Engineering Maths First Aid Kit

7.2

Complex arithmetic

Introduction.

This leaflet describes how complex numbers are added, subtracted, multiplied and divided.

1. Addition and subtraction of complex numbers.

Given two complex numbers we can find their sum and difference in an obvious way.

If $z_1 = a_1 + b_1 j$ and $z_2 = a_2 + b_2 j$ then

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)j$$

$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)j$$

So, to add the complex numbers we simply add the real parts together and add the imaginary parts together.

Example
If $z_1 = 13 + 5j$ and $z_2 = 8 - 2j$ find a) $z_1 + z_2$, b) $z_2 - z_1$.

Solution
a) $z_1 + z_2 = (13 + 5j) + (8 - 2j) = 21 + 3j$.

b) $z_2 - z_1 = (8 - 2j) - (13 + 5j) = -5 - 7j$

2. Multiplication of complex numbers.

To multiply two complex numbers we use the normal rules of algebra and also the fact that $j^2 = -1$. If z_1 and z_2 are the two complex numbers their product is written $z_1 z_2$.

Example
If $z_1 = 5 - 2j$ and $z_2 = 2 + 4j$ find $z_1 z_2$.

Solution

$$z_1 z_2 = (5 - 2j)(2 + 4j) = 10 + 20j - 4j - 8j^2$$

Replacing j^2 by -1 we obtain

$$z_1 z_2 = 10 + 16j - 8(-1) = 18 + 16j$$

In general we have the following result:

If $z_1 = a_1 + b_1 j$ and $z_2 = a_2 + b_2 j$ then

$$\begin{aligned} z_1 z_2 = (a_1 + b_1 j)(a_2 + b_2 j) &= a_1a_2 + a_1b_2j + b_1a_2j + b_1b_2j^2 \\ &= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \end{aligned}$$

3. Division of complex numbers.

To divide complex numbers we need to make use of the **complex conjugate**. Given a complex number, z, its conjugate, written $\bar{z}$, is found by changing the sign of the imaginary part. For example, the complex conjugate of $z = 3 + 2j$ is $\bar{z} = 3 - 2j$. Division is illustrated in the following example.

Example
Find $\dfrac{z_1}{z_2}$ when $z_1 = 3 + 2j$ and $z_2 = 4 - 3j$.

Solution
We require

$$\frac{z_1}{z_2} = \frac{3 + 2j}{4 - 3j}$$

Both numerator and denominator are multiplied by the complex conjugate of the denominator. Overall, this is equivalent to multiplying by 1 and so the fraction remains unaltered, but it will have the effect of making the denominator purely real, as you will see.

$$\begin{aligned} \frac{3+2j}{4-3j} &= \frac{3+2j}{4-3j} \times \frac{4+3j}{4+3j} \\ &= \frac{(3+2j)(4+3j)}{(4-3j)(4+3j)} \\ &= \frac{12 + 9j + 8j + 6j^2}{16 + 12j - 12j - 9j^2} \\ &= \frac{6 + 17j}{25} \qquad \text{(the denominator is now seen to be real)} \\ &= \frac{6}{25} + \frac{17}{25}j \end{aligned}$$

Exercises
1. If $z_1 = 1 + j$ and $z_2 = 3 + 2j$ find a) $z_1 z_2$, b) $\overline{z_1}$, c) $\overline{z_2}$, d) $z_1\overline{z_1}$, e) $z_2\overline{z_2}$
2. If $z_1 = 1 + j$ and $z_2 = 3 + 2j$ find: a) $\frac{z_1}{z_2}$, b) $\frac{z_2}{z_1}$, c) $z_1/\overline{z_1}$, d) $z_2/\overline{z_2}$.
3. Find a) $\frac{7-6j}{2j}$, b) $\frac{3+9j}{1-2j}$, c) $\frac{1}{j}$.

Answers
1. a) $1 + 5j$, b) $1 - j$, c) $3 - 2j$, d) 2, e) 13
2. a) $\frac{5}{13} + \frac{j}{13}$, b) $\frac{5}{2} - \frac{j}{2}$, c) j, d) $\frac{5}{13} + \frac{12}{13}j$.
3. a) $-3 - \frac{7}{2}j$, b) $-3 + 3j$, c) $-j$.

