Linearity rules of integration

Introduction

To enable us to find integrals of a wider range of functions than those normally given in a Table of Integrals we can make use of two rules known as linearity rules.

1. The integral of a constant multiple of a function

A constant factor in an integral can be moved outside the integral sign in the following way.

\[ \int k f(x) \, dx = k \int f(x) \, dx \]

This is only possible when \( k \) is a constant, and it multiplies some function of \( x \).

Example

Find \( \int 11x^2 \, dx \).

Solution

We are integrating a multiple of \( x^2 \). The constant factor, 11, can be moved outside the integral sign.

\[ \int 11x^2 \, dx = 11 \int x^2 \, dx = 11 \left( \frac{x^3}{3} + c \right) = \frac{11x^3}{3} + 11c \]

where \( c \) is the constant of integration. Because 11\( c \) is a constant we would normally write the answer in the form \( \frac{11x^3}{3} + K \) where \( K \) is another constant.

Example

Find \( \int -5 \cos x \, dx \).

Solution

We are integrating a multiple of \( \cos x \). The constant factor, \(-5\), can be moved outside the integral sign.

\[ \int -5 \cos x \, dx = -5 \int \cos x \, dx = -5 (\sin x + c) = -5 \sin x + K \]

where \( K \) is a constant.
2. The integral of the sum or difference of two functions

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

\[
\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx
\]
\[
\int (f(x) - g(x)) \, dx = \int f(x) \, dx - \int g(x) \, dx
\]

Example
Find \( \int (x^3 + \sin x) \, dx \).

Solution
\[
\int (x^3 + \sin x) \, dx = \int x^3 \, dx + \int \sin x \, dx = \frac{x^4}{4} - \cos x + c
\]
Note that only a single constant of integration is needed.

Example
Find \( \int e^{3x} - x^7 \, dx \).

Solution
\[
\int e^{3x} - x^7 \, dx = \int e^{3x} \, dx - \int x^7 \, dx = \frac{e^{3x}}{3} - \frac{x^8}{8} + c
\]

Exercises
1. a) Find \( \int 8x^5 + 3x^2 \, dx \), b) \( \int \frac{2}{3}x \, dx \).
2. Find \( \int 3 \cos x + 7x^3 \, dx \).
3. Find \( \int 7x^{-2} \, dx \).
4. Find \( \int \frac{5}{x} \, dx \).
5. Find \( \int \frac{x + \cos 2x}{3} \, dx \).
6. Find \( \int 5e^{4x} \, dx \).
7. Find \( \int \frac{e^x - e^{-x}}{2} \, dx \).

Answers
1. a) \( \frac{8x^6}{6} + x^3 + c \), b) \( \frac{1}{3}x^2 + c \),
2. \( 3 \sin x + \frac{7x^4}{4} + c \),
3. \(-\frac{7}{x} + c \),
4. \( 5 \log_e |x| + c \),
5. \( \frac{x^2}{6} + \frac{\sin 2x}{6} + c \),
6. \( \frac{5e^{4x}}{4} + c \),
7. \( \frac{e^x + e^{-x}}{2} + c \).