The laws of logarithms

Introduction

There are a number of rules known as the laws of logarithms. These allow expressions involving logarithms to be rewritten in a variety of different ways. The laws apply to logarithms of any base but the same base must be used throughout a calculation.

The laws of logarithms

The three main laws are stated here:

**First Law**

\[ \log A + \log B = \log AB \]

This law tells us how to add two logarithms together. Adding \( \log A \) and \( \log B \) results in the logarithm of the product of \( A \) and \( B \), that is \( \log AB \).

For example, we can write

\[ \log_{10} 5 + \log_{10} 4 = \log_{10}(5 \times 4) = \log_{10} 20 \]

The same base, in this case 10, is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

**Second Law**

\[ \log A - \log B = \log \frac{A}{B} \]

So, subtracting \( \log B \) from \( \log A \) results in \( \log \frac{A}{B} \).

For example, we can write

\[ \log_e 12 - \log_e 2 = \log_e \frac{12}{2} = \log_e 6 \]

The same base, in this case \( e \), is used throughout the calculation. You should verify this by evaluating both sides separately on your calculator.

**Third Law**

\[ \log A^n = n \log A \]

So, for example

\[ \log_{10} 5^3 = 3 \log_{10} 5 \]

You should verify this by evaluating both sides separately on your calculator.

Two other important results are...
The logarithm of 1 to any base is always 0, and the logarithm of a number to the same base is always 1. In particular,
\[ \log_{10} 10 = 1, \quad \text{and} \quad \log_{e} e = 1 \]

**Exercises**

1. Use the first law to simplify the following.
   
   a) \( \log_{10} 6 + \log_{10} 3 \),  
   b) \( \log x + \log y \),  
   c) \( \log 4x + \log x \),  
   d) \( \log a + \log b^2 + \log c^3 \).

2. Use the second law to simplify the following.

   a) \( \log_{10} 6 - \log_{10} 3 \),  
   b) \( \log x - \log y \),  
   c) \( \log 4x - \log x \).

3. Use the third law to write each of the following in an alternative form.

   a) \( 3 \log_{10} 5 \),  
   b) \( 2 \log x \),  
   c) \( \log (4x)^2 \),  
   d) \( 5 \ln x^4 \),  
   e) \( \ln 1000 \).

4. Simplify \( 3 \log x - \log x^2 \).

**Answers**

1. a) \( \log_{10} 18 \),  
   b) \( \log xy \),  
   c) \( \log 4x^2 \),  
   d) \( \log ab^2 c^3 \).

2. a) \( \log_{10} 2 \),  
   b) \( \log \frac{y}{x} \),  
   c) \( \log 4 \).

3. a) \( \log_{10} 5^3 \) or \( \log_{10} 125 \),  
   b) \( \log x^2 \),  
   c) \( 2 \log (4x) \),  
   d) \( 20 \ln x \) or \( \ln x^{20} \),  
   e) \( 1000 = 10^3 \) so \( \ln 1000 = 3 \ln 10 \).

4. \( \log x \).