Introduction

Suppose we have the first order differential equation

\[ P(y) \frac{dy}{dx} = Q(x) \]

where \( Q(x) \) and \( P(y) \) are functions involving \( x \) and \( y \) only respectively. For example

\[ y^2 \frac{dy}{dx} = \frac{1}{x^3} \quad \text{or} \quad \frac{1}{y^2} \frac{dy}{dx} = \frac{x - 3}{x^3}. \]

We can solve these differential equations using the technique of separating variables.

General Solution

By taking the original differential equation

\[ P(y) \frac{dy}{dx} = Q(x) \]

we can solve this by separating the equation into two parts. We move all of the equation involving the \( y \) variable to one side and all of the equation involving the \( x \) variable to the other side, then we can integrate both sides. Although \( \frac{dy}{dx} \) is not a fraction, we can intuitively treat it like one to move the "\( dx \)" to the right hand side. So

\[ P(y) \frac{dy}{dx} = Q(x) \iff \int P(y) \, dy = \int Q(x) \, dx. \]

Example

Let us find the general solution of the differential equation

\[ y^2 \frac{dy}{dx} = \frac{1}{x^3}. \]

\[ y^2 \frac{dy}{dx} = \frac{1}{x^3} \iff \int y^2 \, dy = \int \frac{1}{x^3} \, dx \]

\[ \iff \int y^2 \, dy = \int x^{-3} \, dx \]

\[ \iff \frac{y^3}{3} = -\frac{2}{3} + c \quad \text{where} \ c \ \text{is a constant} \]

\[ \iff y^3 = -\frac{3}{2} + 3c \]

\[ \iff y = \sqrt[3]{-\frac{3}{2} + 3c} \]
Example

To find the general solution of the differential equation

\[
\frac{dy}{dx} = \frac{y^2(x - 3)}{x^3}
\]

we first need to move the \(y^2\) to the left hand side of the equation. Then we move the \(dx\) to the right hand side of the equation and integrate both sides.

\[
\frac{dy}{y^2} = \frac{x - 3}{x^3} \iff \int \frac{1}{y^2} dy = \int \frac{x - 3}{x^3} dx
\]

\[
\iff \int \frac{1}{y^2} dy = \int \frac{x}{x^3} - \frac{3}{x^3} dx
\]

\[
\iff \int y^{-2} dy = \int \frac{1}{x^2} - \frac{3}{x^3} dx
\]

\[
\iff -y^{-1} = -x^{-1} + \frac{3x^{-2}}{2} + c \quad \text{where} \ c \ \text{is a constant}
\]

\[
\iff -\frac{1}{y} = -\frac{1}{x} + \frac{3}{2x^2} + c
\]

\[
\iff \frac{1}{y} = \frac{1}{x} - \frac{3}{2x^2} - c
\]

\[
\iff \frac{1}{y} = \frac{2x}{2x^2} - \frac{3}{2x^2} - \frac{2cx^2}{2x^2}
\]

\[
\iff y = \frac{2x^2}{2x - 3 - 2cx^2}
\]

\[
\iff y = \frac{2x}{2x - 3 - 2cx^2}
\]

Exercises

Find the general solution of

1. \(\frac{dy}{dx} = y(1 + e^x)\)  
2. \(\frac{dy}{dx} = \frac{x}{y}\)  
3. \(\frac{dy}{dx} = 9x^2y\)  
4. \(\frac{4}{y^3} \frac{dy}{dx} = \frac{1}{x}\)

Answers

1. \(y = e^{x+x^e+c}\)  
2. \(y = \pm \sqrt{x^2 + 2c}\)  
3. \(y = e^{x^3+c}\)  
4. \(y = \pm \sqrt{-\frac{2}{\ln |x| + c}}\)

Note that the \(\pm\) symbol like in Exercise 2 means that the differential equation has two sets of solutions, \(y = \sqrt{x^2 + 2c}\) and \(y = -\sqrt{x^2 + 2c}\).