Rules of arithmetic

Evaluating expressions involving numbers is one of the basic tasks in arithmetic. But if an expression is complicated then it may not be clear which part of it should be evaluated first, and so some rules must be established. There are also rules for calculating with negative numbers.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- evaluate an arithmetic expression using the correct order of precedence;

- add and subtract expressions involving both positive and negative numbers;

- multiply and divide expressions involving both positive and negative numbers.

Contents

1. Introduction

In this unit we are going to recall the precedence rules of arithmetic which allow us to work out calculations which involve brackets, powers, $+$, $-$, $\times$ and $\div$ and let us all arrive at the same answer. Then we will go on to calculations involving positive and negative numbers, and generate and use the rules for adding, subtracting, multiplying and dividing them.

2. Order of precedence

Suppose we have this expression:

$$2 + 4 \times 3 - 1\,.$$

To work this out we can work from left to right:

First add, then multiply, and finally subtract:	$(+ \times -)$ to get	17.
Or we could:	$(+ - \times)$ to get	12.
Or we could:	$(\times + -)$ to get	13.

And so on. As you can see, you get different answers according to the order in which the operations are carried out. To prevent this from happening, there is an established order of precedence in which the operations must be done. The order that most people follow is this:

Anything in brackets must be done first. Then we evaluate any powers. Next we do any divisions and multiplications, working from left to right. And finally we do the additions and subtractions, again working from left to right.

This is hard to remember so here's an acronym to help you — BODMAS. It means

$$
\begin{array}{l}
\textbf{B}\text{rackets} \\
\text{p}\textbf{O}\text{wers} \\
\textbf{D}\text{ivision} \\
\textbf{M}\text{ultiplication} \\
\textbf{A}\text{ddition} \\
\textbf{S}\text{ubtraction}
\end{array}
$$

where division and multiplication have the same priority, and also addition and subtraction have the same priority, so in each case we have bracketed them together. You should remember BODMAS, and this will give you the precedence rules to work out calculations involving brackets, powers, $\div$, $\times$, $+$ and $-$. So if we go back to our original expression $2 + 4 \times 3 - 1$, using BODMAS we can evaluate the expression and get a standard answer.

$$
\begin{aligned}
2 + 4 \times 3 - 1 &= 2 + 12 - 1 \qquad (\times \text{ first}) \\
&= 14 - 1 \qquad (+ \text{ and } - \text{ next, so we do } + \text{ first}) \\
&= 13\,.
\end{aligned}
$$

Example

$$
\begin{aligned}
2 \times (3 + 5) &= 2 \times 8 \qquad \text{(brackets first, then multiply)} \\
&= 16\,.
\end{aligned}
$$

Example

$$9 - 6 + 1 = 3 + 1 \quad \text{(left to right, as } + \text{ and } - \text{ have the same priority)}$$
$$= 4.$$

Example

$$3 + 2^2 = 3 + 4 \quad \text{(power then add)}$$
$$= 7.$$

Example

$$(3 + 2)^2 = 5^2 \quad \text{(brackets, then power)}$$
$$= 25.$$

Notice that the final two examples are very similar, but having the brackets in the last one made a big difference to the answer. The square in the earlier one applies only to the 2, whereas the square in the later one applies to the $(3 + 2)$ because of the brackets.

Key Point

BODMAS is an acronyn that serves as a reminder of the order in which operations have to be carried out when working with equations and formulas:

Brackets p**O**wers **D**ivision **M**ultiplication **A**ddition **S**ubtraction

where division and multiplication have the same priority, and so do addition and subtraction. If you have several operations of the same priority then you work from left to right.


Exercises

1. Find the value of the following expressions:

(a) $2 \times 5 + 4$ (b) $2 \times (5 + 4)$ (c) $24 - 6 \div 2$ (d) $3 + 4 \times (7 + 1)$
(e) $(3 + 4) \times 7 + 1$ (f) $5 + 2^2 \times 3$ (g) $5 \times 2 - 4 \div 2$ (h) $(3 + 2)^2$
(i) $(5 + 4)^2 \times 4 \div 2$ (j) $4 \times 2^2 - 12 \div 4$

# 3. Adding and subtracting with negative numbers

We now move on to other rules we use when working out calculations. What happens if we have calculations which involve positive and negative numbers? Are there any rules which help us?

