \text{(a)} \quad & \int_0^\pi x \cos \frac{1}{2}x \, dx & \text{(b)} \quad & \int_0^1 x^2 e^x \, dx & \text{(c)} \quad & \int_1^2 x^3 \ln |x| \, dx & \text{(d)} \quad & \int_0^{\pi/4} x^2 \sin 2x \, dx \\ \text{(e)} \quad & \int_0^1 x \tan^{-1} x \, dx \end{aligned}$$

Answers

1.

$$\begin{aligned} \text{(a)} \quad & -x \cos x + \sin x + C & \text{(b)} \quad & \frac{1}{4}x \sin 4x + \frac{1}{16} \cos 4x + C \\ \text{(c)} \quad & -x e^{-x} - e^{-x} + C & \text{(d)} \quad & x^2 \sin x + 2x \cos x - 2 \sin x + C \\ \text{(e)} \quad & 2x^2 e^x - 4x e^x + 4e^x + C & \text{(f)} \quad & \frac{1}{3}x^3 \ln |x| - \frac{1}{9}x^3 + C \\ \text{(g)} \quad & x \tan^{-1} x - \frac{1}{2} \ln |1 + x^2| + C & \text{(h)} \quad & x \sin^{-1} x + \sqrt{1 - x^2} + C \\ \text{(i)} \quad & \frac{1}{2}e^x(\cos x + \sin x) + C & \text{(j)} \quad & -\frac{1}{3}(\cos x \sin^2 x + 2 \cos x) + C \end{aligned}$$

2.

$$\text{(a)} \quad 2\pi - 4 \quad \text{(b)} \quad e - 2 \quad \text{(c)} \quad 4 \ln 2 - \frac{15}{16} \quad \text{(d)} \quad \frac{\pi}{8} - \frac{1}{4} \quad \text{(e)} \quad \frac{\pi}{4} - \frac{1}{2}$$