

Module 4.**Understanding and using logs and exponential equations****4M What is e? Why natural logs?**

“e” is just a number, in fact it is roughly 2.7182818.....

You know about π , e is just a number just like π is a number.

But, like π , e is a pretty special number. The reason e is special (I think) is because it can describe the properties of all sorts of things in nature.

Whenever you have a process where the rate of loss is proportional to the amount of the substance,

i.e. rate of loss of something \propto amount of something
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which can be written as

rate of loss of something = k x amount of something

then that process follows exponential decay with the formula

$N = N_0 e^{-kt}$

For example, the decay of radioactivity

Let's look at radioactive decay to understand more about e.

Each type of radioactive isotope, decays with a particular characteristic half-life. That is, the time taken for half of the isotope to decay is a characteristic for that isotope. So, iodine-125 has a half-time of about 61 days. Other isotopes such as hydrogen-3 are slower whilst others are faster. What this means is that the number of isotope atoms decaying in a minute is greater when there are more isotope atoms and less when there are fewer isotope atoms. More exactly, the rate of decay is proportional to the number of atoms waiting to decay. Or put it mathematically,

The rate of loss of $N \propto N$ (or if you like calculus $dN/dt \propto N$).

To put it another way, if $y = e^x$ then $dy/dx = e^x$.

and if $y = e^{-x}$, then $dy/dx = -e^{-x}$

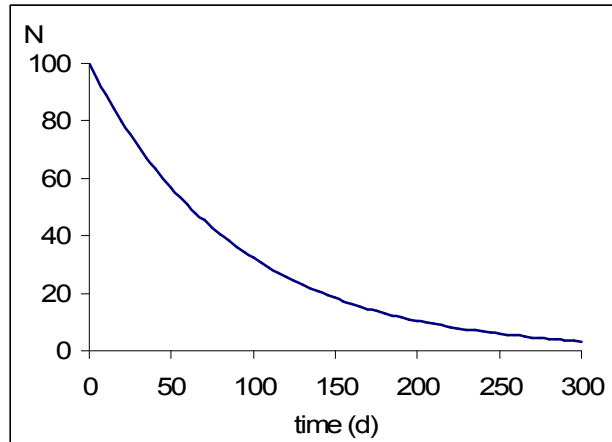
If you think about a lot of natural physical processes, then wherever the rate of change of something is proportional to the amount of that something, then e is going to be very useful.

So getting back to radioactive decay,
the rate of change of radioisotope atoms = k x number of radioisotope atoms
where k is a constant for a particular isotope.



If N = number of radioisotope atoms, then

$$N = N_0 e^{-kt}$$



Now take logs of both sides, note that on your calculator you should have a button labelled "ln". This is the button for natural logs, or logs to the base "e".

$$e^a = x \quad \leftrightarrow \quad \ln(x) = a$$

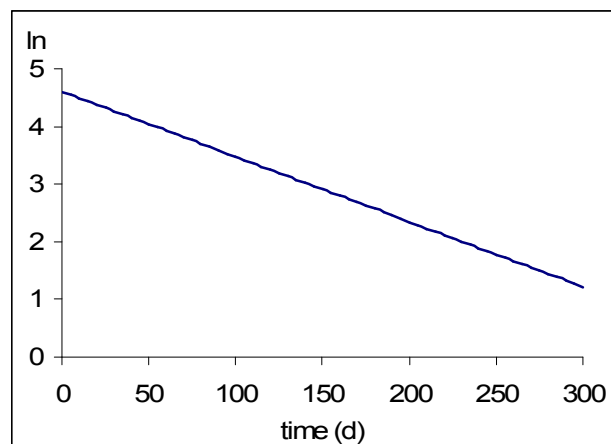
$$N = N_0 e^{-kt}$$

taking logs of both sides gives...

$$\ln N = \ln N_0 - kt$$

this is the equation of a straight line
 $y = c - mx$
 with slope = $-k$ and intercept = $\ln N_0$

$$\ln N = \ln N_0 - kt$$



The half-life ($t_{0.5}$) is the time taken to get from N_0 to $\frac{1}{2}N_0$

$$\frac{N}{N_0} = e^{-kt}$$

$$2 = e^{-kt_{0.5}}$$

$$\ln 2 = -kt_{0.5}$$

$$0.69 = -kt_{0.5}$$

$$t_{0.5} = \frac{-0.69}{k}$$

4N Using natural logs – calculating half time and using log graph paper

Let's take some example data and work through plotting this data in a variety of forms to see the differences between the different representations.

The elimination of drug from the blood follows exponential decay according to the formula

$$C = C_0 e^{-kt}$$

where

C is the concentration in the blood at time t.

C_0 is the starting concentration (at time = 0).

k is the rate constant for elimination.

