Matrix inversion of a 3×3 matrix

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The adjoint and inverse of a matrix

In this leaflet we consider how to find the inverse of a 3×3 matrix. Before you work through this leaflet, you will need to know how to find the **determinant** and **cofactors** of a 3×3 matrix. If necessary you should refer to previous leaflets in this series which cover these topics.

Here is the matrix A that we saw in the leaflet on finding cofactors and determinants. Alongside, we have assembled the matrix of cofactors of A.

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \qquad C = \begin{pmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{pmatrix}$$

In order to find the inverse of A, we first need to use the matrix of cofactors, C, to create the **adjoint** of matrix A. The adjoint of A, denoted adj(A), is the transpose of the matrix of cofactors:

$$\text{adj}(A) = C^T$$

Remember that to find the transpose, the rows and columns are interchanged, so that

$$\text{adj}(A) = C^T = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$$

Then the formula for the inverse matrix is

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

where det(A) is the determinant of A.

Given a matrix A, its inverse is given by

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

where det(A) is the determinant of A, and adj(A) is the adjoint of A.

The inverse has the special property that

$$A\,A^{-1} = A^{-1}\,A = I \qquad \text{(an identity matrix)}$$

Example

Find the inverse of $A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$.

Solution

We already have that $\mathrm{adj}(A) = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$.

In an earlier leaflet, the determinant of this matrix A was found to be 1. So

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{1}\begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix} = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$$

You should verify this is correct by showing that $A\,A^{-1} = A^{-1}\,A = I$, the 3×3 identity matrix.

Solving a set of simultaneous equations

We now show how the inverse is used to solve the simultaneous equations:

$$\begin{aligned} 7x + 2y + z &= 21 \\ 3y - z &= 5 \\ -3x + 4y - 2z &= -1 \end{aligned}$$

In matrix form these equations can be written

$$\begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} 21 \\ 5 \\ -1 \end{pmatrix}$$

Recall that when $AX = B$, then $X = A^{-1}B$ so

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}\begin{pmatrix} 21 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -42 + 40 + 5 \\ 63 - 55 - 7 \\ 189 - 170 - 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

So $x = 3$, $y = 1$ and $z = -2$.

These values should be checked by substituting them back into the original equations.

Finally, note that if the determinant of the coefficient matrix A is zero, then it will be impossible to find the inverse of A, and this method will not be applicable.

Note that a video tutorial covering the content of this leaflet is available from **sigma**.