We start with some revision — all real numbers are either positive or negative (or, of course, zero). The positive numbers are those greater than zero, and the negative ones are those less than zero. We can easily see this if we draw a number line and position zero in the middle. The numbers to the right are the *positive* numbers and the numbers to the left are *negative*.
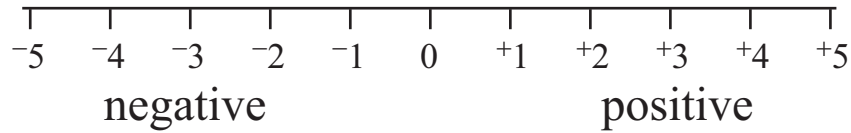
We have written positive three as $^{+}3$, and negative four as $^{-}4$. We use superscripts $^{+}$ and $^{-}$ so that they are not confused with the operations add $(+)$ and subtract $(-)$. So the superscripts help our understanding of what is going on, but with practice the standard notation is used and the superscripts are no longer needed. Also, as positive numbers are the most frequently used numbers it is not always necessary to include the positive sign — it can be omitted. So $^{+}3$ can be written as 3, and we know that it is 'positive three'.

So how do we add and subtract positive and negative numbers? Let's take some examples, using a number line. What is $^{-}4 + {}^{+}5$?
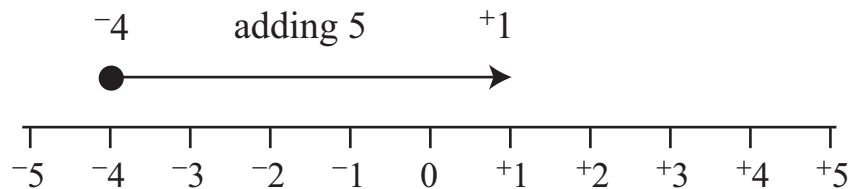
If we start at $^{-}4$ and count on five steps, you see that we get to $^{+}1$. And we use a similar idea for subtraction. What is $^{+}4 - {}^{+}9$?
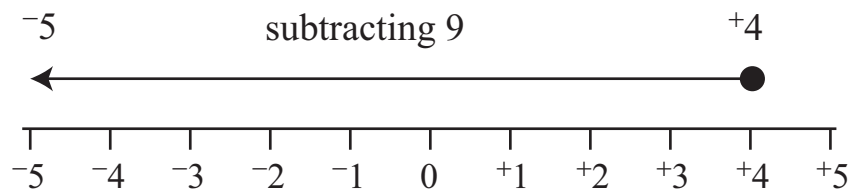
Here, we start at $^{+}4$ and count back by nine steps, reaching $^{-}5$.

In these two examples we added and subtracted positive numbers. What happens if we add or subtract negative numbers? We can see by looking at the pattern in these additions.

$$
\begin{aligned}
5 + 2 &= 7 \\
5 + 1 &= 6 \\
5 + 0 &= 5 \\
5 + {}^{-}1 &= 4 \\
5 + {}^{-}2 &= 3 \\
5 + {}^{-}3 &= 2 \\
5 + {}^{-}4 &= 1 \qquad \text{etc.}
\end{aligned}
$$

Notice that the answers decrease by one, and the numbers added decrease by one each time. Also, look at the four last additions. Here we are adding negative numbers, but we can write these calculations as subtractions. They are subtractions of positive numbers, and give us the same answers.

