mathcentre

Volumes of solids of revolution

mc-TY-volumes-2009-1

We sometimes need to calculate the volume of a solid which can be obtained by rotating a curve about the x-axis. There is a straightforward technique which enables this to be done, using integration.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

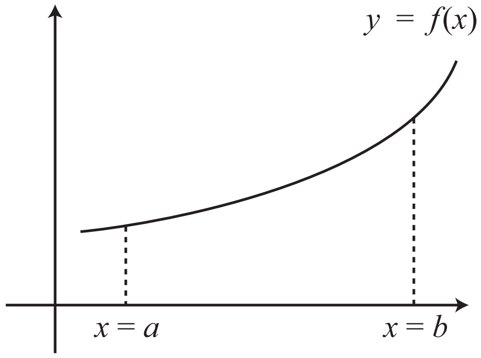
After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- find the volume of a solid of revolution obtained from a simple function $y = f(x)$ between given limits $x = a$ and $x = b$;
- find the volume of a solid of revolution obtained from a simple function $y = f(x)$ where the limits are obtained from the geometry of the solid.

Contents

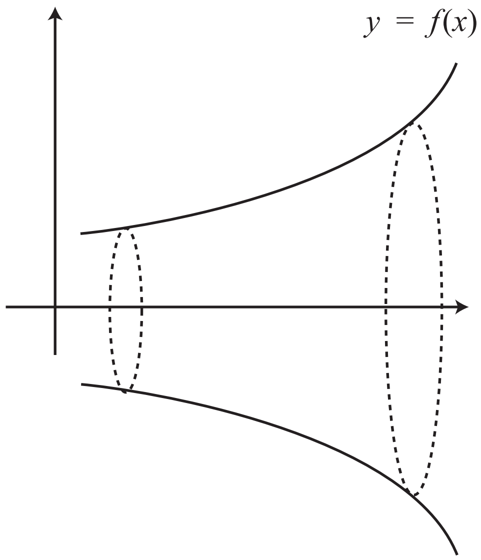
1. Introduction

Suppose we have a curve, $y = f(x)$.

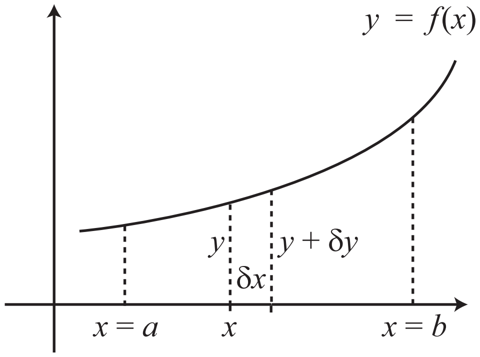


Imagine that the part of the curve between the ordinates $x = a$ and $x = b$ is rotated about the x-axis through $360°$. The curve would then map out the surface of a solid as it rotated. Such solids are called *solids of revolution*. Thus if the curve was a circle, we would obtain the surface of a sphere. If the curve was a straight line through the origin, we would obtain the surface of a cone. Now we already know what the formulae for the volumes of a sphere and a cone are, but where did they come from? How can they calculated? If we could find a general method for calculating the volumes of the solids of revolution then we would be able to calculate, for example, the volume of a sphere and the volume of a cone, as well as the volumes of more complex solids.

To see how to carry out these calculations we look first at the curve, together with the solid it maps out when rotated through $360°$.



Now if we take a cross-section of the solid, parallel to the y-axis, this cross-section will be a circle. But rather than take a cross-section, let us take a thin disc of thickness δx, with the face of the disc nearest the y-axis at a distance x from the origin.



The radius of this circular face will then be y. The radius of the other circular face will be $y+\delta y$, where δy is the change in y caused by the small positive increase in x, δx. The disc is not a cylinder, but it is very close to one. It will become even closer to one as δx, and hence δy, tends to zero. Thus we approximate the disc with a cylinder of thickness, or height, δx, and radius y. The volume δV of the disc is then given by the volume of a cylinder, $\pi r^2 h$, so that

$$\delta V = \pi y^2 \delta x\,.$$

So the volume V of the solid of revolution is given by

$$\begin{aligned} V &= \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \delta V \\ &= \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x \\ &= \int_a^b \pi y^2 \mathrm{d}x\,, \end{aligned}$$

where we have changed the limit of a sum into a definite integral, using our definition of integration. This formula now gives us a way to calculate the volumes of solids of revolution about the x-axis.

