Introduction to differentiation

Introduction

This leaflet provides a rough and ready introduction to **differentiation**. This is a technique used to calculate the gradient, or slope, of a graph at different points.

1. The gradient function

Given a function, for example, \( y = x^2 \), it is possible to derive a formula for the gradient of its graph. We can think of this formula as the **gradient function**, precisely because it tells us the gradient of the graph. For example,

\[
\text{when } y = x^2 \quad \text{the gradient function is } \quad 2x
\]

So, the gradient of the graph of \( y = x^2 \) at any point is twice the \( x \) value there. To understand how this formula is actually found you would need to refer to a textbook on calculus. The important point is that using this formula we can calculate the gradient of \( y = x^2 \) at different points on the graph. For example,

\[
\text{when } x = 3, \quad \text{the gradient is } 2 \times 3 = 6.
\]

\[
\text{when } x = -2, \quad \text{the gradient is } 2 \times (-2) = -4.
\]

How do we interpret these numbers? A gradient of 6 means that values of \( y \) are increasing at the rate of 6 units for every 1 unit increase in \( x \). A gradient of -4 means that values of \( y \) are decreasing at a rate of 4 units for every 1 unit increase in \( x \).

Note that when \( x = 0 \), the gradient is \( 2 \times 0 = 0 \).

Below is a graph of the function \( y = x^2 \). Study the graph and you will note that when \( x = 3 \) the graph has a positive gradient. When \( x = -2 \) the graph has a negative gradient. When \( x = 0 \) the gradient of the graph is zero. Note how these properties of the graph can be predicted from knowledge of the gradient function, \( 2x \).
Example
When \( y = x^3 \), its gradient function is \( 3x^2 \). Calculate the gradient of the graph of \( y = x^3 \) when
a) \( x = 2 \), b) \( x = -1 \), c) \( x = 0 \).

Solution
a) when \( x = 2 \) the gradient function is \( 3(2)^2 = 12 \).

b) when \( x = -1 \) the gradient function is \( 3(-1)^2 = 3 \).

c) when \( x = 0 \) the gradient function is \( 3(0)^2 = 0 \).

2. Notation for the gradient function
You will need to use a notation for the gradient function which is in widespread use.

\[
\frac{dy}{dx}, \ \text{pronounced ‘dee } y \text{ by dee } x \text{’, is not a fraction even though it might look like one! This notation can be confusing. Think of } \frac{dy}{dx} \text{ as the ‘symbol’ for the gradient function of } y = f(x).
\]

The process of finding \( \frac{dy}{dx} \) is called \textbf{differentiation with respect to} \( x \).

Example
For any value of \( n \), the gradient function of \( x^n \) is \( nx^{n-1} \). We write:

\[
\text{if } y = x^n, \quad \text{then } \frac{dy}{dx} = nx^{n-1}
\]

You have seen specific cases of this result earlier on. For example, if \( y = x^3 \), \( \frac{dy}{dx} = 3x^2 \).

3. More notation and terminology
When \( y = f(x) \) alternative ways of writing the gradient function, \( \frac{dy}{dx} \), are \( y' \), pronounced ‘\( y \) dash’, or \( \frac{df}{dx} \), or \( f' \), pronounced ‘\( f \) dash’. In practice you do not need to remember the formulas for the gradient functions of all the common functions. Engineers usually refer to a table known as a \textit{Table of Derivatives}. A \textbf{derivative} is another name for a gradient function. Such a table is available on leaflet 8.2. The derivative is also known as the \textbf{rate of change} of a function.

Exercises
1. Given that when \( y = x^2 \), \( \frac{dy}{dx} = 2x \), find the gradient of \( y = x^2 \) when \( x = 7 \).

2. Given that when \( y = x^n \), \( \frac{dy}{dx} = nx^{n-1} \), find the gradient of \( y = x^4 \) when a) \( x = 2 \), b) \( x = -1 \).

3. Find the rate of change of \( y = x^3 \) when a) \( x = -2 \), b) \( x = 6 \).

4. Given that when \( y = 7x^2 + 5x \), \( \frac{dy}{dx} = 14x + 5 \), find the gradient of \( y = 7x^2 + 5x \) when \( x = 2 \).

Answers
1. 14.  2. a) 32, b) –4.  3. a) 12, b) 108.  4. 33.