Introduction to differentiation

Introduction

This leaflet provides a rough and ready introduction to differentiation. This is a technique used to calculate the gradient, or slope, of a graph at different points.

The gradient function

Given a function, for example, \( y = x^2 \), it is possible to derive a formula for the gradient of its graph. We can think of this formula as the gradient function, precisely because it tells us the gradient of the graph. For example,

\[
\text{when } y = x^2 \quad \text{the gradient function is } 2x
\]

So, the gradient of the graph of \( y = x^2 \) at any point is twice the \( x \) value there. To understand how this formula is actually found you would need to refer to a textbook on calculus. The important point is that using this formula we can calculate the gradient of \( y = x^2 \) at different points on the graph. For example,

\[
\begin{align*}
\text{when } x = 3, \text{ the gradient is } 2 \times 3 &= 6. \\
\text{when } x = -2, \text{ the gradient is } 2 \times (-2) &= -4.
\end{align*}
\]

How do we interpret these numbers? A gradient of 6 means that values of \( y \) are increasing at the rate of 6 units for every 1 unit increase in \( x \). A gradient of \(-4\) means that values of \( y \) are decreasing at a rate of 4 units for every 1 unit increase in \( x \).

Note that when \( x = 0 \), the gradient is \( 2 \times 0 = 0 \).

Below is a graph of the function \( y = x^2 \). Study the graph and you will note that when \( x = 3 \) the graph has a positive gradient. When \( x = -2 \) the graph has a negative gradient. When \( x = 0 \) the gradient of the graph is zero. Note how these properties of the graph can be predicted from knowledge of the gradient function, \( 2x \).

Example

When \( y = x^3 \), its gradient function is \( 3x^2 \). Calculate the gradient of the graph of \( y = x^3 \) when a) \( x = 2 \), b) \( x = -1 \), c) \( x = 0 \).
Solution

a) when $x = 2$ the gradient function is $3(2)^2 = 12$.
b) when $x = -1$ the gradient function is $3(-1)^2 = 3$.
c) when $x = 0$ the gradient function is $3(0)^2 = 0$.

Notation for the gradient function

You will need to use a notation for the gradient function which is in widespread use.

If $y$ is a function of $x$, that is $y = f(x)$, we write its gradient function as $\frac{dy}{dx}$.

$\frac{dy}{dx}$, pronounced ‘dee $y$ by dee $x$’, is not a fraction even though it might look like one! This notation can be confusing. Think of $\frac{dy}{dx}$ as the ‘symbol’ for the gradient function of $y = f(x)$. The process of finding $\frac{dy}{dx}$ is called differentiation with respect to $x$.

Example

For any value of $n$, the gradient function of $x^n$ is $nx^{n-1}$. We write:

\[ \text{if } y = x^n \text{, then } \frac{dy}{dx} = nx^{n-1} \]

You have seen specific cases of this result earlier on. For example, if $y = x^3$, $\frac{dy}{dx} = 3x^2$.

More notation and terminology

When $y = f(x)$ alternative ways of writing the gradient function, $\frac{dy}{dx}$, are $y'$, pronounced ‘$y$ dash’, or $\frac{df}{dx}$, or $f'$, pronounced ‘$f$ dash’. In practice you do not need to remember the formulas for the gradient functions of all the common functions. Engineers usually refer to a table known as a Table of Derivatives. A derivative is another name for a gradient function. The derivative is also known as the rate of change of a function.

Exercises

1. Given that when $y = x^2$, $\frac{dy}{dx} = 2x$, find the gradient of $y = x^2$ when $x = 7$.
2. Given that when $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$, find the gradient of $y = x^4$ when a) $x = 2$, b) $x = -1$.
3. Find the rate of change of $y = x^3$ when a) $x = -2$, b) $x = 6$.
4. Given that when $y = 7x^2 + 5x$, $\frac{dy}{dx} = 14x + 5$, find the gradient of $y = 7x^2 + 5x$ when $x = 2$.

Answers

1. 14. 2. a) 32, b) -4. 3. a) 12, b) 108. 4. 33.