Module 4. Understanding and using logs and exponential equations

4A Rationale – why logs?.

Logarithms appear in a variety of situations – for example the strength of earthquakes (the Richter scale), the strength of an acid (pH), the intensity of noise (decibels), bacterial growth and radioactive decay. Whenever you have data that cover a very wide range of numbers, it often helps to use a logarithmic scale to express and graph the data. We will look at some of these examples in more detail later but first of all we need to look at the notation and understand how logarithms arose.

4B, C Introducing logarithms

You should be familiar with how to multiply and divide large and small numbers using scientific notation. For multiplication, you just add the exponents or powers and for division you subtract.

$$10^a \times 10^b = 10^{a+b} \qquad \frac{10^a}{10^b} = 10^{a-b}$$

In this case, the **base is 10** and the **power is a** or **b**.

Back in the 16th century long before calculators were invented, mathematicians realised that the easy way to do long multiplications and divisions was to express every large number in scientific notation and then you just had to add or subtract the powers which was much easier.

say you had to calculate	156890	express this as	10^a
	33489 x	express this as	10^b

	???????	express this as	10^{a+b}

So if you can find the number *a* that would give $10^a = 156890$,
and the number *b* that would give $10^b = 33489$
Then you could multiply them just by adding *a* + *b*.

So this led to the idea of logarithms. The language is a bit odd, but with practice it will become second nature.

How to write numbers in logarithm notation.

Take an example, you know that

$10^3 = 1000$... saying that in words is "10 raised to the power 3 is 1000".

Writing this in logarithms is

$\log_{10}1000 = 3$... saying that in words is "the logarithm to the base 10 of 1000 is 3"

These two notations are saying the same thing...

$10^3 = 1000$	is the same as	$\log_{10}1000 = 3$

Take another example, in scientific notation, $10^{-1} = 0.1$.

Writing that into logs becomes $\log_{10}0.1 = -1$

$10^{-1} = 0.1$	is the same as	$\log_{10}0.1 = -1$

Although we have only used integers until now, it is possible for *a* to be any number.

For example,

$10^{-0.45} = 0.3548$	is the same as	$\log_{10}(0.3548) = -0.45$

In general then, the logarithm to the base 10 of a number x is the power to which 10 must be raised in order to equal x.

If $10^a = x$	then	$\log_{10}x = a$.

Get some practice with your calculator and make sure you know how it works.

- Input the number 1000
- Now do log(1000) with your calculator.
- Calculators are different so you need to find out how yours works.
- your answer should be 3
- now do the reverse process – i.e. you're doing 10^3 – sometimes this is done as INV LOG or 2nd function LOG. Sometimes you input 3 and do "inverse" then "log". You should get 1000 back again.

The reverse process to taking a logarithm is sometimes called antilogarithm.

Let's explore this a little further - try this on your calculator –

10^6	$= 1000000$	$\leftrightarrow$	$\log_{10}1000000$	$= 6$
$10^{0.1}$	$= 1.259$	$\leftrightarrow$	$\log_{10}(1.259)$	$= 0.1$
$10^{0.001}$	$= 1.002$	$\leftrightarrow$	$\log_{10}(1.002)$	$= 0.001$
10^0	$= 1$	$\leftrightarrow$	$\log_{10}(1)$	$= 0$
$10^{-0.1}$	$= \frac{1}{10^{0.1}} = \frac{1}{1.259} = 0.794$	$\leftrightarrow$	$\log_{10}(0.794)$	$= -0.1$
10^{-6}	$= 0.000001$	$\leftrightarrow$	$\log_{10}(0.000001)$	$= -6$

Note that there is **no** number x for which $10^x \le 0$. That is, it is impossible to multiply 10 by itself how ever many times and get a number which is zero or less. So if you input -1 or 0 or any negative number into your calculator and press "log" you will get an error message.

4D Using logs - the pH scale

The acidity of a solution is a reflection of its hydrogen ion concentration [H+].
HCl is a strong acid which is completely dissociated in water.
HCl(gas) dissolved in water becomes H^+ and Cl^- $HCl \rightarrow H^+ + Cl^-$
(actually H^+ is H_30^+ because it interacts with a water molecule so strictly speaking we should write $HCl + H_2O \rightarrow H_3O^+ + Cl^-$ but in practice we don't)

Therefore 1M HCl = 1M H^+ and 1M Cl^-

The pH of a solution is defined as the negative logarithm of its $[H^+]$
(where the hydrogen ion concentration is given in M).
So the pH of a 1M HCl solution is $-\log_{10}(10^0) = 0$ (since $1 = 10^0$)

1 M HCl	$[H^+] = 10^0$ M	pH = 0
vinegar, beer, cola	$[H^+] = 10^{-3}$ M	pH = 3
urine	$[H^+] = 10^{-6}$ M	pH = 6
pure water	$[H^+] = 10^{-7}$ M	pH = 7
seawater	$[H^+] = 10^{-8}$ M	pH = 8
bleach	$[H^+] = 10^{-12}$ M	pH = 12
caustic soda (1 M NaOH)	$[H^+] = 10^{-14}$ M	pH = 14

Hydrogen ion concentrations in water can vary over a very wide range. Each pH unit represents a tenfold difference H+ concentration. It is this mathematical feature that makes the pH scale so useful. For example a solution of pH 2 is not twice as acidic as a solution of pH 4, but a hundred times more acidic. So when the pH of a solution changes slightly, it actually changes the concentrations of H^+ and OH^- substantially.