**Proof by Induction : Further Examples**

**Example**

Prove by induction that $11^n - 6$ is divisible by 5 for every positive integer $n$.

**Solution**

Let $P(n)$ be the mathematical statement

$$11^n - 6 \text{ is divisible by 5.}$$

**Base Case:** When $n = 1$ we have $11^1 - 6 = 5$ which is divisible by 5. So $P(1)$ is correct.

**Induction hypothesis:** Assume that $P(k)$ is correct for some positive integer $k$. That means $11^k - 6$ is divisible by 5 and hence $11^k - 6 = 5m$ for some integer $m$. So $11^k = 5m + 6$.

**Induction step:** We will now show that $P(k + 1)$ is correct. Always keep in mind what we are aiming for and what we know to be true. In this case we want to show that $11^{k+1} - 6$ can be expressed as a multiple of 5, so we will start with the formula $11^{k+1} - 6$ and we will rearrange it into something involving multiples of 5. At some point we will also want to use the assumption that $11^k = 5m + 6$.

\[
11^{k+1} - 6 = (11 \times 11^k) - 6
\]

by the laws of powers

\[
= 11(5m + 6) - 6
\]

by the induction hypothesis

\[
= 11(5m) + 66 - 6
\]

by expanding the bracket

\[
= 5(11m) + 60
\]

\[
= 5(11m + 12)
\]

since both parts of the formula have a common factor of 5.

As $11m + 12$ is an integer we have that $11^{k+1} - 6$ is divisible by 5, so $P(k + 1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers $n$.

**Example**

Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer $n$.

**Solution**

Let $P(n)$ be the statement $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

**Base Case:** When $n = 1$ the left hand side of the equation is 1 and the right hand side is $\frac{1(1+1)(2\times 1+1)}{6} = \frac{6}{6} = 1$. So $P(1)$ is correct.
**Induction hypothesis:** Assume that $P(k)$ is correct for some positive integer $k$. That means that the left hand side of the equation equals the right hand side, so $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

**Induction step:** We will now show that $P(k+1)$ is correct. Keep in mind what we are aiming for, so in this case the right hand side of the equation should be $\frac{(k+1)(k+2)(2k+3)}{6}$.

So starting with the left hand side we have

\[
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by the induction hypothesis}
\]

\[
= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \quad \text{by making each part a fraction over 6}
\]

\[
= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \quad \text{by making it a single fraction over 6}
\]

\[
= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \quad \text{by taking out the common factor}
\]

\[
= \frac{(k+1)(2k^2 + 7k + 6)}{6} \quad \text{by expanding out the square brackets}
\]

\[
= \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{by factorizing}
\]

\[
= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \text{this is the right hand side.}
\]

So $P(k+1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers $n$.

**Example**

Prove by induction that $2^n > 2n$ for every positive integer $n > 2$.

**Solution**

Let $P(n)$ be the mathematical statement $2^n > 2n$.

**Base Case:** When $n = 3$ we have $2^3 = 8 > 6 = 2 \times 3$. So $P(3)$ is correct.

**Induction hypothesis:** Assume that $P(k)$ is correct for some positive integer $k$. That means that $2^k > 2k$.

**Induction step:** We will now show that $P(k+1)$ is correct.

\[
2^{k+1} = 2 \times 2^k > 2 \times 2k \quad \text{by the induction hypothesis}
\]

\[
= 2(k+1).
\]

So $P(k+1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers $n > 2$.

**Exercises**

Prove by induction that

1. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$ for all positive integers.

2. $n^3 - n$ is divisible by 6 for all positive integers.

3. $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all positive integers.