The polar form

Introduction.

From an Argand diagram the modulus and the argument of a complex number, can be defined. These provide an alternative way of describing complex numbers, known as the polar form. This leaflet explains how to find the modulus and argument.

1. The modulus and argument of a complex number.

The Argand diagram below shows the complex number $z = a + bj$. The distance of the point $(a, b)$ from the origin is called the modulus, or magnitude of the complex number and has the symbol $r$. Alternatively, $r$ is written as $|z|$. The modulus is never negative. The modulus can be found using Pythagoras' theorem, that is

$$|z| = r = \sqrt{a^2 + b^2}$$

The angle between the positive $x$ axis and a line joining $(a, b)$ to the origin is called the argument of the complex number. It is abbreviated to $\text{arg}(z)$ and has been given the symbol $\theta$.

We usually measure $\theta$ so that it lies between $-\pi$ and $\pi$, (that is between $-180^\circ$ and $180^\circ$). Angles measured anticlockwise from the positive $x$ axis are conventionally positive, whereas angles measured clockwise are negative. Knowing values for $a$ and $b$, trigonometry can be used to determine $\theta$. Specifically,

$$\tan \theta = \frac{b}{a} \quad \text{so that} \quad \theta = \tan^{-1} \left( \frac{b}{a} \right)$$

but care must be taken when using a calculator to find an inverse tangent that the solution obtained is in the correct quadrant. Drawing an Argand diagram will always help to identify the correct quadrant. The position of a complex number is uniquely determined by giving its modulus and argument. This description is known as the polar form. When the modulus and argument of a complex number, $z$, are known we write the complex number as $z = r \angle \theta$.

Polar form of a complex number with modulus $r$ and argument $\theta$:

$$z = r \angle \theta$$
Example
Plot the following complex numbers on an Argand diagram and find their moduli.

a) $z_1 = 3 + 4j$,  b) $z_2 = -2 + j$,  c) $z_3 = 3j$

Solution
The complex numbers are shown in the figure below. In each case we can use Pythagoras’ theorem to find the modulus.

a) $|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$,  
b) $|z_2| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ or 2.236,  
c) $|z_3| = \sqrt{3^2 + 0^2} = 3$.

Example
Find the arguments of the complex numbers in the previous example.

Solution
a) $z_1 = 3 + 4j$ is in the first quadrant. Its argument is given by $\theta = \tan^{-1} \frac{4}{3}$. Using a calculator we find $\theta = 0.927$ radians, or 53.13°.

b) $z_2 = -2 + j$ is in the second quadrant. To find its argument we seek an angle, $\theta$, in the second quadrant such that $\tan \theta = \frac{1}{-2}$. To calculate this correctly it may help to refer to the figure below in which $\alpha$ is an acute angle with $\tan \alpha = \frac{1}{2}$. From a calculator $\alpha = 0.464$ and so $\theta = \pi - 0.464 = 2.678$ radians. In degrees, $\alpha = 26.57^\circ$ so that $\theta = 180^\circ - 26.57^\circ = 153.43^\circ$.

c) $z_3 = 3j$ is purely imaginary. Its argument is $\frac{\pi}{2}$, or 90°.

Exercises
1. Plot the following complex numbers on an Argand diagram and find their moduli and arguments.

a) $z = 9$,  b) $z = -5$,  c) $z = 1 + 2j$,  d) $z = -1 - j$,  e) $z = 8j$,  f) $-5j$.

Answers
1. a) $|z| = 9$, $\arg(z) = 0$,  
b) $|z| = 5$, $\arg(z) = \pi$, or 180°,  
c) $|z| = \sqrt{5}$, $\arg(z) = 1.107$ or 63.43°,  
d) $|z| = \sqrt{2}$, $\arg(z) = \frac{-3\pi}{4}$ or $-135^\circ$,  
e) $|z| = 8$, $\arg(z) = \frac{\pi}{2}$ or 90°,  
f) $|z| = 5$, $\arg(z) = -\frac{\pi}{2}$ or $-90^\circ$.  

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