

# Differentiation of $x^n$ .

In this unit we derive the result, from first principles, that if  $y = x^n$  then  $\frac{dy}{dx} = nx^{n-1}$ .

The result is then illustrated with several examples.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate  $y = x^n$ , when  $n$  is a positive integer, from first principles.
- use the result to differentiate a variety of functions of this form.

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## 1. Introduction

In this unit we look at how to differentiate the function  $y = x^n$  from first principles. The number  $n$  can be a positive integer, a negative integer, a fraction or even zero. We start by considering cases in which  $n$  is a positive integer as in the functions  $x^2$ ,  $x^7$  and so on.

## 2. Differentiation of $y = x^n$ when $n$ is a positive integer

Recall the definition of the derivative of a function  $y = f(x)$ .

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

We apply this definition to the function  $y = x^n$ .

We have

$$f(x) = x^n$$

and so

$$f(x + \delta x) = (x + \delta x)^n$$

In order to expand this we make use of the binomial theorem in the following form:

$$(a + b)^n = a^n + na^{n-1}b + \dots + b^n$$

(You may wish to refer to the unit on Pascal's triangle and the binomial theorem to revise this if necessary.) Applying the binomial theorem to  $(x + \delta x)^n$  we find

$$(x + \delta x)^n = x^n + nx^{n-1}\delta x + \dots + (\delta x)^n$$

Then, from the formula for the derivative

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \frac{x^n + nx^{n-1}\delta x + \dots + (\delta x)^n - x^n}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{nx^{n-1}\delta x + \dots + (\delta x)^n}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{(\delta x)(nx^{n-1} + \dots + (\delta x)^{n-1})}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} nx^{n-1} + \dots + (\delta x)^{n-1} \end{aligned}$$

Note that all the terms on the right, apart from the first, contain the term  $\delta x$ . In the limit as  $\delta x$  tends to zero, all these become zero. We are left with the result that

$$\frac{dy}{dx} = nx^{n-1}$$



### Key Point

When  $n$  is a positive integer,

$$\text{if } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}$$

We have proved the result for the case when  $n$  is a positive integer. In fact, the result is true when  $n$  is a negative integer and when  $n$  is a fraction although we will not prove this here. Nevertheless we will use the result in the following examples.

### Example

Suppose we want to differentiate  $y = x^{\frac{1}{2}}$ .

We use the result with  $n = \frac{1}{2}$ .

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

### Example

Suppose we want to differentiate  $y = x^{-1}$ .

We use the result with  $n = -1$ .

$$\frac{dy}{dx} = -1x^{-1-1} = -x^{-2}$$

### Example

Suppose we want to differentiate  $y = x^2$ .

We use the result with  $n = 2$ .

$$\frac{dy}{dx} = 2x^{2-1} = 2x^1 = 2x$$

### Example

Suppose we want to differentiate  $y = x^7$ .

We use the result with  $n = 7$ .

$$\frac{dy}{dx} = 7x^{7-1} = 7x^6$$

### Example

Suppose we want to differentiate  $y = x$ .

Note that  $x = x^1$  and so we use the result with  $n = 1$ .

$$\frac{dy}{dx} = 1x^{1-1} = 1x^0 = 1$$

We obtain the result that the derivative of  $y = x$  is simply 1, as we would expect since  $y = x$  is the equation of a straight line with gradient 1.

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What happens when  $n = 0$  ?

### Example

When  $n = 0$  we are interested in the function  $y = x^0$ . Applying the result obtained above we find

$$\frac{dy}{dx} = 0 \times x^{0-1} = 0$$

So the derivative of  $y = x^0$  is zero. Note also that  $x^0 = 1$  and so we are really considering the constant function  $y = 1$ . Because the graph of this function is a horizontal straight line the result that its derivative is zero should come as no surprise.