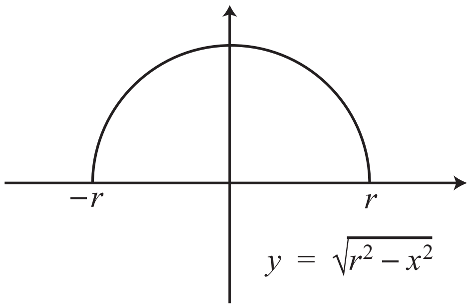
Key Point

If y is given as a function of x, the volume of the solid obtained by rotating the portion of the curve between $x = a$ and $x = b$ about the x-axis is given by

$$V = \int_a^b \pi y^2 \mathrm{d}x\,.$$

2. The volume of a sphere

The equation $x^2 + y^2 = r^2$ represents the equation of a circle centred on the origin and with radius r. So the graph of the function $y = \sqrt{r^2 - x^2}$ is a semicircle.



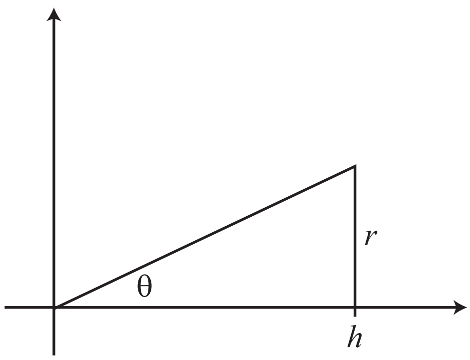
We rotate this curve between $x = -r$ and $x = r$ about the x-axis through $360°$ to form a sphere. Now $x^2 + y^2 = r^2$, and so $y^2 = r^2 - x^2$. Therefore

$$\begin{aligned} V &= \int_a^b \pi y^2 \mathrm{d}x \\ &= \int_{-r}^{r} \pi(r^2 - x^2)\,\mathrm{d}x \\ &= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} \\ &= \pi\left\{\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right\} \\ &= \frac{4\pi r^3}{3}\,. \end{aligned}$$

This is the standard result for the volume of a sphere.

3. The volume of a cone

Suppose we have a cone of base radius r and vertical height h. We can imagine the cone being formed by rotating a straight line through the origin by an angle of $360°$ about the x-axis.



The gradient of the straight line is $\tan\theta$, and from the right-angled triangle we see that $\tan\theta = r/h$. Thus the equation of the line is $y = rx/h$, and the limits of integration are from $x = 0$ to $x = h$. So

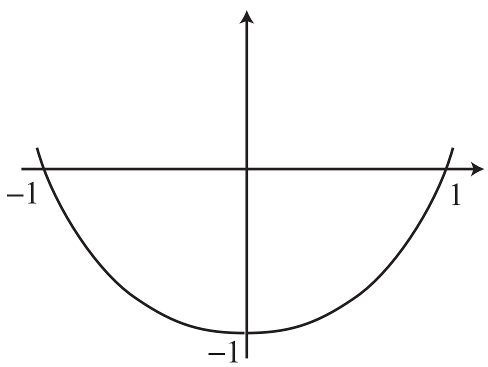
$$\begin{aligned} V &= \int_a^b \pi y^2 \mathrm{d}x \\ &= \int_0^h \pi \left(\frac{rx}{h}\right)^2 \mathrm{d}x \\ &= \int_0^h \pi \frac{r^2x^2}{h^2} \mathrm{d}x \\ &= \pi \left[\frac{r^2x^3}{3h^2}\right]_0^h \\ &= \pi \left(\frac{r^2h^3}{3h^2} - 0\right) \\ &= \frac{\pi r^2 h}{3}. \end{aligned}$$

This is the standard result for the volume of a cone.