The table below gives data for elimination of an antibiotic from the blood.

time (h)	concentration in blood (ng/ml)
2	52.5
4	39.6
8	22.9
12	13.1
16	7.5
20	4.3
24	2.5

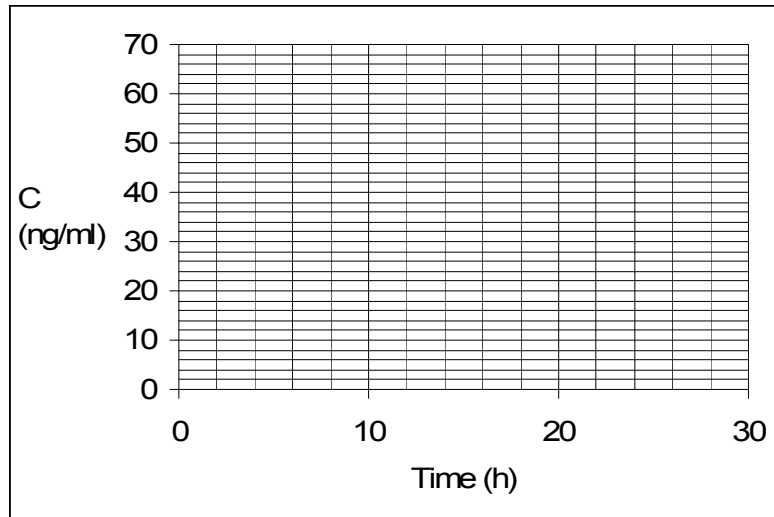
C vs t

$\ln C$ vs t

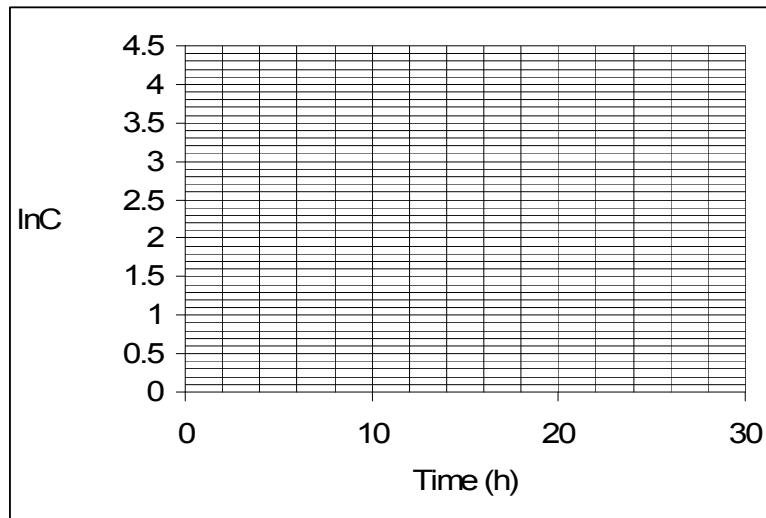
C vs t on log-linear paper.

now plot this data on the following pieces of graph paper

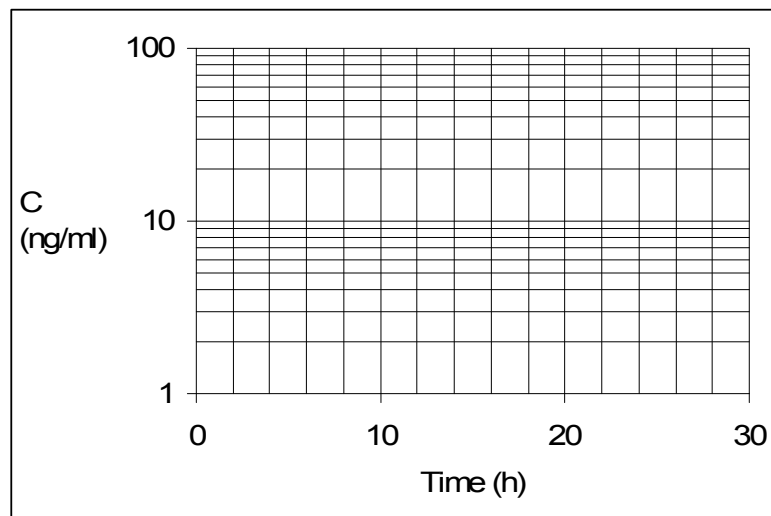
Plot the data on this linear-linear graph paper. Notice the shape of the exponential decay. Estimate the half-time and k . Estimate the value for C_0



Calculate $\ln C$. Then plot $\ln C$ against time. Notice how the shape of the curve has changed. Now calculate the slope of the line (=the rate constant k). Calculate $t_{1/2}$ from the formula $t_{1/2} = \frac{-0.69}{k}$. Estimate C_0



Plot the data for concentration against time on this log – linear graph paper. Notice how the lines are spaced unevenly. Notice that the distance from 1 to 2 is the same as from 2 to 4 and is the same as from 4 to 8. i.e. when you multiply by two, you go up by the same amount, no matter what number you started with. Now calculate the $t_{1/2}$, k and C_0 .



