Introduction

Sigma notation, \( \sum \), provides a concise and convenient way of writing long sums. This leaflet explains how.

1. Sigma notation

The sum
\[
1 + 2 + 3 + 4 + 5 + \ldots + 10 + 11 + 12
\]
can be written very concisely using the capital Greek letter \( \sum \) as
\[
\sum_{k=1}^{12} k
\]
The \( \sum \) stands for a sum, in this case the sum of all the values of \( k \) as \( k \) ranges through all whole numbers from 1 to 12. Note that the lower-most and upper-most values of \( k \) are written at the bottom and top of the sigma sign respectively. You may also see this written as \( \sum_{k=1}^{12} k \), or even as \( \sum_{k=1}^{12} k \).

Example
Write out explicitly what is meant by
\[
\sum_{k=1}^{5} k^3
\]
Solution
We must let \( k \) range from 1 to 5, cube each value of \( k \), and add the results:
\[
\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3
\]

Example
Express \( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \) concisely using sigma notation.

Solution
Each term takes the form \( \frac{1}{k} \) where \( k \) varies from 1 to 4. In sigma notation we could write this as
\[
\sum_{k=1}^{4} \frac{1}{k}
\]
Example
The sum

\[ x_1 + x_2 + x_3 + x_4 + \ldots + x_{19} + x_{20} \]

can be written

\[ \sum_{k=1}^{20} x_k \]

There is nothing special about using the letter \( k \). For example

\[ \sum_{n=1}^{7} n^2 \]  stands for  \( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \)

We can also use a little trick to alternate the signs of the numbers between + and −. Note that \((-1)^2 = 1, (-1)^3 = -1\) and so on.

Example
Write out fully what is meant by

\[ \sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i + 1} \]

Solution

\[ \sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i + 1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} \]

Exercises

1. Write out fully what is meant by
   a) \( \sum_{i=1}^{5} i^2 \)
   b) \( \sum_{k=1}^{4} (2k + 1)^2 \)
   c) \( \sum_{k=0}^{3} (2k + 1)^2 \)

2. Write out fully what is meant by

\[ \sum_{k=1}^{3} (\bar{x} - x_k) \]

3. Sigma notation is often used in statistical calculations. For example the **mean**, \( \bar{x} \), of the \( n \) quantities \( x_1, x_2 \ldots \text{and} x_n \), is found by adding them up and dividing the result by \( n \). Show that the mean can be written as

\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \]

4. Write out fully what is meant by \( \sum_{i=1}^{4} \frac{i}{i+1} \).

5. Write out fully what is meant by \( \sum_{k=1}^{3} \frac{(-1)^k}{k} \).

Answers

1. a) \( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \), \hspace{1cm} b) \( 3^2 + 5^2 + 7^2 + 9^2 \), \hspace{1cm} c) \( 1^2 + 3^2 + 5^2 + 7^2 + 9^2 \).

2. \( (\bar{x} - x_1) + (\bar{x} - x_2) + (\bar{x} - x_3) \), \hspace{1cm} 4. \( \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \), \hspace{1cm} 5. \( \frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} \) which equals \(-1 + \frac{1}{2} - \frac{1}{3} \).