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$$5 + {}^-1 = 4 \text{ is the same as } 5 - 1 = 4$$
$$5 + {}^-2 = 3 \text{ is the same as } 5 - 2 = 3$$
$$5 + {}^-3 = 2 \text{ is the same as } 5 - 3 = 2$$
$$5 + {}^-4 = 1 \text{ is the same as } 5 - 4 = 1$$

So if we take two examples, $8 + {}^-10$ and ${}^-9 + {}^-5$, we can write them as subtractions of positive numbers and then calculate the answers by counting back:

$$8 + {}^-10 = 8 - 10 = {}^-2,$$
$${}^-9 + {}^-5 = {}^-9 - 5 = {}^-14$$

What about subtraction of negative numbers? Again we can use the number patterns to help us.

$$
\begin{aligned}
4 - 2 &= 2 \\
4 - 1 &= 3 \\
4 - 0 &= 4 \\
4 - {}^-1 &= 5 \\
4 - {}^-2 &= 6 \\
4 - {}^-3 &= 7 \\
4 - {}^-4 &= 8
\end{aligned}
$$

So here we have similar number patterns as before. As we subtract one less each time, the answers increase by one. But look at the last four subtractions. These are the same as additions of positive numbers.

$$4 - {}^-1 = 5 \text{ is the same as } 4 + 1 = 5$$
$$4 - {}^-2 = 6 \text{ is the same as } 4 + 2 = 6$$
$$4 - {}^-3 = 7 \text{ is the same as } 4 + 3 = 7$$
$$4 - {}^-4 = 8 \text{ is the same as } 4 + 4 = 8$$

So if we take two examples, $8 - {}^-10$ and ${}^-6 - {}^-12$, we can write them as additions of positive numbers and then calculate the answers by counting on:

$$8 - {}^-10 = 8 + 10 = 18,$$
$${}^-6 - {}^-13 = {}^-6 + 13 = 7.$$

So to add and subtract positive and negative numbers, here is the rule to remember. If the operation and the sign are the same, they work like the addition of a positive number. So

$$+{}^+ \text{ and } -{}^- \text{ work like the addition of a positive number.}$$

If the operation and the sign are different, they work like the subtraction of a positive number. So

$$+{}^- \text{ and } -{}^+ \text{ work like the subtraction of a positive number.}$$

Key Point

All real numbers are either positive or negative. Number lines can be used to show the positions of positive and negative numbers.

Positive three may be written as $^+3$ and negative three may be written as $^-3$ to distinguish the signs from the operations add $(+)$ and subtract $(-)$. Note that positive numbers are the most commonly used, and so the positive sign is usually omitted, so that 3 is usually written instead of $^+3$.

When adding and subtracting positive and negative numbers it is useful to remember the following rules. If the operation and the sign are the same, they work like adding a (positive) number, so that

$$-^- \text{ works like } +^+.$$

If the operation and the sign are different, they work like subtracting a (positive) number, so that

$$+^- \text{ works like } -^+.$$

Exercises

2. Complete the following:

(a) $^-5 + {}^+8 =$ (b) $^-4 - 3 =$ (c) $6 + {}^-3 =$ (d) $^-6 - {}^-3 =$
(e) $13 + {}^-3 =$ (f) $^-20 - {}^-42 =$ (g) $8 - 16 =$ (h) $10 - {}^-23 =$
(i) $^-4 + {}^-2 - 5 =$ (j) $9 - {}^-4 + {}^-5 =$

4. Multiplying and dividing with negative numbers

Now that we have used addition and subtraction with both positive and negative numbers, what happens when we multiply or divide them? Are there any rules to help us?

We know how to multiply and divide pairs of positive numbers, for example

$$5 \times 5 = 25$$

and

$$5 \div 5 = 1.$$

So when we multiply and divide pairs of positive numbers, the answer is always a positive number. But what happens when we multiply and divide using negative numbers? What are the rules?

Let us look at some patterns again.

$$5 \times 4 = 20$$
$$5 \times 3 = 15$$
$$5 \times 2 = 10$$
$$5 \times 1 = 5$$
$$5 \times 0 = 0$$

So, using the pattern,

$$5 \times {}^{-}1 = {}^{-}5$$
$$5 \times {}^{-}2 = {}^{-}10$$
$$5 \times {}^{-}3 = {}^{-}15$$
$$5 \times {}^{-}4 = {}^{-}20$$

Therefore when we multiply a positive number by a negative number, the answer is negative. As it doesn't matter which way round you multiply the numbers, we can also say that if we multiply a negative number by a positive number then the answer is negative. In short, when multiplying, if the two signs are different then the answer is negative.