### Exercise 1

Find the derivative of each of the following:

- a)  $x^6$    b)  $x^{10}$    c)  $x^4$    d)  $x^{12}$   
e)  $x^{-2}$    f)  $x^{-8}$    g)  $x^{-5}$    h)  $x^{-234}$   
i)  $x^{13/2}$    j)  $x^{7/4}$    k)  $x^{3/5}$    l)  $x^{2/3}$   
m)  $\frac{1}{x^4}$    n)  $\sqrt{x}$    o)  $x^{-3/2}$    p)  $x^{-1/5}$

## 3. Some examples requiring the linearity rules

We now look at some more examples which assume that you already know the following rules: if

$$y = f(x) \pm g(x)$$

then

$$\frac{dy}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$$

This means that we can differentiate sums (and differences) of functions, term by term. Also, if

$$y = kf(x) \quad \text{where } k \text{ is a constant}$$

then

$$\frac{dy}{dx} = k \frac{df}{dx}$$

This means that we can differentiate a constant multiple of a function, simply by differentiating the function and multiplying by the constant.

Together with rules are known as **linearity rules**.

### Example

Suppose we want to differentiate  $y = 6x^3 - 12x^4 + 5$ .

We find

$$\frac{dy}{dx} = 6(3x^2) - 12(4x^3) + 0 = 18x^2 - 48x^3.$$

### Example

Suppose we want to differentiate  $y = x - 5x^5 + 6x^7 + 25$ .

By now, you should be getting sufficiently confident to condense several of the stages involved. For example, when differentiating  $6x^7$  you should recognise that the derivative of  $x^7$  is  $7x^6$  and so write down the derivative of  $6x^7$  immediately, as  $42x^6$ :

We find

$$\frac{dy}{dx} = 1 - 25x^4 + 42x^6.$$

### Example

Suppose we want to differentiate:  $y = \frac{1}{x} + 6x - 4x^{3/2} + 8$ .

Recall that  $\frac{1}{x} = x^{-1}$ .

We find

$$\frac{dy}{dx} = -x^{-2} + 6 - 6x^{1/2}$$

### Example

Suppose we want to differentiate  $y = 4x^{1/3} - 5x + 6/x^3$ .

Recall that  $\frac{1}{x^3} = x^{-3}$ .

We find

$$\frac{dy}{dx} = \frac{4}{3}x^{-2/3} - 5 - 18x^{-4}$$

### Exercise 2

Find the derivative of each of the following:

- a)  $5x^2$       b)  $8x^6$       c)  $3x^{-2}$       d)  $4x^{-3/5}$   
e)  $x^6 + x^8$       f)  $x^{10} + 1$       g)  $x^{-3} + x$       h)  $5 + x^{9/2}$   
i)  $12x^3 - 3x^2$       j)  $5x + 3x^{-2}$       k)  $6x^2 + 3x^{3/4}$       l)  $7x^{-3} - 9x$   
m)  $4x^{1/4} - 3x^{-1/3}$       n)  $2 + 7x^6$       o)  $3x - 4x^{-1/2}$       p)  $\frac{2}{x} - \frac{3}{x^2}$

### Answers

#### Exercise 1

- a)  $6x^5$     b)  $10x^9$     c)  $4x^3$     d)  $12x^{11}$     e)  $-2x^{-3}$     f)  $-8x^{-9}$   
g)  $-5x^{-6}$     h)  $-234x^{-235}$     i)  $\frac{13}{2}x^{11/2}$     j)  $\frac{7}{4}x^{3/4}$     k)  $\frac{3}{5}x^{-2/5}$     l)  $\frac{2}{3}x^{-1/3}$   
m)  $-4x^{-5} = -\frac{4}{x^5}$     n)  $\frac{1}{2}x^{-1/2}$     o)  $-\frac{3}{2}x^{-5/2}$     p)  $-\frac{1}{5}x^{-6/5}$

#### Exercise 2

- a)  $10x$     b)  $48x^5$     c)  $-6x^{-3}$     d)  $-\frac{12}{5}x^{-8/5}$     e)  $6x^5 + 8x^7$     f)  $10x^9$   
g)  $-3x^{-4} + 1$     h)  $\frac{9}{2}x^{7/2}$     i)  $36x^2 - 6x$     j)  $5 - 6x^{-3}$     k)  $12x + \frac{9}{4}x^{-1/4}$     l)  $-21x^{-4} - 9$   
m)  $x^{-3/4} + x^{-4/3}$     n)  $42x^5$     o)  $3 + 2x^{-3/2}$     p)  $-2x^{-2} + 6x^{-3} = -\frac{2}{x^2} + \frac{6}{x^3}$