4. Another example

The curve $y = x^2 - 1$ is rotated about the x-axis through $360°$. Find the volume of the solid generated when the area contained between the curve and the x-axis is rotated about the x-axis by $360°$.

From the wording of the question, a portion of the curve traps an area between itself and the x-axis. Hence the curve must cross the x-axis. To find the points where this happens we need to set $y = 0$. So we need to solve the equation $x^2 - 1 = 0$. Factorising, $(x - 1)(x + 1) = 0$, and therefore $x = 1$ or $x = -1$. Here is a sketch of the curve.



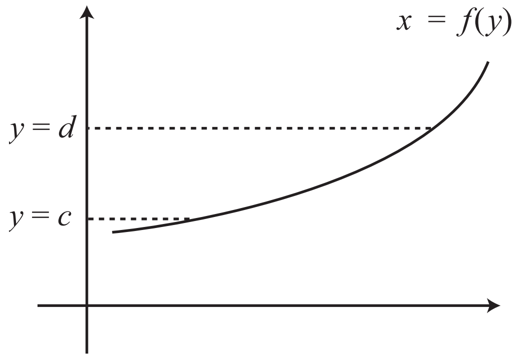
The graph of $y = x^2 - 1$

We calculate the volume as follows.

$$\begin{aligned}
V &= \int_a^b \pi y^2 \mathrm{d}x \\
&= \int_{-1}^{1} \pi (x^2-1)^2 \mathrm{d}x \\
&= \pi \int_{-1}^{1} (x^4 - 2x^2 + 1)\, \mathrm{d}x \\
&= \pi \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_{-1}^{1} \\
&= \pi \left\{ \left(\frac{1}{5} - \frac{2}{3} + 1 \right) - \left(-\frac{1}{5} + \frac{2}{3} - 1 \right) \right\} \\
&= \frac{16\pi}{15} \,.
\end{aligned}$$

5. Rotating a curve about the y-axis

We have looked at how to find the volume of a solid created by rotating an area about the x-axis. But we can also rotate an area about the y-axis. How can we find the volume in this case?



To carry out such a calculation, we must interchange the rôles of x and y. First, the equation of the curve must be given as $x = f(y)$, rather than as $y = f(x)$. And secondly, the limits must be given in terms of y, as $y = c$ and $y = d$. Thus the formula for the volume becomes

$$V = \int_c^d \pi x^2 \, \mathrm{d}y \,.$$

Exercises

1. Find the volume of the solid of revolution generated when the area described is rotated about the x-axis.

(a) The area between the curve $y = x$ and the ordinates $x = 0$ and $x = 4$.

(b) The area between the curve $y = x^{3/2}$ and the ordinates $x = 1$ and $x = 3$.

(c) The area between the curve $x^2 + y^2 = 16$ and the ordinates $x = -1$ and $x = 1$.

(d) The area between the curve $x^2 - y^2 = 9$ and the ordinates $x = -4$ and $x = -3$.

(e) The area between the curve $y = (2 + x)^2$ and the ordinates $x = 0$ and $x = 1$.

2. The area between the curve $y = 1/x$, the y-axis and the lines $y = 1$ and $y = 2$ is rotated about the y-axis. Find the volume of the solid of revolution formed.

3. The area between the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 2$ is rotated about the y-axis. Find the volume of the solid of revolution formed.

4. The area cut off by the x-axis and the curve $y = x^2 - 3x$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

5. Sketch the curve $y^2 = x(x - 4)^2$ and find the volume of the solid of revolution formed when the closed loop of the curve is rotated about the x-axis.

6. A conical funnel is formed by rotating the curve $y = \frac{1}{3}x$ about the y-axis. The radius of the rim of the funnel is to 6 cm. Find the depth of the funnel and its volume.

Answers

1. (a) $21\frac{1}{3}\pi$ (b) 20π (c) $31\frac{1}{3}\pi$ (d) $3\frac{1}{3}\pi$ (e) $\frac{211}{5}\pi$

2. $\frac{1}{2}\pi$

3. 2π

4. $\frac{81}{10}\pi$

5. $21\frac{1}{3}\pi$

6. 24π

