Second Order Ordinary Differential Equations

Introduction

Prerequisites: In order to make the most of this resource, you need to know about trigonometry, differentiation, integration and complex numbers.

We are looking at equations involving a function \( y(x) \), its first derivative and second derivative:

\[
 a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)
\]  

(1)

We will only look at equations where the coefficients \( a, b \) and \( c \) are constant; we will not treat in this handout the case of coefficients which are functions of \( x \).

Homogeneous Equations

If \( f(x) = 0 \) then the ODE is called an homogeneous equation. To solve a second order homogeneous ODE, we look at the characteristic equation, obtained by replacing \( \frac{d^2y}{dx^2}, \frac{dy}{dx} \) and \( y \) by \( r^2, r \) and \( 1 \) in the ODE:

\[
 ar^2 + br + c = 0
\]

We distinguish between 3 cases: the case when the roots of the characteristic equation are distinct and real, complex or equal.

Case 1: real and distinct roots \( r_1 \) and \( r_2 \)

Then the solutions of the homogeneous equation are of the form:

\[
 y(x) = Ae^{r_1x} + Be^{r_2x}
\]

The constants \( A \) and \( B \) can be anything you like if there are no boundary conditions. If you have boundary conditions, e.g. you know that \( y(x_0) = \alpha \) and \( y'(x_0) = \beta \), then \( A \) and \( B \) will be uniquely defined by:

\[
 Ae^{r_1x_0} + Be^{r_2x_0} = \alpha \\
 Ar_1e^{r_1x_0} + Br_2e^{r_2x_0} = \beta
\]

Example

\[
 \frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0
\]

The characteristic equation is: \( r^2 + 5r + 6 = 0 \) and the roots are \( \frac{-5 \pm \sqrt{25 - 4 \times 6}}{2} = -3 \) or \(-2\). Therefore the solutions of the ODE are:

\[
 y(x) = Ae^{-3x} + Be^{-2x}
\]
Case 2: complex roots

If the roots are complex then they can be written as \( r + js \) and \( r - js \) (with \( j \) the imaginary number, \( j^2 = -1 \)) and the solutions of the homogeneous equations are of the form:

\[
y(x) = e^{rx}(Ae^{jsx} + Be^{-jsx})
\]

which can also be written as:

\[
y(x) = e^{rx}(C\cos(sx) + D\sin(sx))
\]

As before, the constants \( A \) and \( B \) (or \( C \) and \( D \)) will be defined by the boundary conditions.

Example

\[
\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y = 0
\]

The characteristic equation is: \( r^2 + 4r + 9 = 0 \) and the roots are:

\[
\frac{-4 \pm \sqrt{16 - 4 \times 9}}{2} = -2 \pm j\sqrt{5}
\]

Therefore the solutions of the ODE are:

\[
y(x) = e^{-2x}(Ae^{j\sqrt{5}x} + Be^{-j\sqrt{5}x})
\]

or

\[
y(x) = e^{-2x}(C\cos(\sqrt{5}x) + D\sin(\sqrt{5}x))
\]

Case 3: equal roots \( r_1 = r_2 = r \)

If the characteristic equation has one root only then the solutions of the homogeneous equation are of the form:

\[
y(x) = Ae^{rx} + Bxe^{rx}
\]

Example

\[
\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0
\]

The characteristic equation is: \( r^2 + 4r + 4 = 0 \) e.g. \( (r + 2)^2 = 0 \) and its root is -2. Therefore the solutions of the ODE are:

\[
y(x) = Ae^{-2x} + Bxe^{-2x}
\]

Second Order ODEs with Right-Hand Side

If the right-hand side in Equation (1) is not 0, then the solutions can be found as follows:

- First, find the form of the solution of the corresponding homogeneous equation keeping the constants \( A \) and \( B \) as such: this is called the complementary solution \( y_c(x) \);

- Second, find a particular integral of the ODE \( y_p(x) \).

Then the solutions of the ODE are of the form: \( y(x) = y_c(x) + y_p(x) \). At this point only, you may determine the constants \( A \) and \( B \) from the boundary conditions.

There are two methods to find a particular integral of the ODE: the method of undetermined coefficients and the method of variation of parameters.
**Undetermined coefficients**

This method consists in making an educated guess as to what the particular integral should look like. The following table can be used:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>particular integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$C$</td>
</tr>
<tr>
<td>$kx$</td>
<td>$Cx + D$</td>
</tr>
<tr>
<td>$kx^2$</td>
<td>$Cx^2 + Dx + E$</td>
</tr>
<tr>
<td>$k \sin x$ or $k \cos x$</td>
<td>$C \cos x + D \sin x$</td>
</tr>
<tr>
<td>$k \sinh x$ or $k \cosh x$</td>
<td>$C \cosh x + D \sinh x$</td>
</tr>
<tr>
<td>$e^{kx}$</td>
<td>$e^{kx}$</td>
</tr>
<tr>
<td>$e^{rx}$, where $r$ is a root of the characteristic equation</td>
<td>$Ce^{rx}$ or $Cx^2e^{rx}$</td>
</tr>
</tbody>
</table>

The constants $C$ and $D$ are found by ‘plugging’ the particular integral in the ODE, which will lead to conditions that define $C$ and $D$.

