Matrix inversion of a $3 \times 3$ matrix

The adjoint and inverse of a matrix

In this leaflet we consider how to find the inverse of a $3 \times 3$ matrix. Before you work through this leaflet, you will need to know how to find the determinant and cofactors of a $3 \times 3$ matrix. If necessary you should refer to previous leaflets in this series which cover these topics.

Here is the matrix $A$ that we saw in the leaflet on finding cofactors and determinants. Alongside, we have assembled the matrix of cofactors of $A$.

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \quad C = \begin{pmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{pmatrix}$$

In order to find the inverse of $A$, we first need to use the matrix of cofactors, $C$, to create the adjoint of matrix $A$. The adjoint of $A$, denoted $\text{adj}(A)$, is the transpose of the matrix of cofactors:

$$\text{adj}(A) = C^T$$

Remember that to find the transpose, the rows and columns are interchanged, so that

$$\text{adj}(A) = C^T = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$$

Then the formula for the inverse matrix is

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

where $\det(A)$ is the determinant of $A$.

Given a matrix $A$, its inverse is given by

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

where $\det(A)$ is the determinant of $A$, and $\text{adj}(A)$ is the adjoint of $A$.

The inverse has the special property that

$$AA^{-1} = A^{-1}A = I$$

(an identity matrix)
Example

Find the inverse of \( A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \).

Solution

We already have that \( \text{adj}(A) = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix} \).

In an earlier leaflet, the determinant of this matrix \( A \) was found to be 1. So

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{1} \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix} = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}
\]

You should verify this is correct by showing that \( AA^{-1} = A^{-1}A = I \), the \( 3 \times 3 \) identity matrix.

Solving a set of simultaneous equations

We now show how the inverse is used to solve the simultaneous equations:

\[
\begin{align*}
7x + 2y + z &= 21 \\
3y - z &= 5 \\
-3x + 4y - 2z &= -1
\end{align*}
\]

In matrix form these equations can be written

\[
\begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} 21 \\ 5 \\ -1 \end{pmatrix}
\]

Recall that when \( AX = B \), then \( X = A^{-1}B \) so

\[
\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix} \begin{pmatrix} 21 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -42 + 40 + 5 \\ 63 - 55 - 7 \\ 189 - 170 - 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}
\]

So \( x = 3, \ y = 1 \) and \( z = -2 \).

These values should be checked by substituting them back into the original equations.

Finally, note that if the determinant of the coefficient matrix \( A \) is zero, then it will be impossible to find the inverse of \( A \), and this method will not be applicable.

Note that a video tutorial covering the content of this leaflet is available from sigma.