Answers for the last section:

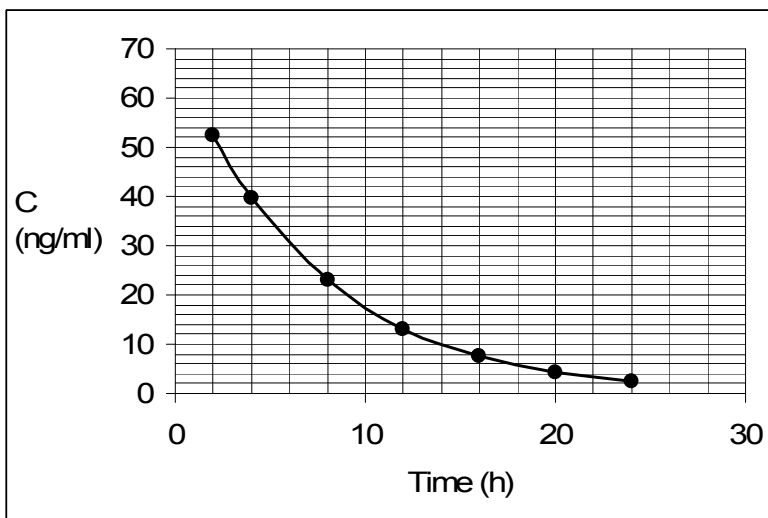
Time taken to decrease by half:

Estimate the time taken to get from 52 to 26 is roughly $7.2 - 2 = 5.2$ h.
 $t_{1/2} \sim 5.2$ h

$$k = \frac{-0.69}{t_{1/2}} = \frac{-0.69}{5.2h} \quad C_0 \text{ is}$$

$$= 0.13h^{-1}$$

hard to get accurately – but can estimate about 70 ng/ml.



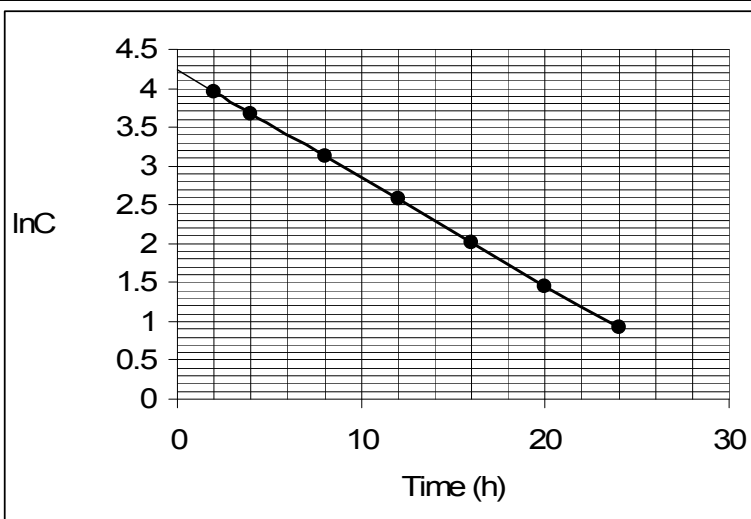
The slope is $-k$. Estimate slope by choosing two points on the line

$$k = \frac{3.96 - 0.916}{24 - 2} = \frac{-3.044}{22}$$

$$k = -0.138h^{-1}$$

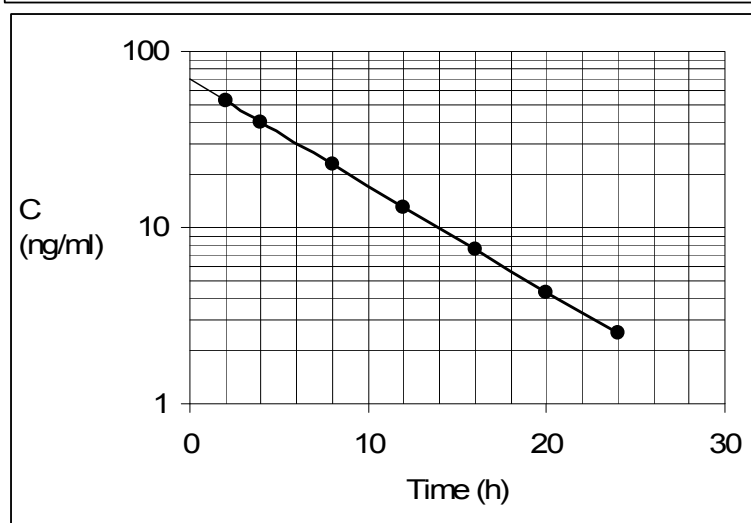
$$t_{1/2} = \frac{-0.69}{-0.138h^{-1}} = 5.0h$$

$$C_0 = e^{4.2} = 66.7 \text{ ng/ml}$$



To get k , you need to remember that this is \log_{10} graph paper so you can't just calculate the slope straight from the numbers on the graph. It is easiest to take two points on the line, calculate the $\ln C$ for each. Then calculate the slope as above.

C_0 can be read off the graph ~ 70 ng/ml



The $\ln C$ vs t plot is the most accurate one for getting the rate constant and starting concentration.

