

# Sigma notation

## Introduction

**Sigma notation**,  $\Sigma$ , provides a concise and convenient way of writing long sums. This leaflet explains how.

## 1. Sigma notation

The sum

$$1 + 2 + 3 + 4 + 5 + \dots + 10 + 11 + 12$$

can be written very concisely using the capital Greek letter  $\Sigma$  as

$$\sum_{k=1}^{k=12} k$$

The  $\Sigma$  stands for a sum, in this case the sum of all the values of  $k$  as  $k$  ranges through all whole numbers from 1 to 12. Note that the lower-most and upper-most values of  $k$  are written at the bottom and top of the sigma sign respectively. You may also see this written as  $\sum_{k=1}^{k=12} k$ , or even as  $\sum_{k=1}^{12} k$ .

### Example

Write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

### Solution

We must let  $k$  range from 1 to 5, cube each value of  $k$ , and add the results:

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

### Example

Express  $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$  concisely using sigma notation.

### Solution

Each term takes the form  $\frac{1}{k}$  where  $k$  varies from 1 to 4. In sigma notation we could write this as

$$\sum_{k=1}^{k=4} \frac{1}{k}$$

### Example

The sum

$$x_1 + x_2 + x_3 + x_4 + \dots + x_{19} + x_{20}$$

can be written

$$\sum_{k=1}^{k=20} x_k$$

There is nothing special about using the letter  $k$ . For example

$$\sum_{n=1}^{n=7} n^2 \quad \text{stands for} \quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

We can also use a little trick to alternate the signs of the numbers between  $+$  and  $-$ . Note that  $(-1)^2 = 1$ ,  $(-1)^3 = -1$  and so on.

### Example

Write out fully what is meant by

$$\sum_{i=0}^5 \frac{(-1)^{i+1}}{2i+1}$$

### Solution

$$\sum_{i=0}^5 \frac{(-1)^{i+1}}{2i+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11}$$

### Exercises

1. Write out fully what is meant by

- a)  $\sum_{i=1}^{i=5} i^2$
- b)  $\sum_{k=1}^4 (2k+1)^2$
- c)  $\sum_{k=0}^4 (2k+1)^2$

2. Write out fully what is meant by

$$\sum_{k=1}^{k=3} (\bar{x} - x_k)$$

3. Sigma notation is often used in statistical calculations. For example the **mean**,  $\bar{x}$ , of the  $n$  quantities  $x_1, x_2 \dots$  and  $x_n$ , is found by adding them up and dividing the result by  $n$ . Show that the mean can be written as

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

4. Write out fully what is meant by  $\sum_{i=1}^4 \frac{i}{i+1}$ .

5. Write out fully what is meant by  $\sum_{k=1}^3 \frac{(-1)^k}{k}$ .

### Answers

- 1. a)  $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ ,    b)  $3^2 + 5^2 + 7^2 + 9^2$ ,    c)  $1^2 + 3^2 + 5^2 + 7^2 + 9^2$ .
- 2.  $(\bar{x} - x_1) + (\bar{x} - x_2) + (\bar{x} - x_3)$ ,    4.  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$ ,    5.  $\frac{-1}{1} + \frac{1}{2} + \frac{-1}{3}$  which equals  $-1 + \frac{1}{2} - \frac{1}{3}$ .