Direct Proof

Introduction

A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form "if \( p \) then \( q \)" or "\( p \) implies \( q \)" which we can write as \( p \Rightarrow q \). The method of the proof is to takes an original statement \( p \), which we assume to be true, and use it to show directly that another statement \( q \) is true. So a direct proof has the following steps:

- Assume the statement \( p \) is true.
- Use what we know about \( p \) and other facts as necessary to deduce that another statement \( q \) is true, that is show \( p \Rightarrow q \) is true.

Example

Directly prove that if \( n \) is an odd integer then \( n^2 \) is also an odd integer.

Solution

Let \( p \) be the statement that \( n \) is an odd integer and \( q \) be the statement that \( n^2 \) is an odd integer. Assume that \( n \) is an odd integer, then by definition \( n = 2k + 1 \) for some integer \( k \). We will now use this to show that \( n^2 \) is also an odd integer.

\[
\begin{align*}
n^2 &= (2k + 1)^2 \\
&= (2k + 1)(2k + 1) \\
&= 4k^2 + 2k + 2k + 1 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1
\end{align*}
\]

since \( n = 2k + 1 \)

by expanding the brackets

Hence we have shown that \( n^2 \) has the form of an odd integer since \( 2k^2 + 2k \) is an integer. Therefore we have shown that \( p \Rightarrow q \) and so we have completed our proof.

Example

Let \( a, b \) and \( c \) be integers, directly prove that if \( a \) divides \( b \) and \( a \) divides \( c \) then \( a \) also divides \( b + c \).

Solution

Let \( a, b \) and \( c \) be integers and assume that \( a \) divides \( b \) and \( a \) divides \( c \). Then as \( a \) divides \( b \), by definition, there is some integer \( k \) such that \( b = ak \). Also as \( a \) divides \( c \), by definition, there is some integer \( l \) such that \( c = al \). Note that we use different letters \( k \) and \( l \) to stand for the integers...
because we do not know if \( b \) and \( c \) are equal or not. We will now use these two facts to get our conclusion. So

\[
b + c = (ak) + (al) = a(k + l)
\]

by our definitions of \( b \) and \( c \) since \( a \) is a common factor.

Hence \( a \) divides \( b + c \) since \( k + l \) is an integer.

**Example**

Directly prove that if \( m \) and \( n \) are odd integers then \( mn \) is also an odd integer.

**Solution**

Assume that \( m \) and \( n \) are odd integers. Then by definition \( m = 2k + 1 \) for some integer \( k \) and \( n = 2l + 1 \) for some integer \( l \). Again note that we have used different integers \( k \) and \( l \) in the definitions of \( m \) and \( n \). We will now use this to show that \( mn \) is also an odd integer.

\[
mn = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1
\]

by our definitions of \( m \) and \( n \) by expanding the brackets since \( 2 \) is a common factor.

Hence we have shown that \( mn \) has the form of an odd integer since \( 2kl + k + l \) is an integer.

**Example**

Let \( m \) and \( n \) be integers. Directly prove that if \( m \) and \( n \) are perfect squares then \( mn \) is also a perfect square.

**Solution**

Recall the definition that an integer \( m \) is a perfect square if \( m = k^2 \) for some integer \( k \). Now assume that \( m \) and \( n \) are integers and are perfect squares. Then by definition \( m = k^2 \) for some integer \( k \) and \( n = l^2 \) for some integer \( l \). We will now use these facts to show that \( mn \) is also a perfect square.

\[
mn = k^2l^2 = (kl)^2
\]

and \( kl \) is an integer, therefore \( mn \) is a perfect square.

**Exercises**

Prove directly that

1. If \( n \) is an even integer then \( 7n + 4 \) is an even integer.
2. If \( m \) is an even integer and \( n \) is an odd integer then \( m + n \) is an odd integer.
3. If \( m \) is an even integer and \( n \) is an odd integer then \( mn \) is an even integer.
4. If \( a, b \) and \( c \) are integers such that \( a \) divides \( b \) and \( b \) divides \( c \) then \( a \) divides \( c \).