The modulus and argument of a complex number

In this unit you are going to learn about the **modulus** and **argument** of a complex number. These are quantities which can be recognised by looking at an Argand diagram. Recall that any complex number, $z$, can be represented by a point in the complex plane as shown in Figure 1.

![Argument and modulus of a complex number](image)

Figure 1. The complex number $z$ is represented by point $P$. Its modulus and argument are shown.

We can join point $P$ to the origin with a line segment, as shown. We associate with this line segment two important quantities. The length of the line segment, that is $OP$, is called the **modulus** of the complex number. The angle from the positive axis to the line segment is called the **argument** of the complex number, $z$.

The modulus and argument are fairly simple to calculate using trigonometry.

**Example.** Find the modulus and argument of $z = 4 + 3i$.

**Solution.** The complex number $z = 4 + 3i$ is shown in Figure 2. It has been represented by the point $Q$ which has coordinates $(4, 3)$. The modulus of $z$ is the length of the line $OQ$ which we can find using Pythagoras’ theorem.

\[(OQ)^2 = 4^2 + 3^2 = 16 + 9 = 25\]

and hence $OQ = 5$.

![Complex number representation](image)

Figure 2. The complex number $z = 4 + 3i$.

Hence the modulus of $z = 4 + 3i$ is 5. To find the argument we must calculate the angle between the $x$ axis and the line segment $OQ$. We have labelled this $\theta$ in Figure 2.
By referring to the right-angled triangle $OQN$ in Figure 2 we see that

\[
\tan \theta = \frac{3}{4}
\]

\[
\theta = \tan^{-1} \frac{3}{4} = 36.97\degree
\]

To summarise, the modulus of $z = 4 + 3i$ is 5 and its argument is $\theta = 36.97\degree$. There is a special symbol for the modulus of $z$; this is $|z|$. So, in this example, $|z| = 5$. We also have an abbreviation for argument: we write $\text{arg}(z) = 36.97\degree$.

When the complex number lies in the first quadrant, calculation of the modulus and argument is straightforward. For complex numbers outside the first quadrant we need to be a little bit more careful. Consider the following example.

**Example.**

Find the modulus and argument of $z = 3 - 2i$.

**Solution.** The Argand diagram is shown in Figure 3. The point $P$ with coordinates $(3, -2)$ represents $z = 3 - 2i$.

![Figure 3. The complex number $z = 3 - 2i$.](image)

We use Pythagoras’ theorem in triangle $ONP$ to find the modulus of $z$:

\[
(\text{OP})^2 = 3^2 + 2^2 = 13
\]

\[
\text{OP} = \sqrt{13}
\]

Using the symbol for modulus, we see that in this example $|z| = \sqrt{13}$.

We must be more careful with the argument. When the angle $\theta$ shown in Figure 3 is measured in a clockwise sense convention dictates that the angle is negative. We can find the size of the angle by referring to the right-angled triangle shown. In that triangle $\tan \alpha = \frac{2}{3}$ so that $\alpha = \tan^{-1} \frac{2}{3} = 33.67\degree$. This is not the argument of $z$. The argument of $z$ is $\theta = -33.67\degree$. We often write this as $\text{arg}(z) = -33.67\degree$.

In the next unit we show how the modulus and argument are used to define the **polar form** of a complex number.