**Example**

\[
\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 2 \sin 4x
\]

We first find the complementary solution of the ODE. The characteristic equation is $r^2 - 5r + 6 = 0$ and the roots are \( \frac{5 \pm \sqrt{25 - 4 \times 6}}{2} = 3 \) or \( 2 \). Therefore the complementary solution is:

\[
y_c(x) = Ae^{3x} + Be^{2x}
\]

Then, we find a particular integral of the ODE. Since the right-hand side contains a $\sin 4x$, we look for a particular integral in the form $y_p(x) = C \cos 4x + D \sin 4x$. We want $y_p$ to be solution of the ODE so we must have:

\[
\frac{d^2y_p}{dx^2} - 5 \frac{dy_p}{dx} + 6y_p = 2 \sin 4x
\]

We have:

\[
\frac{dy_p}{dx} = -4C \sin 4x + 4D \cos 4x
\]

\[
\frac{d^2y_p}{dx^2} = -16C \cos 4x - 16D \sin 4x
\]

Putting back in the ODE:

\[
(-16C \cos 4x - 16D \sin 4x) - 5 (-4C \sin 4x + 4D \cos 4x) + 6 (C \cos 4x + D \sin 4x) = 2 \sin 4x
\]

Re-arranging cos and sin:

\[
(-16C - 20D + 6C) \cos 4x + (-16D + 20C + 6D) \sin 4x = 2 \sin 4x
\]

\[
(-10C - 20D) \cos 4x + (-10D + 20C) \sin 4x = 2 \sin 4x
\]

The last equation must be true for any value of $x$, so we must have:

\[
\begin{cases}
-10C - 20D = 0 \\
20C - 10D = 2
\end{cases}
\]

So:

\[
\begin{align*}
C &= \frac{2}{25} \\
D &= -\frac{1}{25}
\end{align*}
\]
So a particular integral of the ODE is \( y_p(x) = \frac{2}{25} \cos 4x - \frac{1}{25} \sin 4x \) and the general solutions of the ODE are of the form:

\[ y(x) = \frac{2}{25} \cos 4x - \frac{1}{25} \sin 4x + Ae^{3x} + Be^{2x} \]

### Variation of parameters

This method is more general and will work for any function \( f(x) \) in the right-hand side of Equation (1), although it may look intimidating at first sight! First let's rewrite the complementary solution of the ODE in the form:

\[ y_c(x) = Ay_1(x) + By_2(x) \]

with \( y_1(x) = e^{r_1x} \), \( y_2(x) = e^{r_2x} \) or \( xe^{r_1x} \) if \( r_1 = r_2 \) with \( r_1, r_2 \) roots of the characteristic equation

Then a particular integral of Equation (1) is:

\[ y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(y_1, y_2)} \, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(y_1, y_2)} \, dx \]

with \( W \) the Wronskian: \( W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \)

**Example**

\[ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2 + 1} \]

First, let's find the complementary solution of the ODE. The characteristic equation is \( r^2 - 2r + 1 = 0 \), e.g. \( (r - 1)^2 = 0 \), so there is one root which is 1. The complementary solution is of the form:

\[ y_c(x) = Ae^x + Bxe^x \]

To find a particular integral of the ODE, we calculate the Wronskian:

with: \( y_1(x) = e^x \) and: \( y_2(x) = xe^x \)

\[ W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = e^x(1 + x)e^x - e^xe^x = e^{2x} \]

Then a particular integral of the ODE is:

\[ y_p(x) = -e^x \int \frac{xe^x}{e^{2x}x^2 + 1} \, dx + xe^x \int \frac{e^x}{e^{2x}x^2 + 1} \, dx \]

\[ \int \frac{x}{1 + x^2} \, dx = \frac{1}{2} \ln (1 + x^2) \quad \text{and} \quad \int \frac{1}{1 + x^2} \, dx = \arctan x \]

\[ = -e^x \cdot \frac{1}{2} \ln (1 + x^2) + xe^x \cdot \arctan x \]

The general solution of the ODE is:

\[ y(x) = Ae^x + Bxe^x - \frac{1}{2}e^x \ln (1 + x^2) + xe^x \arctan x \]
Exercises

(a) \( \frac{d^2y}{dx^2} + 7y = 0 \) 

(b) \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-2x} \) 

(c) \( 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 2x - 3 \) 

(d) \( \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2e^{-2x} \) with \( y(0) = 1, \frac{dy}{dx}(0) = -2 \) 

(e) \( \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2\cos^2 x \) 

(f) \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4 \sinh x \) 

Answers

(a) \( y = Ae^{\sqrt{7}x} + Be^{-\sqrt{7}x} \) or \( y = C \cos (\sqrt{7}x) + D \sin (\sqrt{7}x) \) 

(b) \( y = Ae^{-x} + Bxe^{-x} + e^{-2x} \) 

(c) \( y = Ae^x + Be^{-1/3x} - 2x + 7 \) 

(d) \( y = e^{-2x}(2 - \cos x) \) 

(e) \( y = (A + Bx)e^{-2x} + \frac{1}{4} + \frac{1}{8} \sin (2x) \) 

(f) \( y = (A + Bx - x^2)e^{-x} + \frac{1}{2}e^x \)