$$(+) \times (-) = (-), \qquad (-) \times (+) = (-).$$

So we can now work out the following examples:

$$6 \times {}^{-}5 = {}^{-}30,$$
$${}^{-}4 \times 3 = {}^{-}12.$$

But what if the two numbers are negative? Look at these multiplications and see if you can work out the rule. We know that

$${}^{-}5 \times 4 = {}^{-}20$$
$${}^{-}5 \times 3 = {}^{-}15$$
$${}^{-}5 \times 2 = {}^{-}10$$
$${}^{-}5 \times 1 = {}^{-}5$$
$${}^{-}5 \times 0 = 0$$

so, using the pattern,

$${}^{-}5 \times {}^{-}1 = 5$$
$${}^{-}5 \times {}^{-}2 = 10$$
$${}^{-}5 \times {}^{-}3 = 15$$
$${}^{-}5 \times {}^{-}4 = 20$$

So we can see from the pattern that, when multiplying two negative numbers together, the answer is always positive. For example,

$${}^{-}6 \times {}^{-}3 = 18,$$
$${}^{-}9 \times {}^{-}2 = 18.$$

But also remember that we get a positive answer when we multiply two positive numbers together:

$$6 \times 3 \;=\; 18,$$
$$9 \times 2 \;=\; 18$$

as well. So when we multiply if the signs are the same the answer is positive.

As division is the inverse of multiplication, the rules for division are the same as the rules for multiplication. So when multiplying and dividing positive and negative numbers remember this:

If the signs are the same the answer is positive,
if the signs are different the answer is negative.

$$\left. \begin{array}{l} (+) \text{ multiply or divide } (+) \\ (-) \text{ multiply or divide } (-) \end{array} \right\} \text{ answer is } (+),$$

$$\left. \begin{array}{l} (+) \text{ multiply or divide } (-) \\ (-) \text{ multiply or divide } (+) \end{array} \right\} \text{ answer is } (-).$$

So, for example,

$$^-6 \div {}^-2 = 3 \qquad \text{(signs the same so positive)},$$
$$^-12 \div 3 = {}^-4 \qquad \text{(signs different, so negative)}.$$

Key Point

When multiplying pairs of positive and negative numbers it is helpful to remember the following rules:

When the signs of the numbers are the same the answer is a positive number,

$$\left. \begin{array}{l} (+) \times (+) \\ (-) \times (-) \\ (+) \div (+) \\ (-) \div (-) \end{array} \right\} \text{ answer is } (+).$$

When the signs of the numbers are different the answer is a negative number,

$$\left. \begin{array}{l} (+) \times (-) \\ (-) \times (+) \\ (+) \div (-) \\ (-) \div (+) \end{array} \right\} \text{ answer is } (-).$$

Exercises

3. Complete the following:

(a) $^-5 \times {}^-8 =$ (b) $^+5 \times {}^-6 =$ (c) $^-6 \times {}^+7 =$ (d) $^-9 \times {}^-3 =$

(e) $8 \times 9 =$ (f) $^+16 \div {}^+2 =$ (g) $^-24 \div 3 =$ (h) $42 \div {}^-6 =$

(i) $^-36 \div {}^-4 =$ (j) $^-12 \div 3 \times {}^-2 =$

Answers

1.

(a) 14   (b) 18   (c) 21   (d) 35   (e) 50

(f) 17   (g) 8   (h) 25   (i) 162   (j) 13

2.

(a) 3 or $^+3$ (b) $^-7$ (c) 3 or $^+3$ (d) $^-3$ (e) 10 or $^+10$

(f) 22 or $^+22$ (g) $^-8$ (h) 33 or $^+33$ (i) $^-11$ (j) 8 or $^+8$

3.

(a) 40 or $^+40$ (b) $^-30$ (c) $^-42$ (d) 27 or $^+27$ (e) 72 or $^+72$

(f) 8 or $^+8$ (g) $^-8$ (h) $^-7$ (i) 9 or $^+9$ (j) 8 or $^+8$